Insecurity of Quantum Secure Computations

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Abstract

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I. INTRODUCTION

Copying of an unknown quantum state (by for example an eavesdropper) is strictly forbidden by the linearity of quantum mechanics [1]. Consequently, quantum cryptography[1] (or more precisely quantum key distribution [3–7]) allows two users to share a common random secret string of information which can then be used to make their subsequent communications totally unintelligible to an eavesdropper. In this paper we are, however, concerned with another class of applications of quantum cryptography—the protection of private information during public decision [8,9]. For instance, two millionaires may be interested in knowing who is richer but neither wishes to disclose the precise amount of money that he/she has. More generally, in a *one-sided* two-party computation, Alice has a private input i and Bob a private input j. Alice would like to help Bob to compute a prescribed function $f(i,j)$ without revealing anything about i more than what is logically necessary. (For a precise definition of a one-sided two-party computation, see Section 2.) In classical cryptography, such two-party computations can be made secure only either 1) through trusted intermediaries or 2) by accepting some unproven cryptographic assumptions.[2] The impossibility of *unconditionally* secure two-party computation in *classical* cryptography had led to much interest in *quantum* cryptographic protocols [2,5,11–18] which are supposed to be unconditionally secure [16–18].

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[1]Quantum Cryptography was first proposed by Wiesner [2] in about 1970 in a manuscript that remained unpublished until 1983.

[2]In the first case, if both Alice and Bob trust Charles, they can simply tell him their private inputs and let Charles perform the computation for them and tell them the result afterwards. The problem here is that Charles can cheat by telling either Alice or Bob the other party's private input. In the second case, assumptions such as the hardness of factoring can be used. However, an adversary with unlimited computing power (or with a quantum computer [10]) can defeat such unproven cryptographic assumptions.

An important primitive in secure computation is the so-called bit commitment.[3] The optimism in unconditional secure quantum two-party computation was largely contributed by well-known claims of unconditional secure quantum bit commitment protocols [16] (and also oblivious transfer [17,18]). However, such optimism has recently been put into serious question due to the surprising demonstration of the insecurity of quantum bit commitment (against an EPR-type of attack with delayed measurements) by Mayers [20,21] and also by Lo and Chau [22,23]. Yet the important question remains: Other than quantum key distribution, can quantum protocols, in particular, two-party computation, be unconditionally secure at all? This is an important question because, in many cases, quantum bit commitment might be thought of as a means to an end—two party secure computation. If secure quantum two-party computation is possible, many applications of quantum cryptography, such as the prevention of frauds due to typing PIN (Personal Identification Number) to dishonest teller machine mentioned in the abstract, will still survive.

Amazingly, one possible viewpoint to take is that there is really nothing to prove because the standard reduction theorems [19,24,25][4] in classical cryptography immediately imply that quantum one-sided two-party computation is impossible: In classical cryptography, an example of one-sided two party computation is one-out-of-two oblivious transfer, which can be used to implement bit commitment. If bit commitment is impossible, one-sided two-party computations must also generally be impossible. Doubt has been expressed in the literature

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[3]It might be useful to note that Yao [18] has shown that any secure quantum bit commitment scheme can be used to implement secure quantum oblivious transfer whereas Kilian [19] has shown that, in classical cryptography, oblivious transfer can be used to implement two-party secure computation. Therefore, this chain of argument appears to suggest that, with quantum bit commitment, quantum cryptography could achieve unconditionally secure two-party computation, thus solving a long standing problem in cryptography.

[4]We thank G. Brassard for helpful discussions about those standard reduction theorems.

concerning the validity of this standard reduction in a quantum model [9,21]. Here we argue that by definition the standard reduction must apply to quantum cryptographic protocols: Bit commitment, oblivious transfer and two-party computations are classical concepts with their security requirements defined in a classical probabilistic language. If there is any sense at all in saying that a quantum protocol can achieve say two-party computation, it is *a matter of definition* that the quantum protocol has to satisfy the classical probabilistic security requirements under all circumstances. In particular, one must be allowed to use a quantum cryptographic protocol as a "black box" primitive in building up more sophisticated protocols and analyze the security of those new protocols with *classical* probability theory.[5]

By adapting this new and, in our opinion, more accurate definition of secure quantum protocols, we see that the impossibility of quantum bit commitment immediately implies the impossibility of quantum one-sided two-party computations (and one-out-of-two oblivious transfer as well as oblivious transfer) and this is the end of the story.

Yet such an ending is disappointing in two aspects. While such a viewpoint is conceptually correct, it is a bit formal and non-constructive. A constructive proof would make things more transparent. A more serious objection is that while such an argument rules out one-out-of-two oblivious transfer and the two-party computation of a general function, there remains the possibility that *some* special class of functions (whose two-party computations cannot be used to implement one-out-of-two oblivious transfer[6]) might still be computed

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[5]One may get the feeling from reading the literature that a quantum protocol should be regarded as secure if it appears to satisfy its security requirements when it is executed *only once* and *in isolation*. This, however, does *not* guarantee that it satisfies the security requirements when it is used as a subroutine of a larger routine because a cheater might defeat the security of the larger routine by performing coherent measurements. Therefore, we think that a more accurate definition of a secure quantum protocol should be much more stringent.

[6]According to Kilian, such functions do exist.

securely in one-sided two-party computations. Here we investigate directly the security of one-sided two-party computation without using the formal standard reduction. Our main result is that one-sided quantum two party secure computation is always impossible.[7] (For its definition, see Section 2.) That is to say that, as far as one-sided two-party computations are concerned, quantum cryptography is absolutely useless. As a corollary, the so-called quantum one-out-of-two oblivious transfer is also impossible. We also present a class of functions that *cannot* be computed in any *two-sided* two-party computation.

II. IDEAL ONE-SIDED TWO-PARTY SECURE COMPUTATION

A. Definition and Security Requirements

Suppose Alice has a private (i.e., secret) input $i \in \{1, 2, \cdots, n\}$ and Bob has a private input $j \in \{1, 2, \cdots, m\}$. An *ideal* one-sided two party secure computation is defined as follows: Alice helps Bob to compute a prescribed function $f(i, j) \in \{1, 2, \cdots, p\}$ in such a way that at the end of the protocol,

(a) Bob learns $f(i, j)$ unambiguously,

(b) Alice learns nothing (about j or $f(i, j)$),

and

(c) Bob knows nothing about i more than what logically follows from the values of j and $f(i, j)$.

Notice that, for a one-sided two-party computation protocol to be secure, Bob is supposed to input a *particular* value of j and to learn the value of $f(i, j)$ for that particular value of

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[7]Remarkably, an alternative proof of the impossibility of *ideal* quantum one-sided two party computation can be made by generalizing Wiesner's [2] early insight on the impossibility of one-way scheme for so-called one-out-of-two oblivious transfer and combining it with the idea of the proof of the impossibility of quantum bit commitment. See Appendix.

j *only*. We show that these three security requirements (a), (b) and (c) are incompatible in the following manner: Assuming that the first two security requirements (a) and (b) are satisfied, we work out a cheating strategy for Bob which would allow him to learn the values of $f(i,j)$ for *all* j's, thus violating security requirement (c).[8] We conclude that ideal quantum one-sided two-party computations are impossible. In Section 4, we will generalize this result to non-ideal protocols (which may violate security requirements (a) and (b) slightly).

B. Bob's cheating strategy

Consider the following cheating strategy by Bob who determines the values of $f(i,j_1), f(i,j_2), \cdots, f(i,j_m)$ successively: Bob first inputs a value j_1 for j and goes through the protocol. At the end of the protocol, he determines the value of $f(i,j_1)$. He then applies a unitary transformation to change the value of j from j_1 to j_2 and determines $f(i,j_2)$. After that, Bob applies a unitary transformation to change j from j_2 to j_3 and determines $f(i,j_3)$ and so on.

C. Key Points of the Proof

The above cheating strategy by Bob works for two reasons. First, using the insight gained from the impossibility of quantum bit commitment [20–23], in Subsection 3B we prove the following: The security requirement (b)—that Alice knows nothing about j—implies that at the end of the protocol, Bob can cheat by changing the value of j from j_1 to j_2 by applying a unitary transformation to his own quantum machine. Consequently, Bob can determine the value of $f(i,j_2)$ instead of $f(i,j_1)$, as long as he has *not* measured $f(i,j_1)$ yet. Of course, Bob would like to learn $f(i,j_1)$ and he *does* measure $f(i,j_1)$ before rotating j_1 to j_2. At first

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[8] A protocol that allows Bob to learn $f(i,j)$ for *all* j's is uninteresting as it can be achieved in classical cryptography simply by having Alice tell Bob those values.

sight, this seems to be a problem because measurements in quantum mechanics generally disturb a signal. Here comes our second point. Measurement of $f(i, j_1)$ does not disturb Bob's state at all for the following reason. Since, by the security requirement (a) of an ideal protocol, Bob can input $j = j_1$ and learn the value of $f(i, j_1)$ unambiguously, the density matrix that Bob has must be an eigenstate of the measurement operator that he uses for determining $f(i, j_1)$. Being an eigenstate, the density matrix is, therefore, undisturbed by Bob's measurement. QED

In effect, we are arguing that the density matrix Bob has is a simultaneous eigenstate of the measurement operators $f(i, j_1), f(i, j_2), \cdots, f(i, j_m)$. See Subsection 3B.

III. DETAILS OF THE PROOF

A. Unitary description

Let us present our result in more detail. It is convenient to use a unitary description of two-party computation [21,23]. Let H_A (H_B respectively) denote the Hilbert space of Alice's (Bob's) quantum machine. Imagine a two-party computation in which both Alice and Bob possess quantum computers and quantum storage devices. By maintaining the quantum coherence of the composite quantum system, $H_A \otimes H_B$, (using external control such as classical computers, assembling of quantum gate arrays and quantum error correction) one can avoid dealing with the collapse of the wavefunction. Alice and Bob's actions on their quantum machines can be summarized[9] as an overall unitary transformation U applied to

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[9]For the basic idea, see [21]. For detailed justification with a concrete model (a variant of Yao's model [18]) see [23]. Of course, in reality the execution of the protocol may not require quantum computers. This is, however, equivalent to a situation when the parties do not make full use of their quantum computers. If we can show that a cheater can cheat successfully against an honest party who has a quantum computer, clearly the cheater can cheat successfully against one without.

the initial state $|u\rangle_{in} \in H_A \otimes H_B$. i.e.,

$$|u\rangle_{fin} = U|u\rangle_{in}. \tag{1}$$

The unitary transformation, U, is known to both Alice and Bob because they know the procedure of the protocol. When both parties are honest, $|u^h\rangle_{in} = |i\rangle_A \otimes |j\rangle_B$ and

$$|u^h\rangle_{fin} = |v_{ij}\rangle \equiv U\left(|i\rangle_A \otimes |j\rangle_B\right). \tag{2}$$

Therefore, the density matrix that Bob has at the end of protocol is simply

$$\rho^{i,j} = \mathrm{Tr}_A |v_{ij}\rangle\langle v_{ij}|. \tag{3}$$

B. Changing j from j_1 to j_2

We have asserted in the last section that, owing to the security requirement (b), at the end of the protocol Bob can change the value of j from j_1 to j_2 by applying a unitary transformation to the state of his quantum machine. Since the value of Alice's input i is unknown to Bob, for such a cheating strategy to work, we need to prove that this unitary transformation can be chosen to be independent of the value of i[10]:

Assertion: Given $j_1, j_2 \in \{1, 2, \cdots, m\}$, there exists a unitary transformation U^{j_1,j_2} such that

$$U^{j_1,j_2} \rho^{i,j_1} \left(U^{j_1,j_2}\right)^{-1} = \rho^{i,j_2} \tag{4}$$

for *all* i.

Therefore, a unitary description is very useful for our purposes.

[10]Using the idea of the impossibility of bit commitment [20–23], it is trivial to prove that, for *each* i, a unitary transformation U^{i,j_1,j_2} that rotates j from j_1 to j_2 exists. What is less trivial to prove is the existence of a unitary transformation U^{j_1,j_2} which works for *all* $[i]$ simultaneously. We thank D. Mayers for enlightening discussions.

Proof: Notice that Bob must allow Alice to choose the value of her input, i, randomly. But then a dishonest Alice may try to learn about j by an EPR-type of attack. i.e., she entangles the state of her quantum machine A with her quantum dice D and prepares the initial state

$$\frac{1}{\sqrt{n}} \sum_i |i\rangle_D \otimes |i\rangle_A. \tag{5}$$

(Recall that n is the cardinality of i.) Instead of measuring the state of her quantum dice D honestly, she may keep D for herself and use the second register, A, to execute the two party protocol honestly from this point on. Suppose Bob's input is j_1. The initial state is, therefore,

$$|u'\rangle_{in} = \frac{1}{\sqrt{n}} \sum_i |i\rangle_D \otimes |i\rangle_A \otimes |j_1\rangle_B. \tag{6}$$

At the end of the protocol, it follows from Eqs. (1) and (6) that the total wave function of the combined system D, A and B is described by

$$|v_{j_1}\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle_D \otimes U\left(|i\rangle_A \otimes |j_1\rangle_B\right). \tag{7}$$

Similarly, if Bob's input is j_2 instead, the total wavefunction at the end of the protocol will be

$$|v_{j_2}\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle_D \otimes U\left(|i\rangle_A \otimes |j_2\rangle_B\right). \tag{8}$$

An ideal protocol should prevent such a dishonest Alice from learning anything about j. Therefore, the reduced density matrices in Alice's hand for the two cases $j = j_1$ and $j = j_2$ must be the same, i.e.,

$$\rho_{j_1}^{Alice} = \mathrm{Tr}_B |v_{j_1}\rangle\langle v_{j_1}| = \mathrm{Tr}_B |v_{j_2}\rangle\langle v_{j_2}| = \rho_{j_2}^{Alice}. \tag{9}$$

Equivalently, the two wavefunctions, $|v_{j_1}\rangle$ and $|v_{j_2}\rangle$ have the same Schmidt decomposition [27]. i.e.,

$$|v_{j_1}\rangle = \sum_k a_k |\alpha_k\rangle_{AD} \otimes |\beta_k\rangle_B \tag{10}$$

and

$$|v_{j_2}\rangle = \sum_k a_k |\alpha_k\rangle_{AD} \otimes |\beta'_k\rangle_B. \tag{11}$$

Here $|\alpha_k\rangle_{AD}$, $|\beta_k\rangle_B$ and $|\beta'_k\rangle_B$ are eigenvectors of the corresponding density matrices and satisfy $\langle\alpha_{k'}|\alpha_k\rangle_{AD} = \delta_{k,k'}$, etc. Notice that Eqs. (10) and (11) contain the same factors a_k and $|\alpha_k\rangle_{AD}$ and the only difference lies in the state of Bob's quantum machine, B. Now, consider the unitary transformation U^{j_1,j_2} that rotates $|\beta_k\rangle_B$ to $|\beta'_k\rangle_B$. Notice that it acts on H_B *alone* and yet, as can be seen from Eqs. (10) and (11), it rotates $|v_{j_1}\rangle$ to $|v_{j_2}\rangle$. i.e.,

$$|v_{j_2}\rangle = U^{j_1,j_2}|v_{j_1}\rangle. \tag{12}$$

Since

$${}_D\langle i|v_j\rangle = \frac{1}{\sqrt{n}}|v_{ij}\rangle \tag{13}$$

(see Eqs. (2), (7) and (8)), by multiplying Eq. (12) by ${}_D\langle i|$ on the left, we find that

$$|v_{ij_2}\rangle = U^{j_1,j_2}|v_{ij_1}\rangle. \tag{14}$$

As we are interested in Bob's reduced density matrix, we take the trace of $|v_{ij_2}\rangle\langle v_{ij_2}|$ over H_A and use Eq. (14) to obtain Eq. (4). This completes the proof of our assertion, Eq. (4).

The implication of Eq. (4) is profound. Independent of the value of Alice's private input, i, at the end of the protocol Bob can change the value of his own input j simply by applying a unitary transformation to his own quantum machine.[11] Therefore, the index j in Bob's density matrix $\rho^{i,j}$ is redundant in the sense that different values of j simply correspond to representing the density matrix ρ^i in different bases.

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[11] A similar idea is used in the proof of the impossibility of bit commitment [20–23]. That Alice knows nothing about Bob's chosen bit automatically implies that Bob can cheat successfully by applying a unitary transformation to change the value of the bit even after the completion of the commitment phase. Thus, the commitment is fake.

With such a simplification, one can essentially argue that ρ^i is a simultaneous eigenstate of $f(i,j_1), f(i,j_2), \cdots, f(i,j_m)$ in the following manner: With an input j_1, Bob can learn $f(i,j_1)$. This implies that ρ^i is an eigenstate of $f(i,j_1)$. But Bob can cheat by changing the value of j from j_1 to j_2 in the last minute to learn $f(i,j_2)$ instead. This means that ρ^i is also an eigenstate of $f(i,j_2)$. By repeating this argument, one sees clearly that ρ^i is a simultaneous eigenstate of all the measuring operators for $f(i,j_1), f(i,j_2), \cdots, f(i,j_m)$. Consequently, Bob can learn the values of $f(i,j)$ for all values of j simultaneously. This is why the cheating strategy that we describe in Subsection 2B works. In the next Section, we generalize our attack to non-ideal protocols.

IV. NON-IDEAL PROTOCOLS

A general non-ideal protocol may violate the security requirements (a) and (b) slightly. In relaxing (b), one would expect that the unitary transformations that Bob uses for changing j from j_i to j_{i+1} would be imperfect. In relaxing (a), the density matrix that Bob has at the end of the protocol will now be slightly different from an eigenstate of the measurement operator that he uses. (This is because Bob will generally be unable to determine the value of $f(i,j_1)$ unambiguously in non-ideal protocols.) Nonetheless, so long as the deviation from idealness is small, one would expect Bob to learn a substantial amount of information about $f(i,j_2)$ even after his honest determination of $f(i,j_1)$. That Bob can learn something about both $f(i,j_1)$ and $f(i,j_2)$ is already a serious violation of the security requirement (c) and there is no need for us to consider the security for $f(i,j_3)$, etc. In other words, one would expect that, for essentially the same reason as the ideal protocol, even non-ideal quantum one-sided two-party computation is impossible. In what follows, we prove that this is indeed the case. Readers who are uninterested in technical details may skip the following and go directly to Subsection A.

More concretely, let us relax security requirement (b) to allow Alice to have a small probability to distinguish between different j's. We mimic the proof of Eq. (4). As before,

consider a dishonest Alice who tries to learn about j by preparing an illegal initial state $\frac{1}{\sqrt{n}} \sum_i |i\rangle_D \otimes |i\rangle_A$ where n is the cardinality of i. She keeps the first register, D, for herself and uses the second register, A, to execute the two party protocol honestly from this point on. Unlike the ideal case, Eq. (9) is violated for non-ideal protocols. i.e., $\rho_{j_1}^{Alice} \neq \rho_{j_2}^{Alice}$. Nonetheless, so long as the probability for Alice to distinguish successfully between the two cases remains small, the two density matrices $\rho_{j_1}^{Alice}$ and $\rho_{j_2}^{Alice}$ must in some sense be close to each other.

Mathematically, the closeness between two density matrices ρ and ρ' of a system S can be described by the *fidelity* [29]. (See also Ref. [28].) Imagine another system E attached to our given system S. There are many pure states $|\psi\rangle$ and $|\psi'\rangle$ on the composite system that satisfy

$$\mathrm{Tr}_E\left(|\psi\rangle\langle\psi|\right) = \rho \qquad \text{and} \qquad \mathrm{Tr}_E\left(|\psi'\rangle\langle\psi'|\right) = \rho'. \tag{15}$$

The pure states $|\psi\rangle$ and $|\psi'\rangle$ are called the *purifications* of the density matrices ρ and ρ'. The fidelity $F(\rho, \rho')$ can be defined as

$$F(\rho, \rho') = max|\langle\psi|\psi'\rangle| \tag{16}$$

where the maximization is over all possible purifications. We remark that[12] for any fixed purification ψ of ρ, there exists a maximally parallel purification ψ' of ρ' that satisfies Eq. (16). Notice that $0 \leq F \leq 1$ and $F = 1$ if and only if $\rho = \rho'$.

Returning to our discussion on non-ideal protocols, the condition that the two density matrices $\rho_{j_1}^{Alice}$ and $\rho_{j_2}^{Alice}$ be close to each other can be specified by the mathematical statement that the fidelity $F(\rho_{j_1}^{Alice}, \rho_{j_2}^{Alice})$ is close to 1. Say

$$F(\rho_{j_1}^{Alice}, \rho_{j_2}^{Alice}) > 1 - \delta \tag{17}$$

where $\delta \ll 1$. It is a common requirement in computer science that

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[12]We thank R. Jozsa for a discussion about this point.

$$\delta \sim e^{-cn} \tag{18}$$

for some $c > 0$. (Recall that n is the cardinality of i.) It follows from the definition of fidelity in Eq. (16) that there exists a unitary transformation U^{j_1,j_2} acting on H_B alone[13] such that

$$\left|\langle v_{j_2}|U^{j_1,j_2}|v_{j_1}\rangle\right| > 1-\delta. \tag{19}$$

Since (from Eqs. (2), (7) and (8)) $|v_j\rangle = \frac{1}{\sqrt{n}}\sum_i |i\rangle \otimes |v_{ij}\rangle$,

$$\left|\langle v_{j_2}|U^{j_1,j_2}|v_{j_1}\rangle\right| = \frac{1}{n}\left|\sum_i \langle v_{ij_2}|U^{j_1,j_2}|v_{ij_1}\rangle\right| > 1-\delta. \tag{20}$$

This implies that, for each i,

$$\left|\langle v_{ij_2}|U^{j_1,j_2}|v_{ij_1}\rangle\right| > 1-n\delta \tag{21}$$

Since we are interested in Bob's density matrix, we take the trace over Alice's quantum machine A and find that

$$F\left(U^{j_1,j_2}\rho^{i,j_1}\left(U^{j_1,j_2}\right)^{-1}, \rho^{ij_2}\right) > 1-n\delta. \tag{22}$$

We now come to the relaxation of security requirement (a). Bob still chooses a j say j_1 and performs a measurement on his quantum state in order to learn the value of $f(i,j_1)$. However, for a non-ideal protocol, Bob's measurement result will not give him full information on $f(i,j_1)$. Nonetheless, for a protocol that is only slightly non-ideal, one may demand that, for each i, Bob's ignorance about $f(i,j_i)$ after his measurement would be much less than one bit. That is to say that Bob's measurement can extract the value of $f(i,j_1)$ from the density matrix ρ^{i,j_1} with a probability close to 1. Therefore, ρ^{i,j_1} can be made to be almost an eigenstate of Bob's measurement and thus the disturbance caused by such measurement is small. Consequently, we must have

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[13]A similar idea was used by Mayers [20] in the discussion of non-ideal bit commitment schemes.

$$F\left(\rho^{i,j_1}, \mathcal{E}\left(\rho^{i,j_1}\right)\right) > 1 - \delta \tag{23}$$

where $\mathcal{E}$ is a linear operator (the so-called super-operator [30]) which represents the action of the (imperfect) measurement of $f(i, j_1)$ by Bob. Since fidelity is preserved by unitary transformations, we have

$$F\left(U^{j_1,j_2}\rho^{i,j_1}\left(U^{j_1,j_2}\right)^{-1}, U^{j_1,j_2}\mathcal{E}\left(\rho^{i,j_1}\right)\left(U^{j_1,j_2}\right)^{-1}\right) > 1 - \delta. \tag{24}$$

From Eqs. (22) and (24), we deduce[14] that

$$F\left(U^{j_1,j_2}\mathcal{E}\left(\rho^{i,j_1}\right)\left(U^{j_1,j_2}\right)^{-1},\ \rho^{i,j_2}\right) > 1 - O(n\delta) = 1 - O(ne^{-cn}), \tag{25}$$

where we have used Eq. (18). Since $ne^{-cn} \ll 1$, the high fidelity of Eq. (25) implies that Bob's cheating strategy—of determining $f(i, j_1)$ approximately first, applying a rotation to his state to change j from j_1 to j_2 and then determining $f(i, j_2)$—will allow him to defeat the security requirement (c) of the protocol by learning substantial information about $f(i, j_2)$. Therefore, even non-ideal protocols are unsafe.

A. Corollaries

Definition: *One-out-of-two oblivious transfer* is an example of two party secure computation in which the sender sends two messages and the receiver chooses to receive either message but cannot read both. More precisely, Alice's input, i, is a pair of messages, (m_0, m_1) and Bob's input, j, is a bit 0 or 1. At the end of the protocol, Bob learns about the message m_j, but not the other message $m_{\bar{j}}$. i.e., $f(m_0, m_1, j = 0) = m_0$ and $f(m_0, m_1, j = 1) = m_1$.

Corollary 1: Quantum one-out-of-two oblivious transfer is impossible.

Remark: This corollary is a generalization of Wiesner's insight [2] which showed that it is impossible to achieve *ideal* quantum one-out-of-two oblivious transfer using only *one-way* communications.

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[14]This follows from the fact that the fidelity is closely related [29] to the Bures metric.

Incidentally, there have been claims that quantum cryptography is useful for one-way oblivious identification [14,15]. Such a protocol would allow the first user Alice to identify herself in front of a second user, Bob, by means of a password, known only to both. The safety requirement is that somebody impersonating Bob, who only pretends to know Alice's password, shall not be able to obtain much information on the password from the exchange. One-way oblivious identification is an example of one-sided two-party secure computation in which the prescribed function $f(i,j) = 1$ if $i = j$ and $f(i,j) = 0$ otherwise. In other words, $f(i,j)$ gives a yes/no answer to the question whether the two persons have the same password. Such oblivious identification scheme is, therefore, very useful for preventing frauds from typing PIN (Personal Identificaton Number) to a dishonest teller machine that steals passwords.

Corollary 2: Quantum one-way oblivious identification is impossible.

V. SECURITY OF TWO-SIDED TWO-PARTY COMPUTATIONS

Definition: Suppose Alice has a private input i and Bob a private input j. A *two-sided two-party secure computation* of a prescribed function $f(i,j)$ is a protocol such that at the end,

(a) both Alice and Bob learn $f(i,j)$,

(b) Alice learns nothing about j more than what logically follows from $f(i,j)$ and her private input i, and

(c) Bob learns nothing about i more than what logically follows from $f(i,j)$ and his private input j.

Notice that in classical cryptography, a one-sided two-party computation of a function $f(i,j)$ can be reduced to a two-sided two-party computation of a function $F(i,j,r) = f(i,j) \ \ XOR \ \ r$ where r is a random string of input chosen by Bob and the XOR is taken

bitwise.[15] At the end of the protocol, both Alice and Bob learn $F(i,j,r)$. While Bob can invert the function to find $f(i,j) = F(i,j,r)\ \ XOR\ \ r$, Alice, being ignorant of Bob's input r, has absolutely no information about $f(i,j)$.

Here we demonstrate explicitly that the quantum two-sided two-party computation of $F(i,j,r)$ is insecure. Alice's density matrix at the end of the protocol should only be a function of i and $F(i,j,r)$. This is because $F(i,j,r)$ is the only piece of information that Alice is supposed to know about Bob's inputs j and r. Let us therefore denote Alice's density matrix by $\rho_{Alice}^{i,F(i,j,r)}$. Suppose a dishonest Bob inputs $|j_1\rangle \otimes \frac{1}{p^{1/2}} \sum_r |r\rangle \otimes |r\rangle_D$ and he keeps the system D for himself. (Here p is the cardinality of $f(i,j)$, as $f(i,j) \in \{1,2,\cdots,p\}$.) In other words, he entangles the state of r with a quantum dice D and performs an EPR-type of cheating. Suppose further that a honest Alice inputs i. The density matrix that Alice has at the end of the protocol will simply be a normalized direct sum, $\frac{1}{p}\sum_r \rho_{Alice}^{i,F(i,j_1,r)}$, of the individual density matrices. For *any* fixed but arbitrary j, as r changes, $F(i,j,r)$ runs over all the p values, $\{1,2,\cdots,p\}$. (Recall that $F(i,j,r) = f(i,j)\ \ XOR\ \ r$.) Consequently, $\frac{1}{p}\sum_r \rho_{Alice}^{i,F(i,j_1,r)} = \frac{1}{p}\sum_r \rho_{Alice}^{i,F(i,j_2,r)}$. i.e., Alice's density matrix is *independent* of the value of j. But then by precisely the same attack as in the one-sided case—by determining the value of $f(i,j_1)$, changing j from j_1 to j_2 by a unitary transformation, determining the value of $f(i,j_2)$ and so on, Bob can determine the value of $f(i,j)$ for all values of j. This violates the security requirement (c) for the two-sided protocol. In conclusion, there are functions, namely $F(i,j_1,r) = f(i,j)\ \ XOR\ \ r$, that cannot be computed securely by any two-sided protocol.

VI. SUMMARIES AND DISCUSSIONS

This paper deals with the applications of quantum cryptography in the protection of private information during public decision (rather than with the most well-known application—

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[15]We thank R. Cleve for enlightening discussions about this point.

so-called quantum key distribution). As an important example, in a one-sided two-party secure computation, one party Alice has a private input, i, and the other party Bob who has a private input, j. Alice helps Bob to compute a prescribed function $f(i,j)$ in such a way that at the end of the protocol,

(a) Bob learns $f(i,j)$,

(b) Alice learns nothing (or almost nothing) about j,

and

(c) Bob knows nothing about i more than what logically follows from the value of j and $f(i,j)$.

(For example, in password identification $f(i,j) = 1$ if $i = j$ and $= 0$ otherwise.) Notice that Bob is supposed to choose a j (say j_1) and learn $f(i,j)$ for that particular value of j only. However, we prove that quantum one-sided two-party computation is always insecure because Bob can learn $f(i,j)$ for *all* values of j. In the cheating strategy that we consider, Bob determines the values of $f(i,j)$ for the various values of j's successively.[16] That is to say that Bob inputs $j = j_1$, determines the value of $f(i,j_1)$, changes j to j_2 and determines $f(i,j_2)$ and so on.

Such a cheating strategy works for two reasons. For simplicity, let us first consider the ideal protocol. Let Bob input $j = j_1$ initially. Using the insight from the impossibility of bit commitment [20–23], we prove that, owing to the security requirement (b), Bob can cheat at the end of the protocol by changing the value of j from j_1 to j_2. Thus he can determine the value of $f(i,j_2)$ instead of $f(i,j_1)$ as long as he has *not* performed a measurement to determine $f(i,j_1)$ yet. Of course, Bob is interested in learning $f(i,j_1)$ as well. So, he must first measure the value of $f(i,j_1)$ before rotating j from j_1 to j_2. If we can show that his measurement of $f(i,j_1)$ does not disturb the quantum state he possesses, it is clear that our cheating strategy will work. This is precisely what we do: Since in an ideal protocol with an

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[16]See footnote 8.

input $j = j_1$, Bob can unambiguously determine the value of $f(i, j_1)$ (security requirement (a)), the density matrix that Bob has must be an eigenstate of the measurement operator that he uses. Consequently, he can measure the value of $f(i, j_1)$ *without* disturbing the quantum state of the signal at all! (Notice that, in effect, we have shown that owing to the security requirements (a) and (b), the density matrix that Bob has is a simultaneous eigenstate of $f(i, j_1), f(i, j_2), \cdots, f(i, j_m)$. This contradicts security requirement (c).)

These two points taken together mean that our cheating strategy beats an ideal protocol for one-sided two-party computation.[17] In Section 4, we generalize our result to show that a similar attack defeats non-ideal protocols as well. Therefore, quantum one-sided two-party secure computation (ideal or non-ideal) is always impossible. As corollaries to our results, contrary to popular belief in earlier literature, quantum one-out-of-two oblivious transfer and one-way oblivious identification are also impossible. We remark that the reduction theorem in classical cryptography can be used to show that quantum (ordinary) oblivious transfer is impossible. In future, it would be interesting to work out a direct attack that defeats quantum oblivious transfer.

Since a one-sided two-party computation of a function can be reduced to a two-sided two-party computation of a related function, there are functions that cannot be computed securely in two-sided two-party computations as well. Can *any* function be computed se-

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[17] As discussed in the introduction, one may also use the classical reduction theorem from bit commitment to one-out-of-two oblivious transfer to argue the impossibility of quantum one-sided two-party computations. Such proof is, however, not transparent at all. Yet another alternative proof of the insecurity of *ideal* quantum one-sided two-party computation will be made in Appendix A by combining the idea of the proof of the impossibility of quantum bit commitment with a generalization of Wiesner's early insight [2] on the insecurity of a subclass of quantum one-out-of-two oblivious transfer schemes. Such proof is, however, non-constructive and does not apply directly to non-ideal protocols.

curely in a quantum two-sided two-party computation? While we do not have a definite answer, the argument for impossibility of ideal quantum coin tossing [23] can be used to prove the impossibility of *ideal* two-sided two-party secure computation (and also ideal so-called zero-knowledge proof). Furthermore, Section 4 of Ref. [23] shows that quantum two-sided two-party secure computation can never be done *efficiently*[18] In conclusion, our results rule out the prefect or nearly perfect protection of private information in one-sided two-party computations by quantum mechanics. The security of the quantum two-sided two-party computation is also shown to be in very serious trouble.

In retrospect, there were good reasons for the reexamination of the foundations of quantum cryptographic protocols such as secure computation: While the security of quantum key distribution can intuitively be attributed to the quantum no-cloning theorem, no simple physical reason has ever been given to the security of other quantum cryptographic protocols such as bit commitment. This is a highly unsatisfactory situation. Besides, most proposed quantum protocols are highly inefficient. From both theoretical and practical points of view, a more fundamental understanding of the issues of security and efficiency of those protocols would therefore be most welcome. In the claimed "secure" quantum bit commitment protocol [16], researchers have implicitly assumed that measurements are made by the two parties. What we have shown is that by using a quantum computer and performing an EPR-type of attack, the party, Bob, can defeat the security requirement of the protocol. This is remarkable because an EPR-type of attack was something that Wiesner [2] noted in

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[18]Let us normalize everything so that Alice and Bob both learn one bit of information from executing a two-sided two-party computation. If both parties are shameless enough to stop running the protocol whenever one of them has an amount of information that is ϵ greater than his/her opponent, it is easy to show [23] that the number N of rounds of communications needed for the protocol to be successful has to satisfy $N\epsilon \geq 1$. An exponentially small ϵ requires an exponentially large N and the scheme is necessarily inefficient.

his pioneering paper more than two decades ago. The sky has fallen because its foundation has been shaky.

The security of other quantum cryptographic protocols such as quantum multi-party computations and non-ideal quantum coin tossing remains to be investigated. A final answer to the usefulness of quantum mechanics in those "post-cold-war" cryptographic applications is of immense conceptual and practical interests. Finally, we remark that quantum cryptography remains useful in providing prefect security in key distribution as well as *partial* security in, for example, "quantum money" proposed by Wiesner. The security of such applications is guaranteed by the quantum no-cloning theorem. It would be interesting to work out how secure quantum money really is. "The big lesson to learn from all this is that quantum information is always more elusive than its classical counterpart: extra care must be taken when reasoning about quantum cryptographic protocols and analyzing them." [9] A thorough reexamination of the foundations of the whole subject may provide us with new insights.

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APPENDIX A: ALTERNATIVE PROOF OF THE INSECURITY OF IDEAL PROTOCOLS

An alternative but non-constructive proof can be made by combining the idea of the impossibility of quantum bit commitment with a generalization of Wiesner's early insight [2] on the impossibility of one-way schemes for quantum one-out-of-two oblivious transfer (defined in Section 4). The idea is the following. Let us consider a fixed but arbitrary j_0. Given a pair of possible Alice's input, i and i', we claim that Bob's density matrices at the end of an ideal one-sided two-party computation protocol, ρ^{i,j_0} and ρ^{i',j_0}, have orthogonal supports whenever there exists a j_1 such that $f(i,j_1) \neq f(i',j_1)$. Here by the support of ρ^{i,j_0} we mean the subspace spanned by the positive eigenvectors of ρ^{i,j_0}, etc.

Proof: If Bob is lucky enough to choose j_1 as the input, since $f(i,j_1) \neq f(i',j_1)$, by security requirement (a) Bob should be able to distinguish between i and i' unambiguously in an ideal protocol. This implies that ρ^{i,j_1} and ρ^{i',j_1} have orthogonal supports [28,29]. (See also [23].) Since orthogonality is preserved by unitary transformations, in applying the unitary transformation U^{j_1,j_0} to rotate j_1 to j_0 (see Eq. (4), we see that ρ^{i,j_0} and ρ^{i',j_0} have orthogonal supports as well (whenever there exists a j_1 such that $f(i,j_i) \neq f(i',j_1)$).

It is now convenient to divide i's up into equivalence classes $[i]$ labelled by $(f(i,1), f(i,2), \cdots, f(i,m))$ and consider the support of each *class*, which is defined to be the space generated by the union of the supports of all i's in a given equivalence class $[i]$. The last paragraph implies that the supports for the various classes are mutually orthogonal. Consequently, there exists a measurement that allows Bob to distinguish unambiguously between all those supports. In other words, Bob can read off unambiguously the set $f(i,1), f(i,2), \cdots, f(i,m)$. This contradicts the security requirement (c) of the protocol—that Bob is supposed to learn $f(i,j)$ for only one value of j chosen by him.

This proof is similar in spirit to the proof given in Section 2 except that we find it less concrete and more difficult to analyze in the non-ideal case.

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