Towards an Applied Mathematics for Computer Science

Jeremy Gunawardena
Basic Research Institute in the Mathematical Sciences
HP Laboratories Bristol
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If you go up to a conventional engineer - someone who designs bridges or aeroplanes or concert halls or communication systems - and ask her what mathematical theorems she uses as part of the design process, you will get a long list. If you go up to a computer engineer - someone who designs microprocessors or operating systems or network protocols or traffic light controllers - and ask him the same question, you are likely to get an uncomprehending stare. Not only will he not be able to answer but he will give the impression that the question itself is ill-posed: what has mathematics got to do with building computer systems? This paradoxical and worrying discrepancy presents mathematics with tremendous challenges and opportunities in the years ahead. The present note expands upon an invited contribution to a panel discussion at the Symposium on Current and Future Directions in Applied Mathematics, held at the University of Notre Dame from 18-21 April 1996.

If you go up to a conventional engineer—someone who designs bridges or aeroplanes or concert halls or communication systems—and ask her what mathematical theorems she uses as part of the design process, you will, with probability 1, get a long list. For instance, a communications engineer might start with the sampling theorem of Shannon and Nyquist: to reconstruct a band limited signal from samples, it is necessary to sample at least at twice the highest frequency.

If you go up to a computer systems engineer—someone who designs microprocessors or operating systems or network protocols or traffic light controllers—and ask him the same question, you will, with probability 1, get an uncomprehending stare. Not only will he not be able to answer but he will give the impression that the question itself is ill-posed: what has mathematics got to do with building computer systems?

This paradoxical discrepancy presents mathematics with tremendous challenges and opportunities in the years ahead. It is also rather worrying: computer systems are increasingly intruding into our lives and it is disturbing to realise that many aspects of their behaviour are horribly unpredictable.

It is hard to explain and justify these assertions in a short paper of this nature. They form a complex set of issues which impinge on the history of computer science and such problems as the nature of mathematics itself and what we can expect from it in giving us knowledge about the "real" world. Instead of getting blown up in these philosophical minefields, let me try and give an existence proof by describing some specific *challenges* and *opportunities*.

I will describe some problems which I have worked on, not because these are necessarily the most important, or even the most representative, but because I can describe the essential issues quickly. I first came across them when studying timing analysis in digital circuits, [4]. I will not discuss the application in detail but will try and motivate it and point out some of the open questions.

Consider $\mathbb{R}^n$ and let $\| x \|$ denote the ℓ^∞ norm. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a nonexpansive function: for all $x, y \in \mathbb{R}^n$,

$$\| F(x) - F(y) \| \leq \| x - y \|.$$

Such functions can be thought of as describing the time evolution of a *discrete event system*. That is, a system consisting of a finite set of events which occur repeatedly, such as a digital circuit, in which an event might be a voltage change on a wire. If there are n wires, and $x_i \in \mathbb{R}$ represents the time of of first occurrence of an event on wire i, then $F_i(x_1, \cdots, x_n)$ represents the time of the next occurrence on the same wire. This time may depend on when events occur on the other wires and this dependence is captured by the function F. It is not obvious why F should be nonexpansive in the ℓ^∞ norm but convincing reasons can be given for why this class is the right one to study, [3, §4].

The main problem is to understand F as a discrete dynamical system. That is, to study the long term behaviour of the sequence $x, F(x), F^2(x), \cdots$. Does the sequence stabilise and attain some periodic regime? If so, what can we say about these regimes? Let x be a periodic point of F and p the period at x: the least integer such that $F^p(x) = x$. The geometry of $\mathbb{R}^n$ with the ℓ^∞ norm constrains the dynamics in an unexpected way: there is a universal bound on the size of p which depends only on n.

Theorem 1 ([1]) *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive in the ℓ^∞ norm and if p is the period of a periodic point of F, then $p \leq (2n)^n$.*

Results of this form originate in the work of Robert Sine, [10]. The bound is not tight: Roger Nussbaum has conjectured that $p \leq 2^n$, and this can be shown to be best possible. The Nussbaum Conjecture has been proved only for $n \leq 3$ and remains one of the outstanding open problems in this area. For a survey of this and related questions, see [9].

Let us consider another question. How fast is the discrete event system, represented by F, operating? One way to answer this is to consider elapsed times from one set of occurrences to the next: $F(x) - x$. This depends dramatically on x but we can take an average over several occurrences:

$$\frac{F^k(x) - F^{k-1}x + \cdots + F(x) - x}{k} .$$

If we now let $k \to \infty$, we get

$$\lim_{k\to\infty} F^k(x)/k .$$

It is not at all clear that this limit exists but suppose it does for some $x \in \mathbb{R}^n$. Then it is a trivial consequence of nonexpansiveness that it must exist everywhere in $\mathbb{R}^n$ and must have the same value. This common value, when it exists, is called the cycle time of F and denoted $\chi(F)$. It is a measure of the asymptotic performance of the system.

When does χ exist? It does sometimes, [5, Proposition 2.1], but not always, [8, Theorem 3.1]. It is an open problem to even formulate a sensible conjecture as to which nonexpansive functions have cycle times, or, to put it another way, which discrete event systems have measurable performance.

The cycle time tells us something about fixed points of F. For certain functions, $F(x) = x$ if and only if $\chi(F) = 0$, [3, §4]. The existence of fixed points for nonexpansive functions on Banach spaces is a very classical problem, [2]. It is an interesting question whether the results in finite dimensions can be extended to infinite dimensions. The problems can be formulated, for instance, for the space of continuous maps on a compact Hausdorff space and such extensions are of interest for other applications, [3]. We know almost nothing about what goes on in infinite dimensions.

I hope this inadequate discussion has given some idea of what can emerge out of the interaction between computer science and mathematics. Let me make a few further remarks.

The kind of problems above are not the same as in classical applied mathematics. They have a discrete or combinatorial aspect but at the same time they are not part of discrete mathematics, combinatorics or mathematical logic, the areas traditionally associated with computer science. They are to do with dynamical systems and functional analysis. I think this is an indication that the interaction between mathematics and computer science is entering a new phase and that this is where the challenges and opportunities will lie in the future. A workshop at the Isaac Newton Institute in Cambridge on *"New Connections between Mathematics and Computer Science"* brought to light a number of other areas where similiar developments are taking place, [7].

It is fashionable these days to talk about *relevance*, *technology transfer*, *wealth creation* and other such buzzwords. In my view, these are neither necessary nor sufficient reasons for a mathematician to study problems such as those above. They should be studied for their intrinsic mathematical interest. The point is that computer science and computer engineering really do give rise to such problems. By solving these, we may be able to provide the computer engineer with some of the theorems that he so badly lacks at present.

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