Bifurcations in the Falkner-Skan Equation.

Colin Sparrow
Basic Research Institute in the Mathematical Sciences
HP Laboratories Bristol
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Bifurcations in the Falkner-Skan equation *

Colin Sparrow †
Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK
&
Basic Research Institute in the Mathematical Sciences
Hewlett-Packard Research Laboratories
Stoke Gifford, Bristol BS12 6QZ, UK

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Abstract

The Falkner-Skan equation is a reversible three dimensional system of ODEs without fixed points. A novel sequence of bifurcations, each of which creates a large set of periodic and other interesting orbits 'from infinity', occurs for each positive integer value of a parameter. Another sequence of bifurcations destroys these orbits as the parameter increases; topological constraints allow us to understand this sequence of bifurcations in considerable detail. While outlining these results, we can also make a number of possibly illuminating remarks connecting parts of the proof, well-known numerical techniques for locating and continuing periodic orbits, and recent ideas in the control of chaos.

1 Introduction

The Falkner-Skan equation is an ordinary differential equation derived originally from a problem in boundary-layer theory [5]. We will not be interested in the derivation of the problem here, and the behaviour we will be discussing is outside the physically interesting region of parameters. Instead we are interested in the bifurcations which create and destroy periodic orbits (and other interesting trajectories) in the flow as a parameter is changed. One sequence of bifurcations, which are the main subject of this paper, is a sequence of 'heteroclinic bifurcations at infinity'. For reasons that will become clear below, we also refer to these as 'gluing' bifurcations. These are analysed in greater detail in Swinnerton-Dyer & Sparrow [13], where the interested reader may look to fill out the details missing from sections 2 – 5 below. Each of the bifurcations in this sequence creates infinitely many periodic orbits and other interesting trajectories as a parameter increases. As far as we are aware, these bifurcations have not been discussed elsewhere in the literature except in the work of Hastings & Troy [8] which is mentioned further below. They would seem, therefore, to be a new mechanism for creating complicated invariant sets in the flows of ordinary differential equations.

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†Permanent Address: DPMMS, 16 Mill Lane, Cambridge CB2 1SB.

There is another sequence of bifurcations in the system that is discussed more briefly in section 6, which will be the subject of another paper [12], and that is still the subject of active research. This second sequence destroys periodic orbits as the parameter increases, and is probably typical of some sequences of bifurcations observed in 3-dimensional *reversible* differential equations (for example, see Roberts, this volume). These two sequences of bifurcations are very different and it is of considerable interest to understand how they are related; periodic orbits created in one sequence are destroyed in the other, and each orbit has various topological invariants associated with it which very much restrict the possibilities for what can occur in between. In particular, we know which orbits are created from the theory, and this gives us detailed insight into the orbits involved in the destruction sequence for orbits of considerably greater complexity than can be examined numerically. This topic is also discussed briefly in section 6.

At first sight this chapter may appear to be a little way away from the main theme of the meeting (and book). However, it is of some interest to note how the proof outlined in section 5 relies on much the same use of hyperbolicity as techniques for the control of chaos described elsewhere in this volume. It is also worth spending a little time thinking about the numerical techniques commonly used to investigate periodic orbits in systems of ordinary differential equations – for example to produce Figs 2 and 3 below. Such thoughts may even suggest refinements of the methods of chaos control currently under investigation, and are discussed further in section 8.

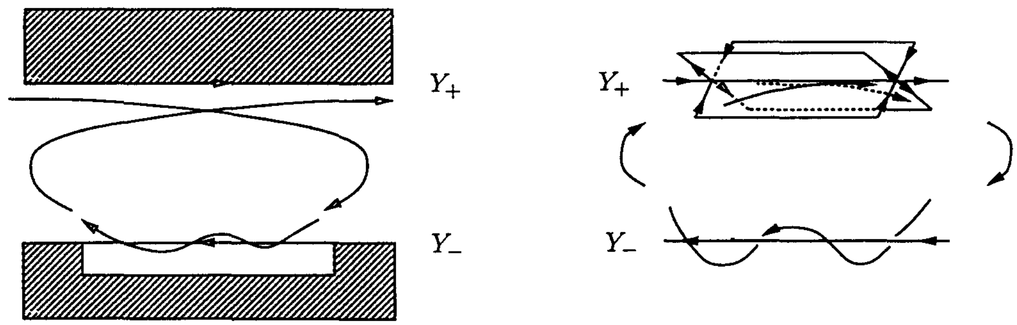
2 The Falkner-Skan Equation; basics

The Falkner-Skan equation [5] is:

$$y''' + y''y + \lambda(1 - y'^2) = 0 \tag{1}$$

with phase space $(y, y', y'') \in \mathbf{R}^3$, and we are concerned with values of the parameter $\lambda > 0$. Differentiation is with respect to an independent variable t that we think of as time-like (though in the derivation of the equation from a model of flow in a boundary layer, the independent variable would be thought of as a space variable). Previous authors who have looked at equation (1) were primarily interested in finding periodic orbits or in finding solutions which exist in $0 < t < \infty$ and satisfy $y(0) = y'(0) = 0$ and $y'(\infty) = 1$. Solutions of the second type are of interest in the physical application, but typically have unbounded behaviour in $t < 0$, and we will have little more to say about them. We do find, however, that it is important for the theory to consider some special unbounded solutions, and so will be interested in a set of solutions to equation (1) which we have called ***admissible solutions***. Admissible solutions are those for which $|y'|$ is bounded for all t, and include, in particular, periodic orbits (P-orbits) and orbits satisfying $y' \to 1$ as $t \to \pm\infty$ (which we call Q-orbits). We do not ask that y be bounded on admissible solutions as this would exclude the unbounded solutions $y' = \pm 1$, $y'' = y''' = 0$ which are crucial in the analysis that follows. We call these two straight-line solutions Y_+ and Y_-; the Q-orbits are the solutions biasymptotic to Y_+.

It is possible to determine some properties of admissible solution by standard methods. It follows from equation (1) that every maximum in y' of a trajectory, where $y'' = 0$ and $y''' < 0$, must lie in $|y'| \leq 1$. Similarly, every minimum of y' on a trajectory lies in $|y'| \geq 1$. As illustrated in Fig 1, these facts are consistent with the



(a) (b)

Figure 1: The phase-space of the Falkner-Skan equation projected onto the (y, y') plane. (a) Admissible trajectories cannot enter the shaded region. (b) Behaviour near $Y_\pm$.

possibility of recurrent behaviour in the system because solutions may travel in the direction of increasing y close to Y_+ for some time before descending to travel back in the direction of decreasing y close to Y_-, and then back to Y_+ again, etc.

For admissible trajectories standard arguments [13] give the boundedness lemma below, which is illustrated in Fig 1(a). The constants N_i in the lemma depend on the integer n considered, but this is not important; the λ-intervals in the lemma are chosen to ensure that there is an overlap between successive intervals.

Lemma 2.1 *For given n and $\lambda \in [n - \frac{3}{4}, n + \frac{3}{4}]$, there are constants $N_1, \ldots, N_4$ such that any admissible trajectory satisfies*

$$1 \geq y' \geq -N_1, \qquad \textit{with} \quad y' \geq -1 \quad \textit{if} \quad |y| > N_2$$

and $|y''| \leq N_3$, $|y'''| \leq N_4$.

Fig 1(b) shows the local behaviour near $Y_\pm$. Trajectories near Y_- spiral around it if $|y|$ is not too large. It is relatively easy to see [13] that the maximum number of turns an admissible solution may make about Y_- while staying close to it is bounded; in fact it follows from results later in this paper that the bound is $[\lambda]$, the integer part of λ. The trajectory Y_+ acts as a 'saddle trajectory' and nearby trajectories move with increasing y ($y' \approx 1$) and approach Y_+ as $t \to \infty$ if they lie on a two-dimensional local stable manifold of Y_+ and will approach Y_+ as $t \to -\infty$ if they lie on a two-dimensional local unstable manifold of Y_+.

Note that equation (1) is *reversible* (see Roberts, this volume, and [11]) as it is invariant under a symmetry that is the composition of an involution on phase space and time reversal. The required symmetry for equation (1) is $(t, y, y', y'') \leftrightarrow (t_0 - t, -y, y', -y'')$. Trajectories may be invariant under this symmetry, as is the case with each of the trajectories shown in Fig 2, or they may occur in pairs, each of which is taken to the other by the symmetry. Most of our discussion below does not make any essential use of the symmetry, though it simplifies many of the arguments. Of course, Roberts' remarks (this volume) on the stability of invariant sets of reversible systems will apply. We are not aware of parameter ranges in which there are symmetric pairs of attractors and repellors in the flow, but we cannot rule out this possibility; numerical solution of the equations from randomly chosen initial conditions leads to unbounded trajectories in all cases that we have tried.

3 P and Q-orbits and topological invariants

Figure 2 shows four numerically computed admissible solutions projected onto the (y, y') plane. Fig 2(a) shows a periodic trajectory which makes a single pass through $y' < 0$, during which it makes 3 turns around Y_-; we call the orbit $P3$, following Botta *et al* [2]. We will see that this orbit is created as λ increases through $\lambda = 3$, though this is not important in this section. Fig 2(b) shows another periodic orbit which makes two passes through the region $y' < 0$ and which makes one and two turns around Y_- during the passages through $y' < 0$; we label it $P12$ (or $P21$ since names of P-orbits can be permuted cyclically). This orbit is created when λ increases through $\lambda = 2$. The total winding number around Y_- (3 in both cases) is an invariant of a periodic orbit, and so cannot change as the orbit is continued for changing λ. Thus, if we continue periodic orbits, the total of the digits in their names cannot change. The names themselves, however, which describe the number of turns around Y_- on successive passes through $y' < 0$, may not be invariant. In fact, the two orbits $P3$ and $P21$ annihilate one another at a larger λ value at which they both make three passes through $y' < 0$ with one turn around Y_- on each pass (and so can legitimately be thought to have both changed into $P111$ orbits) [2, 12].

Periodic orbits are important in the study of equation (1), but an even more central role is played by the Q-orbits. (The importance of the Q-orbits was also noted in [7].) Two of these are shown in Figs 2(c) and (d). We call the orbit shown in Fig 2(d) $Q11$, as it makes two passes into the region $y' < 0$ and on each occasion it turns once around Y_-. This orbit is created as λ increases through $\lambda = 1$. The orbit shown in Fig 2(c) is called $Q2$ and is created as λ increases through $\lambda = 2$.

It is important to note that Q-orbits, like periodic orbits, cannot just disappear as λ varies without their being involved in a bifurcation with one or more other Q-orbits. In fact there is a local bifurcation theory for Q-orbits like the more familiar bifurcation theory for periodic orbits, but simpler because there are no phenomena analogous to period multiplication. We can, in fact, expect saddle-node and pitchfork bifurcations to occur in a generic way between appropriate Q-orbits. As with P-orbits, the total winding number of a Q-orbit around Y_- will be an invariant as the orbit is continued for changing λ. In addition, there is extra information about Q-orbits. Since all Q-orbits approach Y_+ eventually on the same two-dimensional manifold, they can be ordered on this manifold by their closeness to Y_+. We have two such orderings, corresponding to the two limits $t \to \infty$ and $t \to -\infty$ and only symmetric orbits will occupy the same place in both orderings. These orderings will give us rather a lot of information about the order in which Q-orbits can bifurcate [12] and so are important in understanding how orbits disappear from the system (section 6). Yet further information comes from the symmetry. Symmetric P-orbits must each intersect the symmetry line $y = y'' = 0$ in exactly two points; symmetric Q-orbits will intersect the line just once (see Roberts, this volume). The ordering of these intersections on the symmetry line will further restrict the order in which bifurcations involving symmetric P and Q-orbits can occur as λ increases.

These topological considerations are not crucial for our study of gluing bifur-

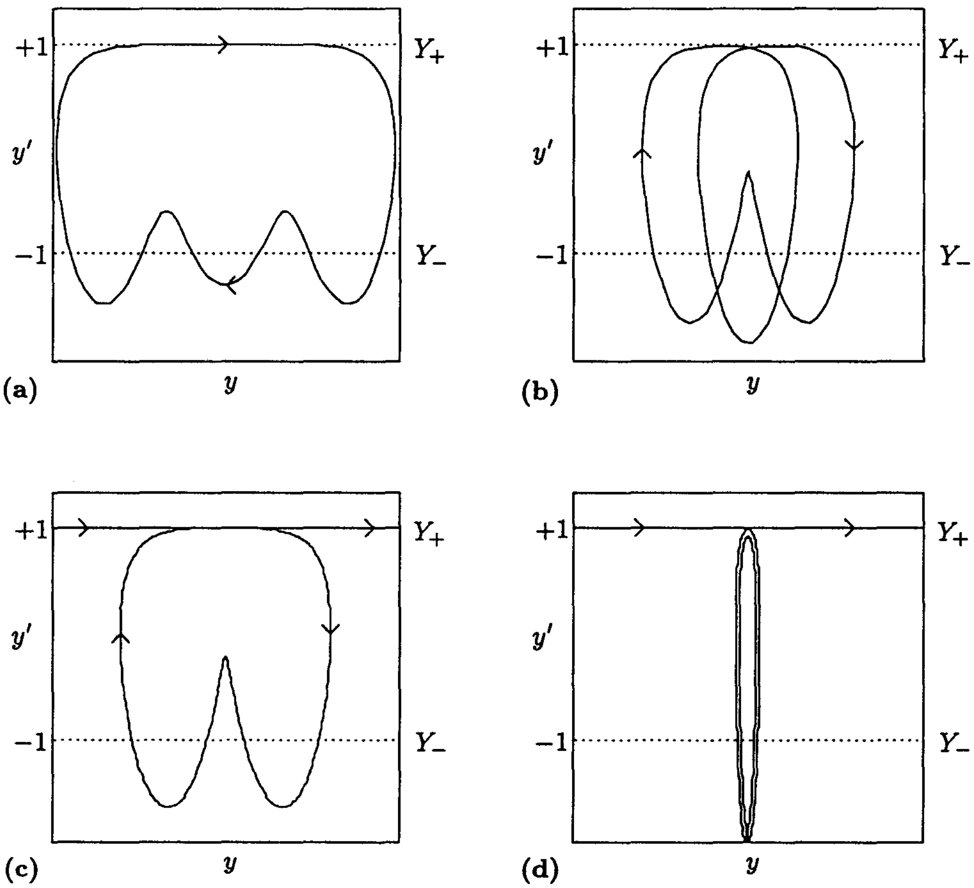


Figure 2: Some admissible solutions for $\lambda = 10$. All orbits are shown at the same scale. (a) Periodic orbit $P3$. (b) Periodic orbit $P21$. (c) The orbit $Q2$. (d) The orbit $Q11$.

cation in sections 4 and 5 below. It is, however, important to note that numerical experiments and theory both show that Q-orbits *are* removed by bifurcation from the set of admissible orbits as λ increases, contrary to the conjecture of Hastings & Troy [8]; this affects the statement of Theorem 5.1 as we cannot in general be precise about the sets of Q-orbits that exist at particular λ values.

4 Creation of Q-orbits

Very crudely, our main result is that large sets of admissible trajectories are created each time λ increases through an integer $n \in \mathbf{Z}$. Slightly more precisely, for integer $\lambda \geq 2$ each of these bifurcations creates an infinity of periodic and other admissible solutions by bifurcation away from the two unbounded trajectories Y_+ and Y_-. When $\lambda = 1$ there is a similar bifurcation that produces many Q-orbits but only one periodic orbit. These bifurcations were discussed by Hastings & Troy [8] who proved some results for the cases $\lambda = 1, 2$ and made some almost correct conjectures about the bifurcations for larger λ values. Our approach is different and more generally applicable than theirs.

The precise statement of the result splits naturally into two parts. The first, Theorem 4.2, states that an orbit Qn is created as λ increases through integer n. Recall that Qn is a Q-orbit which makes n turns around Y_- during a single pass through $y' < 0$. The second, Theorem 5.1, follows from the first, and gives details of the other P and Q-orbits created.

A major step in the proof of Theorem 4.2 is Lemma 4.1 below, which gives a reasonable precise and necessary condition on λ for an admissible trajectory to cross $y' = 0$ with $|y|$ large. This is important, since it will transpire that new admissible solutions created as λ increases through n consist of the union of segments lying very close to Y_+, segments lying very close to Y_-, and nearly vertical segements (on which y is approximately constant) joining the two. In particular, Qn has just two such nearly vertical segments, and as as $\lambda \downarrow n$ the y-values where these segments cross $y' = 0$ satisfy $|y| \to \infty$.

Lemma 4.1 *For λ in a similar interval as in Lemma 2.1, there are constants N_5 and N_6 with the following properties. If (y_0, y_0', y_0'') is a point of an admissible solution with $|y_0| > N_5$ and $y_0' = 0$ then $\lambda > n$, and*

$$\left| (\lambda - n)|y_0|^{2n} - \frac{n^2(2n!)^2}{4(n+1)(n!)^3} \right| < N_6(\lambda - n)^{\frac{1}{(n+1)}} . \qquad (2)$$

Moreover, if $y_0 > 0$ the trajectory dips below $y' = -1$ exactly n times before it next crosses $y' = 0$, and it crosses $y' = 0$ upwards through the gateway which satisfies (2) with $y < -N_5$. A similar result holds with time reversed if $y_0 < 0$.

Notice that the gateways are only far out if λ is near an integer. Therefore, the lemma shows that if new admissible orbits are to be created as described above, then this can only occur as λ increases through an integer n.

We omit most of the details of the proof of the lemma, but it is worth mentioning that an important step in the proof involves studying the behaviour of trajectories moving very close to Y_-. For this purpose we take y as the independent variable and

$z = 1 + y'$ as the dependent one. Using dots to denote differentiation with respect to y, (1) now becomes

$$\ddot{z} - y\dot{z} + 2\lambda z = Z(y). \tag{3}$$

This equation describes the evolution of the distance from Y_- (given by z) as the trajectory moves along near Y_-, and $Z(y)$ is small compared with z so long as z and its derivatives are all small. The associated linear equation

$$\ddot{z} - y\dot{z} + 2\lambda z = 0 \tag{4}$$

is Weber's equation and the fact that integers are the important parameter values for equation (1) ultimately follows from the fact that the solutions of equation (4) which grow slowly as $t \to \infty$ and as $t \to -\infty$ coincide at these parameter values, but the intervening calculations are non-trivial.

Once the lemma is proved, the remainder of the proof of the creation of Qn is easier and involves matching together several pieces of approximate solution. These include a piece close to Y_-, pieces near to Y_+, and some nearly vertical pieces with large $|y|$. The 'matching' involves showing there is a true trajectory close to the pseudo-trajectory formed of the union of these pieces; we will have more to say about this sort of argument in sections 5 and 8 below. The precise statement of the theorem is:

Theorem 4.2 *In some interval to the right of $\lambda = n$ there is a unique Q-orbit which enters $y' < 0$ just once and does so through the gateway defined by (2). It is symmetric and makes n twists around Y_-.*

5 Gluing Q-orbits

We can next show that the existence of the Qn-orbit in $\lambda > n$ implies the existence of a whole new set of admissible solutions formed by 'gluing' segments of the Qn-orbit to each other, and to segments of other Q-orbits which already exist when $\lambda = n$. A precise statement of the theorem we can prove about the creation of a large set of admissible solutions is:

Theorem 5.1 *Let $Q\alpha_0, Q\alpha_1, \ldots$ be a finite set of Q-orbits which exist and are regular at $\lambda = n$ and let $m_0, m_1, \ldots$ be a finite set of positive integers. Then there exists an $\epsilon > 0$ such that in the interval $n < \lambda < n + \epsilon$ there is a unique Q-orbit $Q\alpha_0 n^{m_0} \alpha_1 n^{m_1} \ldots$ close to the pseudo-orbit obtiained by gluing together in sequence segments of $Q\alpha_0$, m_0 copies of Qn, $Q\alpha_1$ and so on. The name can finish with α_r or n^m and a similar result holds for $Qn^{m_0}\alpha_0 n^{m_1} \ldots$. Furthermore, for any cyclic alternating sequence as above (of minimal period) there exists a unique periodic orbit $P\alpha_0 n^{m_0} \alpha_1 \ldots$.*

[The technical condition that the $Q\alpha$ be regular means, essentially, that $\lambda = n$ cannot be a bifurcation point for $Q\alpha$.]

The theorem and its proof is illustrated in Figs 3 and 4. Given a finite collection of regular $Q\alpha$'s, we first choose an M with the property that every one of our collection remains close to Y_+ in $|y| > M$. For an example, see Fig 3(a) where $n = 2$, the points B and C are at $y = \pm M$, and our finite collection of pre-existing Q-orbits consists of the single orbit $Q1$. We then pick ϵ small enough that in $n < \lambda < n + \epsilon$ the new trajectory Qn that exists in $\lambda > n$ has all its interesting behaviour outside the region

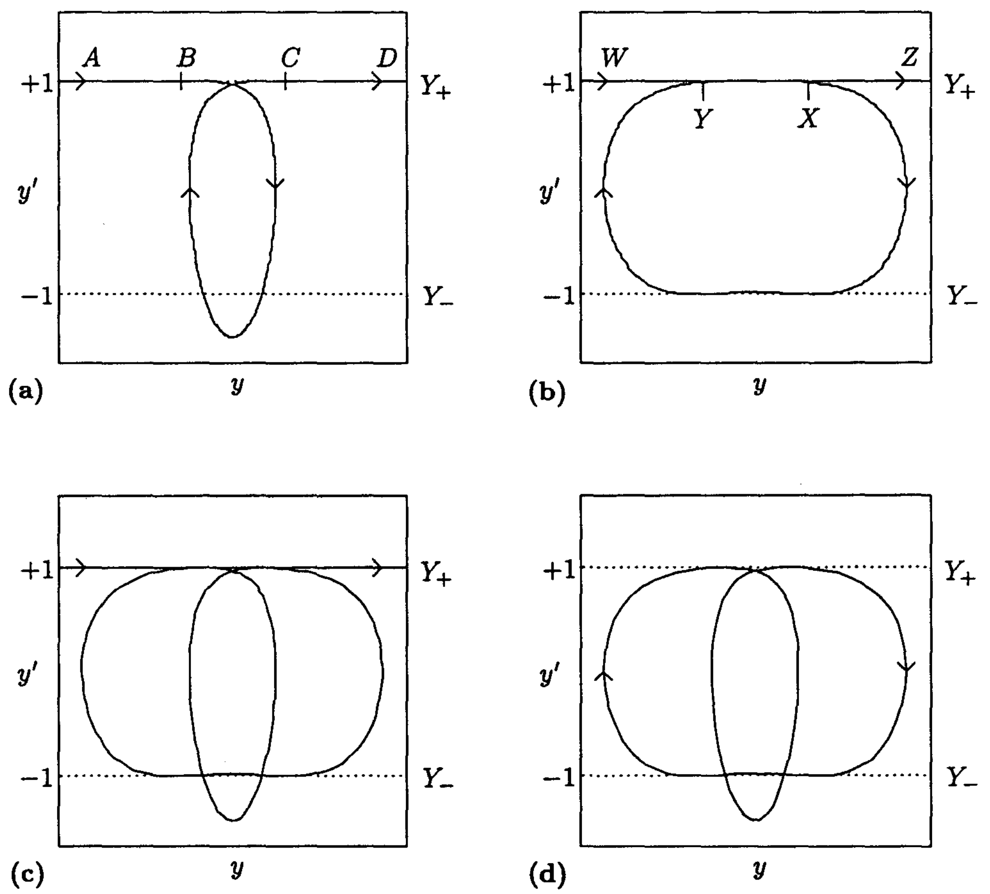


Figure 3: Gluing illustrated at $\lambda = 2.1$. All orbits are shown at the same scale; the y scale is 2.5 times that in Fig 2. (a) A segment ABCD from $Q1$; B and C are at $y = \pm M$. (b) A segment WXYZ from $Q2$; Y and X are at $y = \pm M$. (c) $Q12$ can be formed by 'gluing' ABC from $Q1$ to XYZ from $Q2$. (d) $P21$ can be formed by 'gluing' BC from $Q1$ into XY from $Q2$.

$|y| < M$ as illustrated for the orbit $Q2$ at $\lambda = 2.1$ in Fig 3(b). In other words, ϵ is chosen so that Qn is still close to Y_+ at $y = M$ when tending away from Y_+, and is again close to Y_+ at $y = -M$ when tending to Y_+ as $t \to \infty$. We can then create pseudo-trajectories, with small discontinuities at $y = \pm M$, with names like those in the theorem, by gluing together appropriate pieces of Qn and $Q\alpha$ at $y = \pm M$. This process can be used to produce pseudo-Q-orbits or pseudo-P-orbits as illustrated in Figs 3(c) and 3(d). We then only need to show that there will be a unique true P or Q-orbit close to any pseudo-trajectory formed as above. In fact, Figs 3(c) and 3(d) show numerically calculated 'true' trajectories $Q12$ and $P12$ at the same scale and at the same parameter value as the orbits $Q1$ and $Q2$ shown in Figs 3(a) and (b); to the naked eye they are indistinguishable from the pseudo-trajectories created by gluing together pieces of $Q1$ and $Q2$ as described in the caption to Fig 3.

The argument to establish the existence of the required true trajectories is fairly standard in dynamical systems theory, and is illustrated in Fig 4 for the particular case where we wish to prove the existence of a true $Q12$ orbit close to the pseudo-orbit constructed in Fig 3. The figure shows the situation on the plane $y = M$ where we desire to glue together a part of $Q1$ which strikes the return plane at point q_1 ($= C$ on Fig 3(a)) on the local stable manifold of Y_+, and a part of $Q2$ which strikes the return plane at the point q_2 ($= X$ on Fig 3(b)) on the local unstable manifold of Y_+. Since both $Q1$ and $Q2$ lie in the interior of both the global stable and unstable manifolds of Y_+ (as Q-orbits are biasymptotic to Y_+), we can expect that some part of the global stable manifold containing $Q2$ will intersect the plane in a 1-dimensional line segment through q_2, and some part of the global unstable manifold containing $Q1$ will intersect the plane in a 1-dimensional line segment through q_1. The proof then consists of establishing estimates that ensure that these segments exist, are transverse, and intersect in a unique point q_{12}. This point q_{12} will then be a point on a true $Q12$-orbit with the desired name and properties. For the details of the proof readers should consult [13]. We will have more to say about the relationship between this type of argument and the control of chaos in section 8 below.

Theorem 5.1 does not, unfortunately, give a complete description of the set of admissible solutions created at each bifurcation, and the restriction to a finite set of pre-existing Q-orbits, which is essential for the choice of M in the proof, suggests that it would be difficult to extend this approach as far as we would like. It would be relatively easy to extend the techniques to prove the existence of admissible trajectories which only go to Y_+ at one end, or which are bounded but not periodic; there are no additional complications provided we use only finitely many distinct $Q\alpha$ in constructing each new trajectory.

This means that for $n = 1$, we could extend the theorem to give a complete description. Since we can deduce from earlier work of Coppel [3] that there are no interesting trajectories when $\lambda < 1$, it is only possible to glue $Q1$ to itself. In this case we therefore obtain just one periodic orbit $P1$, an infinity of Q-orbits $Q1^m = Q1\ldots1$, $m \geq 2$ (of which the orbit $Q11$ of Fig 2(d) is the simplest) and exactly two other admissible orbits, each of which is asymptotic to Y_+ at one end and to $P1$ at the other.

If $n > 1$ the set of new admissible trajectories created is large and complicated, and the theorem only gives an exact description of the P and Q-orbits within them, there being infinitely many of each of these. Even if we extend our results as described above, our methods do not allow us to make complete statements about the set of

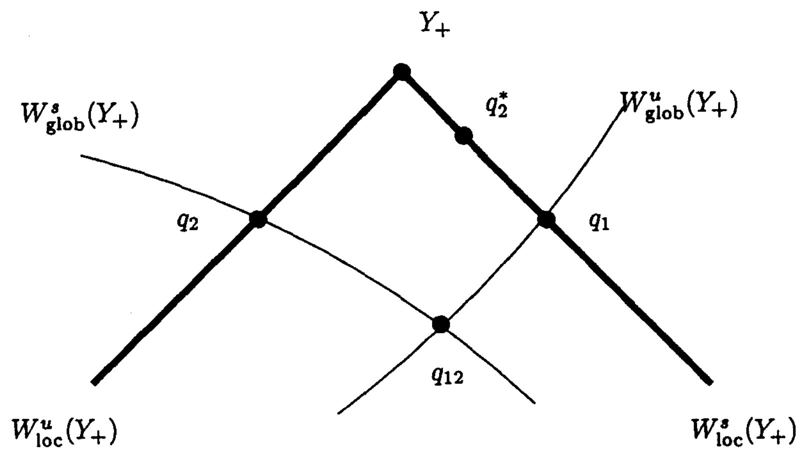


Figure 4: Gluing illustrated on the plane $y = M$ near Y_+. Coordinates are y' vertically and y'' horizontally. Bold lines show parts of the local stable and unstable manifolds of Y_+. Point $q_2 = X$ is on the orbit $Q2$ and $q_1 = C$ is on the orbit $Q1$ (see Figs 2(a) and 2(b) for the points X and C). $Q2$ also intersects the plane again at a point q_2^* (lying between Y and Z on Fig 2(b)), but this is not important here. There is an arc segment through q_2 such that orbits started at these points remain close to $Q2$ as $t \to \infty$, and so make two turns around Y_- and then tend to Y_+. Similarly, there is a segment through q_1 such that orbits through these points remain close to $Q1$ for $t \to -\infty$, and so came from Y_+ and made one turn around Y_- before hitting the return plane. The two segments intersect at a point q_{12} which therefore lies on an orbit $Q12$.

admissible trajectories in these cases. In addition, there is some necessary vagueness in the statement of the theorem because we do not know precisely which Q-orbits exist at $\lambda = n$. As we shall see, with the exception of $Q1$, any Q-orbit eventually disappears as λ increases; however, numerical evidence indicates that most trajectories created at one integer λ value persist over a large range of λ values, and so are available for gluing at many subsequent integer values. For example, it seems that the first Q-orbits to disappear are $Q11$ and $Q2$, which annihilate one another at $\lambda \approx 250$. If this is correct, then the set of admissible solutions for $n < \lambda < n+1$ will contain at least a set equivalent to a full-shift on n symbols for all $n < 250$, as conjectured by Hastings & Troy [8]. For larger n values some of the trajectories will be missing as Q-orbits disappear from the admissible set.

6 Destruction of P and Q-orbits

There is theoretical and numerical evidence that P and Q-orbits created in the sequence of bifurcations described above are destroyed at larger λ values as $\lambda \to \infty$.

First, we have proved results [12] on the behaviour of the equation (1) for large λ. These results show that $P1$ is the only periodic orbit to exist for all sufficiently large values of λ, and that $Q1$ is similarly unique amongst Q-orbits. This implies that all other P and Q-orbits are ultimately destroyed as λ increases; since we know from the results of the previous sections that infinitely many P and Q orbits are produced as λ increases through each integer $n \geq 2$, this means that the sequence of 'destroying' bifurcations must be relatively complicated and continue for all values of λ however large. It is also relevant here that there are no admissible orbits for $\lambda < 1$ [3]; if we follow orbits created in gluing bifurcations with changing λ we know they cannot just 'turn back' in saddle-node bifurcations and disappear into $\lambda < 0$; something must actually destroy them in the parameter range $1 < \lambda < \infty$.

It is course possible that orbits created as above come together at some λ value larger than the ones at which they were created and annihilate themselves in pairs in saddle-node bifurcations or in larger groupings in more complicated bifurcations. This does actually occur and accounts for all orbit destructions, but there are severe topological restrictions on which orbits can bifurcate with which others, and these can be used to gain a deep understanding of the bifurcation sequences which are possible in the system. For example, two P or two Q orbits can only annihilate one another in a saddle-node bifurcation if they wind the same number of times around Y_-. Furthermore, two Q-orbits can only so annihilate each other if all intermediate Q-orbits in the ordering of Q-orbits on the two-dimensional stable and unstable manifolds of Y_+ (see section 2) have been removed at lower λ values. Similarly, the ordering of intersections of symmetric P and Q orbits with the symmetry line $y = y'' = 0$ places yet further restrictions on the possible bifurcations and their order with λ increasing. Indeed, this last invariant ensures that P and Q-orbit bifurcations cannot happen entirely independently; some P orbits must disappear before certain Q-orbits can bifurcate, and *vice versa*. Further topological invariants concern the number of windings that orbits have around each other, and around $P1$ in particular, but these will not concern us further here.

It would not be useful to describe in great detail all we know about the sequence of bifurcations that destroys P and Q orbits as λ increases. In part this is because the results will appear elsewhere [12], and in part because the research is still in progress and the results are not yet in their final form. It is useful, however, to give some flavour of the argument and to describe the fate of a few example orbits. This is more especially true because the reversibility of the system allows generic period n-tupling bifurcations to occur near symmetric periodic orbits (which must be neutrally stable because of the time reversal symmetry) rather like those that occur in Hamiltonian systems. These bifurcations also play a crucial role in the sequence of bifurcations destroying orbits, and whilst such bifurcations are well-known in Hamiltonian systems, or in reversible systems of even dimension, it is useful to have a further well understood example in an odd dimensional system. Of course, the order in which these period n-tupling bifurcations can occur also suffers some restrictions. For example, orbits undergoing n-tupling bifurcations with the symmetric orbit $P1$ must do so at λ values where the Floquet multipliers of $P1$ have the form $e^{i\theta}$ where $\theta = \pm\frac{2\pi q}{n}$ where q and n are coprime. As the Floquet multipliers of $P1$ (and other symmetric periodic orbits) move continuously round the unit circle with increasing λ, the order in which the rational values of θ are visited orders the infinite sequence of n-tupling bifurcations. [When $n = 2, \theta = \pi$ we expect to see a normal period-doubling bifurcation where one orbit meets up with another of twice the period. For $n > 2$, bifurcations generically involve one orbit of some period T which persists, and two orbits of period nT which we can, in an appropriate orbit-counting interpretation, think of as annihilating each other at the bifurcation.]

That there is some bifurcation scheme which matches the many orbits produced in the gluing bifurcations with the destroying bifurcations in a way that is compatible with all these restrictions is obvious; after all, the Falkner-Skan equation does something that fulfills all the requirements. We could just try to investigate this scheme numerically, though this is difficult for anything other than the shorter periodic orbits, and is in any case not enormously rewarding. It would be much more interesting if we could determine which types of bifurcation scheme are compatible with all the restrictions, and we are certainly close to having results of this kind. It seems likely, however, that in trying to determine what actually does occur, there will always be some assumptions that need to be checked numerically to supplement the rigorous theory.

For example, it seems from numerical experiment that the Floquet multipliers for $P1$ are initially real, but meet at -1 for $\lambda \approx 340$; they then seem to travel round the unit circle towards $+1$ with the complex argument θ decreasing monotonically from π to 0 as $\lambda \to \infty$. If we accept this observation, which is the product of very stable numerical computations and which may be provable, we can argue as follows.

There are only one P and one Q orbit which wind only once around Y_- and these are $P1$ and $Q1$. The large λ theory tells us that orbits with these names continue to exist for all λ. It follows from the 'shooting' results of Hastings & Troy [8], though not directly from our own large λ results, that the orbit $P1$ can be continuously and monotonically followed from its creation at $\lambda = 1$ up to the large λ regime. A similar result is doubtless true for $Q1$. Thus, we have accounted for the creation and ultimate fate of all P and Q-orbits which wind only once around Y_-. The only periodic orbit which winds twice around Y_- is $P2$, and precisely one such orbit is required to be destroyed in a period-doubling bifurcation with $P1$ at $\lambda \approx 340$ when its Floquet multipliers are -1. There are two Q-orbits which wind twice around Y_-, $Q11$ and $Q2$, and these must ultimately disappear; it turns out that the ordering on the stable and unstable manifolds of Y_+, and on the symmetry line, are such as to permit them to destroy each other in a saddle-node bifurcation without requiring any other bifurcation to take place first (i.e. these orbits are 'next to each other' in all these various orderings). Thus we account for all P and Q-orbits which wind twice around Y_-. Continuing, we find that the two periodic orbits $P12$ and $P3$ which wind 3 times round Y_- are just what is needed to meet for mutual annihilation in the vicinity of $P1$ when $\theta = \frac{2\pi}{3}$ at a period-tripling bifurcation. And so it continues. The first element of flexibility in the scheme appears only to occur for Q-orbits winding at least 8 times around Y_-, and even there we believe that slighly more complicated topological argument will reduce all the choices to one. Simple numerical experiments serve to confirm our understanding (at least for the topologically less complicated orbits which are accessible to numerical work), and – in particular – seem to indicate that orbits behave as simply as possible in the intermediate parameter range (between creation and destruction) where we have little theoretical control over their behaviour.

The elements of internal self-consistency in the scheme give us reassurance that our understanding the behaviour of P and Q-orbits in the system is likely to be correct, even where this cannot be checked in detail by numerical experiments or theory. More importantly, in some sense which we have yet to make precise, an accounting scheme of this type comes very close to establishing that any system with the same basic topological features as the Falkner-Skan equation must have the 'same' sequences of bifurcations.

7 Further remarks on the Falkner-Skan equation

It is important to understand how much of the analysis above may apply to perturbations of the Falkner-Skan equation, or to other systems of ordinary differential equations.

First, it is possible to add a further parameter to the Falkner-Skan equation which destroys neither the symmetry, nor the trajectories $Y_\pm$, nor the gluing bifurcations at integer values of λ, nor the large λ analysis. In this case, everything above remains valid and changing the new parameter should only alter the precise λ values at which the destroying bifurcations occur. Of course, this would imply that once the sequence of Q-orbit destructions begins, presumably at a different λ value, the collections of Q-orbits available for gluing at integer λ values will differ, and some details of the bifurcation scheme will change. This implies only that our bifurcation scheme described above cannot be absolutely fixed, and must allow for this kind of flexibility.

Second, we can consider more general perturbations. First let us consider the role of the reversible symmetry. This does not in fact play a crucial role in the analysis of the gluing bifurcations – the essential process in the gluing bifurcation is the creation from two unbounded trajectories of a single new Q-orbit at discrete values of a parameter. Neither this process, nor the gluing from which we then deduce the existence of a much larger set of interesting trajectories, depends on the symmetry. The details of the orbit destroying bifurcations relied more heavily on the symmetry; however, in the usual way we would expect to be able to unfold this scheme and generalise it to non-symmetric but nearby systems.

Another way of looking at the system, pointed out to us by Robert Mackay (personal communication), is illuminating for considering symmetry-breaking and more general perturbations. By doing a standard Poincaré compactification we can convert the Falkner-Skan equation into a form where it is easier to study the 'behaviour at infinity'. In this new form, there are two fixed points (at infinity in the original form), and the transformed trajectories $Y_\pm$ form heteroclinic connections between the two points (as do all the Q-orbits). Analysis of the bifurcations in this system appears to be no easier than the analysis described above (Mackay, personal communication) and the usual theorems about homoclinic and heteroclinic bifurcations in flows do not apply directly because of the very non-generic nature of the compactified fixed points. However, this set-up is more convenient for describing the effects of perturbation. What is important for our analysis is that perturbations should not destroy the fixed points at infinity nor the heteroclinic loops ($Y_\pm$) between them. Providing we keep within the space of systems with these properties our analysis will apply, and the gluing bifurcations will be a co-dimension one phenomenon. We have not yet analysed precisely what this statement means in terms of perturbations to the original Falkner-Skan equation. Certainly it is not clear how, when given an ordinary differential equation to study, one should go about noticing the existence of or locating important unbounded trajectories like $Y_\pm$; in the case of the Falkner-Skan equation their existence is obvious from a cursory examination of the equation, but this is presumably not always going to be the case.

Finally, some similar but essentially different analysis does appear to be useful in the analysis of at least one other set of equations, as noticed by Paul Glendinning. These are the Nosé equations [9, 10] and an analysis of this system is underway [6].

8 Numerical methods, Control of Chaos, and Conclusion

In this section, I would like to make some remarks that arose as a result of hearing other presentations and discussions at the workshop on which this volume is based. There are striking and moderately obvious connections between:

- ideas in the literature about controlling chaotic systems to lock onto unstable periodic orbits;
- the activity of those who use the location and continuation of periodic orbits in flows as tool in understanding the bifurcations and attractors occurring in differential equations and maps;
- the algorithms used in numerical packages to locate and continue periodic solutions
- elements in the proof of the theorems presented here, and in other 'shadowing of pseudo-orbit' type proofs in dynamical systems.

These connections include the following. There are obvious parallels between the way in which chaos-controlling algorithms (such as those described by Glass *et al*, Kostelich & Barreto, and Ott & Hunt, this volume) wait for a nearly periodic orbit to occur before switching on the control, and the type of procedure used by some investigators to find initial conditions for their orbit continuing packages. There is also a close match between the strategy then used to control a chaotic system, and the type of relatively crude (but often effective) numerical continuation algorithm which takes an initial point, computes a numerical trajectory and its Jacobian round to a nearby point, and then uses a Newton method or similar to calculate a new guess for a point on the periodic orbit. In a system like the Falkner-Skan equation, such one-shot numerical methods do not work well, at least for periodic orbits that come close to $Y_{\pm}$, or for Q-orbits. In this case it is better to use a package that employs a more sophisticated numerical method – in my case I used the package AUTO developed by Doedel [4]. This uses a collocation method and simultaneously manipulates many intermediate points on the orbit instead of merely adjusting an initial point. To put this another way, a whole sequence of points on a pseudo-orbit are adjusted to produce a better pseudo-orbit (closer to a true orbit). This method is actually spectacularly well-adjusted to work on the Falkner-Skan equation, since the easiest way to find more complicated P and Q orbits appears to be to build pseudo-orbits out of pieces from previously located but simpler orbits exactly as suggested by the proof of Theorem 5.1, and to feed these in as initial data for AUTO. The output from AUTO is then a sequence of improving pseudo-orbits, and the algorithm is stable precisely because the hyperbolicity conditions needed in the proof of the theorem hold and cause numerical convergence.

This suggests that in investigating chaos-controllers, there might be much that can be usefully learnt from the numerical procedures and algorithms used and developed over many years by many researchers. In particular, the improved performance obtained from methods like AUTO in some systems, suggests that control of chaos in these systems may be best achieved by locating many short and simple pieces of trajectory and then driving the system to follow desired pseudo-orbits formed from gluing these pieces together. This is a similar idea to that used in targetting as described by Kostelich (this volume), though in his case the short pieces of trajectory were only used to get onto the orbit, rather than while travelling around it. In circumstances where there is a convenient symbolic dynamics, there may be optimal collections of short segments that through gluing provide good pseudo-orbits to shadow all more complicated orbits. For more on this, though it is not what they had in mind, see [1].

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References

[1] Artuso R, Aurell E & Cvitanović P, 1990. Recycling of strange sets: I. Cycle expansions. *Nonlinearity* **3**:325-359.

[2] Botta E F F, Hut F J & Veldman A E P, 1986. The rôle of periodic solutions in the Falkner-Skan problem for $\lambda > 0$. *J. Eng. Math.,* **20**:81-93.

[3] Coppel W A, 1960. On a differential equation of boundary layer theory. *Philos. Trans. Roy. Soc. London Ser. A* **253**:101-136.

[4] Doedel E, 1986. *AUTO: Software for continuation and bifurcation problems in ordinary differential equations.* C.I.T. Press, Pasadena.

[5] Falkner V M & Skan S W, 1931. Solutions of the boundary layer equations. *Phil. Mag.*, **7**(12):865-896.

[6] Glendinning P & Swinnerton-Dyer H P F, 1996. Bifurcations in the Nosé equation. In preparation.

[7] Hastings S P, 1993. Use of simple shooting to obtain chaos. *Physica D* **62**:87-93.

[8] Hastings S P & Troy W, 1988. Oscillating solutions of the Falkner-Skan equation for positive beta. *J. Diff. Eqn.*, **71**:123-144. [Note: their β is our λ.]

[9] Nosé S, 1984. A unified formulation of the constant temperature molecular-dynamics methods. *J. Chem. Phys.* **81**: 511-519.

[10] Posch H A, Hoover W G & Vesely F J, 1986. Canonical dynamics of the Nosé oscillator: stability, order and chaos. *Phys. Rev. A* **33**: 4253-4265.

[11] Sevryuk M B, 1986. *Reversible systems.* Lect. Notes Math. **1211**, Springer-Verlag, Berlin.

[12] Sparrow C and Swinnerton-Dyer H P F, 1996. The Falkner-Skan Equation II: The bifurcations of P and Q-orbits in the Falkner-Skan equation. *In preparation.*

[13] Swinnerton-Dyer H P F & Sparrow C, 1995. The Falkner-Skan Equation I: The creation of strange invariant sets. *J. Diff. Eqn.*, **119**(2): 336-394.