



Existence of Cycle-Time Vectors for Minmax Functions in Dimension 3

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Abstract

We show that for all minmax functions $F : \mathbf{R}^3 \rightarrow \mathbf{R}^3$, the cycle time vector $\chi = \lim_{k \rightarrow \infty} F^k(\mathbf{x})/k$ exists. Our proof is dynamical in nature, and gives considerable insight into the asymptotic dynamics of minmax functions. It makes substantial use of a previous result of Gunawardena & Keane that $\bar{\chi}$ and $\underline{\chi}$ exist for a class of functions that includes minmax functions. An indication is given of how the argument may be extended to show the existence of the cycle-time vector for dimensions $n > 3$.

1 Introduction

This is a preliminary report, and as such does not contain complete references, definitions, or background information; these may be found in [5] or [1, 2, 3, 4] or the references contained therein. Of essential relevance here are the following facts. For topical (non-expansive in the supnorm and homogenous) functions (and even more generally), it is known that if χ exists it is independent of \mathbf{x} . For topical functions it is known that $\bar{\chi}$ and $\underline{\chi}$, the limits of the maximal and minimal coordinates of $F^k(\mathbf{x})/k$ exist; it is not known that there are particular coordinates achieving these limits, though this is conjectured in [5]. The existence of χ is known for max only functions [6], and in the case $n = 2$ for all topical functions. Non-existence of χ for general topical functions is known for $n \geq 3$ [5]. Existence of χ (and much else) for minmax functions would follow from the Duality Conjecture of Gunawardena [2], but even so this paper gives an instructive direct method of proof for the case $n = 3$ that we hope to extend to cover all n . We will actually prove:

Theorem 1.1 *Let $\chi(F)$, the cycle time vector, be defined by*

$$\chi(F) = \lim_{k \rightarrow \infty} F^k(\mathbf{x})/k.$$

Then for all minmax functions in dimension $n = 3$, χ exists (independent of \mathbf{x}).

2 Preliminaries

Throughout, when necessary, we think of each coordinate function F_i of a minmax function F as written as the min over a collection of max only terms. We also assume for simplicity (and w.l.o.g.) that all the constants appearing in the coordinate functions are greater than or equal to zero, and that the maximum of them has value C . Thus if $\mathbf{y} = F(\mathbf{x})$ we have that for each $1 \leq i \leq n$,

$$y_i = x_j + c$$

for some j and $0 \leq c \leq C$ both depending on the relative values of the coordinates of \mathbf{x} .

First let us consider general topical functions. If $\bar{\chi} = \underline{\chi}$ then there is nothing to prove. So we assume that $\bar{\chi} > \underline{\chi}$.

Pick some initial point \mathbf{x}^0 which will remain fixed throughout the argument. Consider the trajectory $\mathbf{x}^0, \mathbf{x}^1, \dots$. By non-expansiveness

$$|\mathbf{x}^{k+1} - \mathbf{x}^k| \leq |\mathbf{x}^1 - \mathbf{x}^0| = D \quad \text{for all } k = 0, 1, \dots$$

for some constant D (that depends on \mathbf{x}^0 but which is therefore also fixed throughout the argument; if we take $\mathbf{x}^0 = \mathbf{0}$ then $D \leq C$). In particular, \mathbf{x}^{k+1} differs from \mathbf{x}^k by at most D in each coordinate.

This is already enough to produce a new proof of the existence of χ for topical functions in $n = 2$ which illustrates the idea of the proof for minmax functions in higher dimensions.

2.1 Proof of existence of χ for topical functions with $n = 2$

Pick $0 < \epsilon < (\bar{\chi} - \underline{\chi})/2$. Then there is an N such that for $k > N$ we have

$$|\mathbf{x}_{t_k}^k - k\bar{\chi}| < k\epsilon$$

for the ‘top’ coordinate $t_k = 1$ or $t_k = 2$ and

$$|\mathbf{x}_{b_k}^k - k\underline{\chi}| < k\epsilon$$

for the ‘bottom’ coordinate $b_k = 1$ or $b_k = 2$ but $b_k \neq t_k$. Now, similar inequalities apply for the coordinates of \mathbf{x}^{k+1} , so that, for example,

$$|\mathbf{x}_{t_{k+1}}^{k+1} - (k+1)\bar{\chi}| < (k+1)\epsilon.$$

Comparing the top coordinate of \mathbf{x}^{k+1} with the bottom coordinate of \mathbf{x}^k we see that

$$|\mathbf{x}_{t_{k+1}}^{k+1} - \mathbf{x}_{b_k}^k| \geq k(\bar{\chi} - \underline{\chi} - 2\epsilon) + \bar{\chi} - \epsilon$$

and that for some $M \geq N$ large enough, $k \geq M$ implies the right-hand side is greater than D . Consequently, for all $k \geq M$, $t_{k+1} \neq b_k$ (i.e. the top and bottom coordinates cannot swap between \mathbf{x}^k and \mathbf{x}^{k+1}) which in turn implies that $t_k = t_M$ and that $b_k = b_M$ for all $k \geq M$. Since our original choice of ϵ can be made as small as we like, this implies that $\mathbf{x}_{t_M}^k \rightarrow \bar{\chi}$ and $\mathbf{x}_{b_M}^k \rightarrow \underline{\chi}$ as $k \rightarrow \infty$, and that χ exists. •

The important point in the proof above was that the difference between $\bar{\chi}$ and $\underline{\chi}$ implies that eventually the difference between coordinates of \mathbf{x}^k grows until it is so large that coordinates, which can each only change by D on each step, cannot ‘jump the gap’. It is helpful to make the idea of a gap precise.

Definition 1 For a point $y \in \mathbb{R}^n$ order the coordinates,

$$y_{i_1} \geq y_{i_2} \geq \dots \geq y_{i_n}$$

and say that for the point y there is gap of size G between coordinates $\{i_1, \dots, i_j\}$ and $\{i_{j+1}, \dots, i_n\}$ if

$$y_{i_j} - y_{i_{j+1}} > G.$$

Gaps are important, since if large enough gaps arise between coordinates of points x^k in our trajectory (as they ultimately must), we may, in the case where F is a minmax function, deduce much about the the form of the function F , as illustrated by the following lemma.

Lemma 2.1 Suppose that F is a minmax function, and that, in the notation introduced above, there is some k for which x^k has a gap of size $C+D$ between coordinates $A = \{i_1, \dots, i_j\}$ and coordinates $B = \{i_{j+1}, \dots, i_n\}$. Then, for coordinates $i \in A$ the coordinate functions F_i have the form:

$$F_i(x) = [(x_A + c) \vee \dots] \wedge [(x_A + c) \vee \dots] \wedge \dots \wedge [(x_A + c) \vee \dots]$$

where each max only term includes at least one of the coordinates from A . [In the expression above x_A refers to one of the coordinates from A (not necessarily the same one each time) and the c 's represent possibly different constants.] For coordinates $i \in B$, on the other hand, the coordinate functions F_i , take the form

$$F_i(x) = [(x_B + c) \vee (x_B + c) \vee \dots \vee (x_B + c)] \wedge [\vee \dots \vee] \wedge \dots \wedge [\vee \dots \vee]$$

where at least one of the max only terms contains only coordinates from the set B .

Proof. This is immediate, since if F does not have this form, we will either have that

$$x_r^{k+1} = x_s^k + c$$

for some $r \in A$ and $s \in B$, or vice versa, and in either case

$$|x_r^{k+1} - x_r^k| \geq D$$

which is a contradiction. •

Remark. For the $n = 2$ case above, if F is a minmax function the asymptotic behaviour of x^k is particularly simple. Assuming that ultimately it is the first coordinate of x^k which is larger and tends to $\bar{\chi}$, the lemma implies that the coordinate function F_1 has x_1 in every max only term, asymptotically the terms in x_2 become irrelevant, and that eventually $x_1 \rightarrow x_1 + \bar{\chi}$ on each step. Similarly, F_2 contains at least one max only term which contains only x_2 , and ultimately the behaviour of this coordinate is $x_2 \rightarrow x_2 + \underline{\chi}$ on each step.

3 Proof of the existence of χ for $n = 3$ and F min-max

As in the $n = 2$ case we choose an ϵ and N such that for $k > N$ we have

$$|x_{i_k}^k - k\bar{\chi}| < k\epsilon$$

for the top coordinate;

$$|\mathbf{x}_{b_k}^k - k\underline{\chi}| < k\epsilon$$

for the bottom coordinate; and

$$k(\overline{\chi} - \underline{\chi} - 2\epsilon) > 2G,$$

so that the top and bottom coordinates cannot swop in one step, and there is at least one gap of size $G = C + D$ between the ‘middle’ coordinate and the top coordinate, or the middle coordinate and the bottom coordinate.

We now consider how it could happen that χ fails to exist. If eventually there is only ever one gap between the three coordinates, then one pair remain a bounded distance apart and are $k\epsilon$ close to $k\overline{\chi}$ while the other coordinate remains $k\epsilon$ close to $k\underline{\chi}$ or *vice versa*. In this case χ clearly exists. So if χ does not exist we must infinitely often have gaps both between top and middle and between middle and bottom coordinates.

Without loss of generality, suppose that for some such k , $t_k = 1$, $b_k = 3$ and the middle coordinate $m_k = 2$. Then F_1 has x_1 in every max only term, F_2 contains at least one max only term which contains x_2 but not x_1 , and F_3 contains at least one max only term containing only x_3 . As long as these gaps persist, therefore, the order $x_1 > x_2 > x_3$ of the three coordinates remains the same, and the dynamics is given by

$$\mathbf{x}_1^{k+1} = \mathbf{x}_1^k + \alpha, \quad \mathbf{x}_2^{k+1} = \mathbf{x}_2^k + \beta, \quad \mathbf{x}_3^{k+1} = \mathbf{x}_3^k + \gamma$$

where α , β and γ are fixed given the gaps and the ordering of the coordinates. Now if $\alpha \geq \beta \geq \gamma$ the gaps will persist forever, and we will have $\chi = (\alpha, \beta, \gamma)$ (so that $\alpha = \overline{\chi}$ and $\gamma = \underline{\chi}$) and have nothing more to prove. Otherwise we must have either $\beta > \alpha$ or $\beta < \gamma$ and the middle coordinate \mathbf{x}_2 moves monotonically closer to either the top or bottom coordinates. Importantly, the same situation must pertain for any k where the three coordinates have the same order, so if we take (again without loss of generality) the case where $\beta > \alpha$, coordinate \mathbf{x}_2^k always moves closer to \mathbf{x}_1^k and away from \mathbf{x}_3^k whenever it lies between top coordinate \mathbf{x}_1^k and bottom coordinate \mathbf{x}_3^k . In particular, it is not possible for the coordinate \mathbf{x}_2 to wander up and down between the top and bottom coordinates as in the example of a non-converging topical function given in [5]; the finite number of terms in the minmax expressions determines that ultimately \mathbf{x}_2 will always increase at the constant rate β whenever it lies between \mathbf{x}_1 as the top coordinate and \mathbf{x}_3 as the bottom coordinate.

Thus, if χ does not exist, there must be a larger k value where \mathbf{x}_2^k becomes the top coordinate (constrained for the time being to lie in the top band $|\mathbf{x}_2^k/k - \overline{\chi}| < \epsilon$) and \mathbf{x}_1^k becomes the middle coordinate separated from \mathbf{x}_2^k by a new gap (the two coordinates cannot remain together, and we have just seen that if \mathbf{x}_1^k remains above \mathbf{x}_2^k the latter will increase towards the former). Given that this new ordering of coordinates has arisen on the orbit, we may deduce more about the coordinate functions F_i , though we do not yet need to spell out the details. More importantly at this stage, we can deduce that either this new set of gaps persists forever and χ exists with each coordinate increasing by a fixed amount each iteration, or the coordinates remain in this new order only until \mathbf{x}_1^k reaches the bottom band. A similar argument to that above then shows that if χ does not exist the coordinate \mathbf{x}_3^k must next become the middle coordinate, moving up towards \mathbf{x}_2^k . Continuing in this way, we either eventually reach the easy situation where two particular gaps

persist forever, or we must have that the order of the coordinates as k increases cycles forever:

$$\begin{aligned} (x_1 > x_2 > x_3) &\rightarrow (x_2 > x_1 > x_3) \rightarrow (x_2 > x_3 > x_1) \rightarrow \\ (x_3 > x_2 > x_1) &\rightarrow (x_3 > x_1 > x_2) \rightarrow (x_1 > x_3 > x_2) \rightarrow (x_1 > x_2 > x_3) \rightarrow \dots \end{aligned}$$

In this case, each such order occurs with two gaps for some k , and so the appropriate deductions may be made about the coordinate functions F_i . Consider F_1 for example. Since x_1 is sometimes top, each max only term contains x_1 , and the behaviour of x_1^k while it is top and separated from the other coordinates by a gap is given by

$$x_1^{k+1} = x_1^k + \alpha_T$$

where α_T is the minimum of all the constants appearing with x_1 in F_1 . Also, since x_1 is sometimes bottom, F_1 contains at least one max only term containing only x_1 , and the behaviour of x_1^k while it is bottom and separated from the other coordinates by a gap is given by

$$x_1^{k+1} = x_1^k + \alpha_B$$

where α_B is the minimum of all the constants appearing with x_1 in these terms only. The impossibility of this sort of behaviour becomes apparent when we note that

$$\alpha_B \geq \alpha_T$$

since α_B is the minimum over a smaller subset of the constants occurring with x_1 in F_1 than α_T . This implies that x_1^k increases more quickly when it is at the bottom than when it is at the top which does not seem reasonable, and will eventually lead us to the contradiction we require. More formally, it is also easy to see that when x_1 is in the middle it has an intermediate rate of increase, leading us to an inequality of the form

$$\alpha_T \leq \alpha_{213}, \alpha_{312} \leq \alpha_B \quad (1)$$

where the α_{213} is, for example, the intermediate rate of increase of x_1 when $x_2 > x_1 > x_3$.

There are two obvious ways of establishing the final contradiction, which are both presented here in case one turns out to be better adjusted to generalization to dimension $n > 3$. Both methods depend on the observation that we must have, for example,

$$\alpha_T \leq \underline{\chi} < \bar{\chi} \leq \alpha_B \quad (2)$$

where there will be similar inequalities for the rates β_T and β_B for x_2 and γ_T and γ_B for x_3 . This follows since it would not be possible for the coordinates x_i^k to wander between the top and bottom bands if their maximal rates of increase were not at least $\bar{\chi}$ and their minimal rate of increase at least as small as $\underline{\chi}$.

Now one method of completing the proof consists of combining the above two inequalities with further inequalities of the kind

$$\alpha_{213} < \beta_T \quad (3)$$

which follows (with intentionally strict inequality) because we have deduced that for $\underline{\chi}$ not to exist we must have that at some stage while x_2 is in the top band, x_1 starts in the bottom band with x_3 and increases until it reaches the top band; if the

inequality were not strict both x_1 and x_3 would need to increase together which would contradict the fact that x_3 must stay in the bottom band. Each order of coordinates in the cycle gives a similar inequality. Combining these we can deduce, for example

$$\alpha_{213} < \beta_T \leq \beta_{321} < \gamma_T \leq \gamma_{132} < \alpha_T \leq \alpha_{213} \quad (4)$$

which is a contradiction.

The other method of completing the proof does not require so many detailed conclusions about the entire sequence of orderings of the coordinates observed along the orbit (and may therefore be more suited to generalization?). Note just that the inequalities above limit the number of steps for which x_1 can remain in the top band whilst separated from the other coordinates by a gap. In particular, if $x_1^k \leq k(\bar{\chi} + \epsilon)$ for some k , then after a number of steps

$$l_1 = \frac{2k\epsilon}{\bar{\chi} - \underline{\chi} - \epsilon}$$

we have $x_1^k + l_1\alpha_T < (k+l_1)(\bar{\chi} - \epsilon)$, so l_1 is an upper bound on the number of steps for which x_1 can remain in the top band whilst separated from the other two coordinates by a gap. Meanwhile, by a very similar argument for coordinate x_2 , x_2 takes at least

$$l_2 \geq \frac{k(\bar{\chi} - \underline{\chi} - 2\epsilon)}{\epsilon}$$

steps to get from the top band at step k to the bottom band at step $k + l_2$. But by choosing ϵ sufficiently small, we can easily ensure that $l_2 > l_1$, so that if coordinates x_1 and x_2 are together at the top for some k then x_1 must drop out of the top band before x_2 can reach the bottom band. This is incompatible with one of the steps in the cycling behaviour described above, and so completes the proof of the theorem. [Note that the final inequality between l_1 and l_2 is very strong for small ϵ , so perhaps this argument still uses more than necessary.]

4 Proof of the existence of χ for $n > 3$ and F min-max

The basic ideas in the proof above should be sufficient to prove the existence of χ for dimension $n > 3$. However, we will not want to have to rule out all the possible cyclic rearrangements of the coordinate orderings individually. (In the case of $n = 3$ we came quite quickly to the conclusion that a particular cycle must occur if χ were to fail to exist.) Some form of induction on the dimension should be sufficient. Certainly we will have that gaps must appear eventually between groups of coordinates. For each such group the inductive hypothesis would imply that left to themselves these coordinates of the cycle-time vector would exist, and in the long-term gaps will get sufficiently large and long lasting that it will be possible to appeal to this behaviour, and to construct appropriate inequalities for the limiting rates obtained for different groupings and orders of coordinates. One particular case of this type of argument would be to consider all those coordinates for which the *lim sup* behaviour goes like $\bar{\chi}$. Assuming the theorem is true, a persistent gap should ultimately open up between these coordinates and all the others, and these will all have limit $\bar{\chi}$ and the remaining coordinates will form a system of lower dimension with maximal rate of increase strictly less than $\bar{\chi}$ which can be dealt with by the inductive hypothesis.

5 Decomposition of eventual dynamics

It is apparent from the above proof, in the spirit of the remark at the end of section 2, that eventually the behaviour of the trajectory can be decomposed by partitioning the coordinates into sets, over each of which χ_i is constant, and where the eventual behaviour of each set of these coordinates is given by a reduced minmax function involving only other coordinates from the same set.

6 Other open questions

Can the argument be sharpened up to show that for topical functions $\bar{\chi}$ and $\underline{\chi}$ are realised as limits by particular coordinates (as conjectured in [5]), or is this is different kind of question?

For minmax functions, can we show that eventually the behaviour of orbits under F is governed exactly (and not just asymptotically) by χ . In other words, given \mathbf{x} does there exist k and p such that

$$F^{k+lp}(\mathbf{x}) = F^k(\mathbf{x}) + lp\chi \quad \text{for all integer } l \geq 0.$$

To prove this, given the eventual decomposition of the dynamics, it is only necessary to show that when $\chi = h$ is the same in each coordinate that every trajectory is eventually periodic. These results would, of course, follow from the Duality Conjecture, but they may be provable in other ways as well.

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