Shilnikov's Saddle-Node Bifurcation

Paul Glendinning[1], Colin Sparrow
Basic Research Institute in the
Mathematical Sciences
HP Laboratories Bristol
HPL-BRIMS-96-07
February, 1996

Shilnikov; bifurcation; homoclinic orbit

In 1969 Shilnikov described a bifurcation for families of ordinary differential equations involving $n \geq 2$ trajectories bi-asymptotic to a non-hyperbolic stationary point. At nearby parameter values the system has trajectories in correspondence with the full shift on n symbols. We investigate a simple (piecewise smooth) example with an infinite number of homoclinic loops. We also present a smooth example which shows how Shilnikov's mechanism is related to the Lorenz bifurcation by considering the unfolding of a previously unstudied codimension two bifurcation point.

[1] Isaac Newton Institute for Mathematical Sciences

To appear: International Journal of Bifurcation and Chaos: issue in honour of Leonid Shilnikov's birthday.

Shilnikov's Saddle-Node Bifurcation

Dedicated to Leonid Shilnikov on his birthday.

Paul Glendinning† and Colin Sparrow*

†*Isaac Newton Institute for Mathematical Sciences
20, Clarkson Road, Cambridge CB3 0EH, U.K.

*Basic Research Institute in the Mathematical Sciences
Hewlett-Packard Laboratories, Filton Road,
Stoke Gifford, Bristol BS12 6QZ, U.K.

November, 1995

Abstract

In 1969 Shilnikov described a bifurcation for families of ordinary differential equations involving $n \geq 2$ trajectories bi-asymptotic to a non-hyperbolic stationary point. At nearby parameter values the system has trajectories in correspondence with the full shift on n symbols. We investigate a simple (piecewise smooth) example with an infinite number of homoclinic loops. We also present a smooth example which shows how Shilnikov's mechanism is related to the Lorenz bifurcation by considering the unfolding of a previously unstudied codimension two bifurcation point.

Keywords Shilnikov, bifurcation, homoclinic orbit

(†) Permanent Address: School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London, U.K.

(*) Permanent Address: Department of Pure Mathematics and Mathematical Statistics, Mill Lane, Cambridge CB2 1SB, U.K.

1. Introduction

Shilnikov [1969] described a bifurcation involving homoclinic orbits to a non-hyperbolic stationary point. If there are $n \geq 2$ homoclinic orbits to the stationary point, and a parameter is varied so that the stationary point undergoes a saddle-node bifurcation, then on the side of the bifurcation where no stationary points exist locally there is an invariant set on which the dynamics is equivalent to a full shift on n symbols. As far as we are aware, there are no examples of this bifurcation discussed in any detail in the literature, though Dombre et al [1986] and Lau et al [1992] suggest the bifurcation may occur in some simple models of incompressible fluid flow.

Shilnikov [1969] considers $(m+n+1)^{th}$ order differential equations of the form

$$\dot{x} = P(x, y, z, \mu), \quad \dot{y} = Q(x, y, z, \mu), \quad \dot{z} = R(x, y, z, \mu) \tag{1.1}$$

where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^p$, $z \in \mathbf{R}$ and μ is a real parameter. The functions P, Q and R are smooth and satisfy further conditions described below. Assume that when $\mu = 0$, the system (1.1) has a non-hyperbolic equilibrium at the origin, O, and that the Jacobian matrix of the linear flow at O has m ($m \geq 1$) eigenvalues with negative real parts, p ($p \geq 1$) eigenvalues with positive real parts and a simple zero eigenvalue. The coordinates can be chosen so that the corresponding eigenspaces are the x, y and z coordinate planes respectively. Assume further that the higher order conditions for a standard saddle-node bifurcation to occur in the z-direction as μ is varied hold and that the sign of μ is chosen so that if $|\mu|$ is sufficiently small then, in a neighbourhood of O, (1.1) has two hyperbolic stationary points if $\mu > 0$ and no stationary points if $\mu < 0$. ¿From standard theory [e.g. Guckenheimer & Holmes, 1983], the stable manifold of the non-hyperbolic stationary point, O, when $\mu = 0$, is a $(m+1)$-dimensional half manifold locally, spanned by the strong stable manifold, tangential to $\{y = z = 0\}$ at O, and the weak stable (half) manifold, tangential to $\{x = y = 0, z > 0\}$ at O. Similarly the unstable manifold is a $(p+1)$-dimensional half manifold locally, spanned by the strong unstable manifold, tangential to $\{x = z = 0\}$ at O, and the weak unstable (half) manifold, tangential to $\{x = y = 0, z < 0\}$ at O. Assume further that when $\mu = 0$ the following conditions hold (conditions (2) and (3) are standard genericity conditions):

1) there exist n ($n \geq 2$) trajectories, Γ_i, $1 \leq i \leq n$, which are bi-asymptotic to O;

2) Γ_i is tangential to the z-axis at O, $1 \leq i \leq n$;

3) the stable and unstable manifolds of O intersect transversely on Γ_i, $1 \leq i \leq n$.

Now let $U(\Gamma_1, \ldots, \Gamma_n, \epsilon)$ be some small ϵ-neighbourhood of the homoclinic loops, Γ_i, and O.

Theorem *(Shilnikov [1969]). Consider equation (1.1) satisfying the conditions described above. For all $\epsilon > 0$ sufficiently small there exists $\mu_0(\epsilon) > 0$ such that if $\mu \in (-\mu_0, 0)$ then the set of trajectories which lie entirely in $U(\Gamma_1, \ldots, \Gamma_n, \epsilon)$ is in one-to-one correspondence with the doubly infinite sequences of n symbols.*

This result is easily interpreted in terms of the return map induced by the flow on a suitably chosen return plane. The dynamics of this return map is equivalent to the full shift on n symbols.

In this paper we present two examples of flows in which the bifurcation occurs, and discuss the circumstances in which we would expect to see the bifurcation in typical one or two parameter families of equations. In section 2 we give a somewhat artificial (non-smooth) example where the appropriate local return maps can be calculated explicitly. This example has infinitely many homoclinic orbits and gives a very direct intuitive sense of how the bifurcation mechanism works. The example of section 3 arises naturally as an adaptation of the examples of Robinson [1989] and Rychlik [1990] who were interested in systems having a Lorenz-like attractor in the vicinity of a co-dimension two bifurcation. In our example we have two homoclinic connections, and Shilnikov's mechanism creates an invariant set very similar to invariant sets occurring in Lorenz-like systems; our analysis shows how these sets are related via a new co-dimension two bifurcation. This co-dimension two bifurcation can be seen as an extension of the standard $n = 1$ case which has been more extensively studied [Schecter, 1987, Kaas-Petersen & Scott, 1988, Glendinning & Proctor, 1993].

2. An example illustrating Shilnikov's Theorem

In this section we will study an example in three dimensions where the main structural conditions and conclusions of the theorem are easily verified. The price that we pay for this simplicity is that our vector field is discontinuous on a two-dimensional subspace, but since the subspace is far from the important stationary point(s) of the system this does not significantly affect the analysis. The example is, for $|\theta| \leq \frac{\pi}{2}$:

$$\begin{aligned} \dot{\theta} &= \mu - 2\sin^2\frac{\theta}{2} \\ \dot{x} &= \lambda x \\ \dot{y} &= -y \end{aligned} \tag{2.1a}$$

and for $|\theta| \geq \frac{\pi}{2}$:

$$\begin{aligned} \dot{\theta} &= \mu - 2\sin^2\frac{\theta}{2} \\ \dot{x} &= -\rho x - \omega y + y(x^2+y^2) \\ \dot{y} &= \omega x - \rho y - x(x^2+y^2) \end{aligned} \tag{2.1b}$$

where $\theta \in S^1$ takes values $-\pi \leq \theta < \pi$ and $x, y \in \mathbf{R}$. The real parameters μ, λ, ρ and ω are all strictly positive. If we preferred to work in Euclidean space we could add another variable $r \in \mathbf{R}^+$ governed by an equation such as:

$$\dot{r} = r(1 - r^2)$$

and then concentrate on the attracting subspace $r = 1$; for simplicity we use the three-dimensional equation (2.1) as it stands.

It is immediate from equation (2.1) that if $\mu < 0$ then $\dot{\theta} < 0$ for all θ, so θ decreases monotonically. For these μ values there can clearly be no stationary points of the system. At $\mu = 0$ there is a saddle-node bifurcation and for $\mu > 0$ there are two stationary points on the circle $x = y = 0$ with θ-values close to zero. The θ motion is independent of the other two coordinates.

The (x, y) coordinates evolve according to two different simple systems, each with a fixed point at the origin. Depending on the value of θ we either have, for $|\theta| < \frac{\pi}{2}$:

$$\dot{x} = \lambda x, \qquad \dot{y} = -y \tag{2.2}$$

which is linear with a saddle at the origin and with solutions expanding in the x-coordinate and contracting in the y-coordinate,

$$x(t) = x(0)e^{\lambda t}, \quad y(t) = y(0)e^{-t},$$

or, for $|\theta| > \frac{\pi}{2}$:

$$\dot{x} = -\rho x - \omega y - y(x^2 + y^2), \qquad \dot{y} = \omega x - \rho y + x(x^2 + y^2) \tag{2.3}$$

which can also be written in polar coordinates (R, ψ) as

$$\dot{R} = -\rho R, \qquad \dot{\psi} = \omega + R^2. \tag{2.4}$$

Again, equation (2.4) can be solved explicitly, but for the moment the important thing to note is that the R-coordinate decays and that the rate of rotation in the angular direction decreases with R.

Fig 1 near here

We now wish to establish that if $\mu = 0$, so that equation (2.1) has a single non-hyperbolic stationary point at $(\theta, x, y) = (0, 0, 0)$, then there are solutions equivalent to the special homoclinic solutions in the conditions of Shilnikov's Theorem. Fig 1 shows the solid torus $R < M$ (for some M which can be as large as we like) with sections at $\Sigma_\pm$, where

$$\Sigma_\pm = \{(\theta, x, y) \mid \theta = \pm\frac{\pi}{2}\}.$$

The local unstable manifold of the stationary point includes the set $y = 0$, $-\frac{\pi}{2} < \theta < 0$ and intersects Σ_- on the line $y = 0$ labelled W^u. Similarly, the set $x = 0$, $0 < \theta < \frac{\pi}{2}$, is part of the local stable manifold of the stationary point and intersects Σ_+ on the line $x = 0$ labelled W^s. There will be homoclinic connections of the type described by Shilnikov's theorem if the image of the line W^u under the flow (2.4) from Σ_- to Σ_+ intersects W^s.

This image is easy to calculate. The time taken to get from one section to the other is

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{2\sin^2\frac{\theta}{2}} = 2 \tag{2.5}$$

independent of where on the section Σ_- the trajectory begins.

During this time the (R, ψ) coordinates evolve under the nonlinear system (2.4), which has solutions

$$R(t) = R(0)e^{-\rho t}, \quad \psi(t) = \psi(0) + \omega t + \frac{R(0)^2}{2\rho}(1 - e^{-2\rho t}). \tag{2.6}$$

Points on the intersection of the local unstable manifold of the stationary point with the surface Σ_- lie on the line $y = 0$, i.e. they have $\psi(0) = 0$ or $\psi(0) = \pi$ and $R(0) = U$, $U \geq 0$ arbitrary. After time 2, given by (2.5), trajectories started at these points strike the surface Σ_+ at (R, ψ) with

$$R = Ue^{-2\rho}, \quad \psi = \psi(0) + 2\omega + \frac{U^2}{2\rho}(1 - e^{-4\rho}). \tag{2.7}$$

This is a double infinite spiral centred on $(x, y) = (0, 0)$ with infinitely many intersections (as $U \to \infty$) with W^s on Σ_+. Each of these intersection points lies on a different homoclinic orbit, bi-asymptotic to the non-hyperbolic stationary point. The intersection points on one arm of the spiral (the image of the half-line $(U, 0)$ on Σ_-) occur at points $(0, U_n)$ in Cartesian coordinates, where we use equation (2.7) to show that U_n is given by

$$U_n^2 = \frac{2\rho(\psi_n - 2\omega)}{e^{4\rho} - 1}, \tag{2.8}$$

where the U_n values alternate in sign, and $\psi_n = (n + \frac{1}{2})\pi$. Loosely speaking, the value of n counts the number of half-turns the corresponding trajectory makes around $x = y = 0$ in passing from Σ_- to Σ_+, and it is therefore necessary to restrict attention to values of $n \geq n_0$ where n_0 is the number of half-turns that occur very near to $x = y = 0$ due to the positive value of ω. From (2.8) we see that n_0 is the smallest integer that makes the right-hand side of (2.8) strictly positive. The intersections of the other arm of the spiral can be similarly labelled as a series of points $(0, U_{-n})$ on W^s with $U_{-n} = -U_n$.

What we expect from Shilnikov's Theorem, therefore, is that for $\mu < 0$ but small there should be trajectories in correspondence with the full shift on some large set of symbols. In fact, given that our system has an infinite number of homoclinic orbits, we can choose any large number M of these and deduce that there is some small interval of μ values in which a subset of our trajectories is in correspondence with a full shift on M symbols.

The following argument, which we run through for our system, (2.1), also illustrates the geometric idea behind Shilnikov's original proof.

Fig 2 near here

Suppose that $\mu < 0$, and let $P_n \in \Sigma_+$ be the intersection points $(0, U_n)$ described above, $|n| \geq n_0$. Consider, as illustrated in Fig 2, the thin strip, B_n, containing P_n defined by

$$B_n = \{(\theta, x, y) \mid \theta = \frac{\pi}{2},\ |x| < \epsilon,\ \frac{1}{2}(U_n + U_{-(n-1)}) < y < \frac{1}{2}(U_n + U_{-(n+1)})\ \}$$

for some sufficiently small $\epsilon > 0$. Since $\dot{\theta} < 0$, trajectories through B_n will move through to the surface Σ_- after a time τ, which depends only on μ, given by

$$\tau(\mu) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{|\mu| + 2\sin^2\frac{\theta}{2}}\ . \tag{2.9}$$

Note that $\tau(\mu)$ tends to infinity as μ tends to zero from below. Thus, from (2.2), the first intersection of trajectories started at points in B_n with the section Σ_- is the strip

$$C_n = \{(\theta, x, y) \mid \theta = -\frac{\pi}{2},\ |x| < \epsilon e^{\lambda\tau},\ \frac{1}{2}(U_n + U_{-(n-1)}) < ye^{\tau} < \frac{1}{2}(U_n + U_{-(n+1)})\ \}.$$

The important thing to note about this set is that it is very close to the unstable manifold of the non-hyperbolic stationary point which exists for $\mu = 0$ (the line $y = 0$ on Σ_-) and the x-coordinate has been stretched by a factor which tends to infinity as μ tends to zero from below. The image of this strip under the flow from Σ_- to Σ_+ will therefore be close to the spiral obtained earlier (equation (2.7)), and hence, as illustrated in Fig 2, will intersect a large number of the original strips, B_k, transversely. More precisely, we have the following result.

Proposition *Let N be any positive integer. There exists $\mu_N > 0$ such that if $-\mu_N < \mu < 0$ then there are trajectories of equation (2.1) in correspondence with the full shift on $2N$ symbols.*

Sketch proof: Let n_0 be the smallest positive integer such that the right hand side of equation (2.8) is strictly positive, and consider the $2N$ sets B_n for $n_0+1 \leq |n| \leq n_0+N+1$, on Σ_+. Suppose we have chosen an $\epsilon > 0$ sufficiently small. Then we can choose $\mu < 0$ sufficiently close to zero, which makes $\tau(\mu)$ sufficiently large, so that:

$$\epsilon e^{-2\rho+\lambda\tau(\mu)} > |U_{n_0+N+2}|.$$

This will ensure that C_n is mapped to Σ_+ by the nonlinear flow (2.4) after time

$$T(\mu) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{|\mu| + 2\sin^2\frac{\theta}{2}}$$

which is close to 2, in such a way that for each m, $n_0 + 1 \leq |m| \leq n_0 + N + 1$, the image of C_n will intersect B_m transversely. Hence the return map on Σ_+ has orbits equivalent to the full shift on $2N$ symbols, recording passages of trajectories through the sets B_m, $n_0 + 1 \leq |m| \leq n_0 + N + 1$.

In fact, both $\tau(\mu)$ and $T(\mu)$ can be calculated explicitly [Prudnikov et al, 1992], but this does not really teach us anything new; $\tau(\mu)$ tends to infinity as μ tends to zero from below proportional to $|\mu|^{-\frac{1}{2}}$, and $T(\mu) = 2 + O(\mu)$.

3. Codimension two bifurcations.

Shilnikov's homoclinic saddle-node bifurcation is of codimension one, so in typical two-parameter families of systems the bifurcation will occur on line segements. It is natural to ask how these line segements can terminate. In other words, what are the codimension two bifurcations which can create or destroy the conditions required for Shilnikov's theorem to hold? There are two obvious answers to this question: either the local conditions described in the theorem break down for some value of the parameters, or one or more of the homoclinic loops is broken. We shall concentrate on the latter possibility here.

Let us first consider the case of a single homoclinic loop. If the codimension two bifurcation does not involve other stationary points of the system, then there are two ways of breaking a homoclinic loop. One of these, which will not concern us, is that the amplitude of the loop may tend to infinity or zero. The other is that there may be a point in the two-parameter space at which a homoclinic loop tends to the non-hyperbolic (saddle-node) stationary point tangential to the strong unstable manifold as $t \to -\infty$ or tangential to the strong stable manifold as $t \to \infty$, which breaks condition (2) of the Introduction. (The normal situation is that homoclinic loops approach stationary points tangential to the weakest contracting and expanding directions.) This second situation has been studied by a number of authors going back to Andronov in the case where there is only a single homoclinic loop which can be realized by a two parameter family of differential equations in the plane (e.g. Schecter, 1987, Kaas-Petersen & Scott, 1988, Glendinning & Proctor, 1993). It will be instructive to consider this simple case first; the associated bifurcation diagram is sketched in Fig 3, which also includes schematic phase plane diagrams of important trajectories in each region of parameter space. The parameter μ controls the saddle-node bifurcation at the origin. We choose this parametrization, as in the previous section, such that if $\mu > 0$ there are two stationary points near the origin (one a saddle, the other a repellor), that these collide in a saddle-node bifurcation at $\mu = 0$ and do not exist if $\mu < 0$. The second parameter, ν, is chosen to control the point at which the stable manifold of the saddle returns (in backwards time) to a neighbourhood of the origin. More concretely, choosing coordinates (x, z) in the plane, we can require the equations to take the form

$$\begin{aligned} \dot{x} &= \lambda x + \dots \\ \dot{z} &= \mu - z^2 + \dots \end{aligned}$$

near to the origin, where the dots indicate terms which do not affect the qualiatative local behaviour of the flow near the origin but are such that the inverse return map from a surface $\{(x, z)|\ z = h\}$ $(h \ll 1)$ to $\{(x, z)|\ x = h\}$ is approximately of the form

$$(x, h) \to (h, \nu + cx + \dots).$$

It is usual to study this system with time running the other way, but for consistency with the rest of the paper we make the choice as above.

Fig 3 near here

Consider the case $\mu = 0$. There is a non-hyperbolic stationary point at the origin which has a two-dimensional local unstable manifold spanned by the strong unstable manifold (approximately the $x-$axis) and the weak unstable manifold (approximately the negative $z-$axis), and a one dimensional (weak) local stable manifold (approximately the positive $z-$axis). In the appropriate coordinates, this local stable manifold intersects $z = h$ at $x = 0$ and, by the assumption made above on the inverse return map, returns (in backwards time) to a small neighbourhood of the origin with $x = h$ at $z = \nu$. So if $\nu = 0$ there is a single homoclinic orbit which is tangential to the strong unstable manifold of the origin. If $\nu < 0$ then there is a single homoclinic loop which is tangential to the weak unstable manifold (the z-axis for $z < 0$) whilst if $\nu > 0$ there is no homoclinic loop.

There are two important curves in the half-plane of parameter space $\mu > 0$. The upper branch of the stable manifold of the upper stationary point, $(0, \sqrt{\mu})$, returns (in backwards time) to a neighbourhood of the origin with $z = \nu$. Hence if $\nu = +\sqrt{\mu}$, there is a standard homoclinic orbit to the upper stationary point; it is well known that in this case a periodic orbit is created as parameters are altered to cross into the region $\nu < \sqrt{\mu}$. Similarly, if $\nu = -\sqrt{\mu}$ the same branch of the stable manifold of the upper stationary point returns tangential to the strong unstable manifold of the lower (repelling) stationary point. In the region $\nu < -\sqrt{\mu}$, $\mu > 0$, there is a C^1 heteroclinic loop consisting of this manifold and the part of the z-axis between the two stationary points. As μ decreases through zero the two stationary points come together, and the saddle-node bifurcation occurs on a periodic orbit, which exists in $\mu < 0$.

The bifurcation described above is the one homoclinic loop analogue of the bifurcation we wish to consider. A standard way of introducing more homoclinic loops is to use symmetry. We can therefore consider systems

$$\begin{aligned} \dot{x} &= \lambda x + f_1(x, y, z; \mu, \nu) \\ \dot{y} &= -y + f_2(x, y, z; \mu, \nu) \\ \dot{z} &= \mu - z^2 + f_3(x, y, z; \mu, \nu) \end{aligned} \tag{3.1}$$

which are invariant under the symmetry $(x, y, z) \to (-x, -y, z)$. Here, $\lambda > 0$ and, as usual, the functions f_i, $i = 1, 2, 3$, contain the higher order terms and satisfy the assumptions of Shilnikov's Theorem. If we now assume that when $\mu = \nu = 0$ one branch of the strong unstable manifold of the origin (locally the positive x-axis, say) is part of a homoclinic orbit which approaches the origin tangential to the weak stable direction (the positive z-axis, say), then, by symmetry, a second homoclinic loop exists (tangential to the negative x-axis as $t \to -\infty$ and to the positive z-axis as $t \to \infty$). The bifurcation diagram of this codimension two configuration, sketched in Fig 4, is essentially the same as the one-loop diagram of Fig 3, but whenever there is one homoclinic loop, a second homoclinic loop exists by symmetry. With analogous dependence of the system on ν, the conditions for Shilnikov's Theorem will hold on the $\mu = 0$ axis with $\nu < 0$; we can arrange that two homoclinic loops to the non-hyperbolic stationary point exist at these parameter values. Hence, as μ decreases through $\mu = 0$ with $\nu < 0$, an invariant set is created which dynamics equivalent to a full shift on two symbols.

Fig 4 near here

In the one loop case a single periodic orbit is created as μ decreases through $\mu = 0$ with $\nu < 0$, and a single periodic orbit is created by the standard homoclinic bifurcation on the curve $\nu = \sqrt{\mu}$ in $\nu > 0$ to produce a consistent bifurcation diagram. In other words, there is a satisfactory explanation for the creation and destruction of the periodic orbit on closed parameter paths enclosing the origin. In the two loop case, the $\nu = \sqrt{\mu}$ bifurcation involves two standard homoclinic orbits in the same configuration as the loops in the Lorenz equations [Lorenz, 1963, Sparrow, 1982]. The modulus of the ratio of the weaker stable eigenvalue of the Jacobian matrix at the stationary point to the unstable eigenvalue is less than one since the weak stable eigenvalue tends to zero as μ tends to zero. Hence this bifurcation also produces an invariant set equivalent to a full shift on two symbols and once again the bifurcation diagram is consistent in the same sense as above. [Sparrow, 1982, Afraimovich et al, 1983, Afraimovich & Shilnikov, 1983]. It is interesting to note that this codimension two bifurcation point can be considered as both an end-point of a bifurcation curve on which Shilnikov's homoclinic saddle-node bifurcation occurs and as the end-point of a curve of Lorenz bifurcations, another bifurcation to which Shilnikov's contribution to our understanding has been considerable.

This connection to Lorenz systems can also be exploited to produce a natural example of Shilnikov's homoclinic saddle-node bifurcation. Recently both Robinson [1989] and

Rychlik [1990] have produced systems of equations for which they can prove the existence of a pair of homoclinic loops and a (co-dimension two) bifurcation in which a Lorenz-like attractor exists arbitrarily close to the bifurcation point. Robinson's equations are:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 - \alpha y + \beta x^2 y - \nu yz \\ \dot{z} &= -\gamma z + \delta x^2 \end{aligned} \tag{3.2}$$

and he showed [Robinson, 1989] computer simulations of homoclinic orbits for these equations with $\alpha = 0.71$, $\beta = 1.8587$, $\delta = 0.1$, $\gamma = 0.7061$ and $\nu = 1$.

By continuously modifying Robinson's equation we were able to show (numerically) that the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 - 0.71y + 1.8587x^2 y - yz \\ \dot{z} &= \mu - z^2 + \delta x^2 \end{aligned} \tag{3.3}$$

has the codimension two bifurcation point of Fig 4 if $(\mu, \delta) = (\mu_0, \delta_0) \approx (0.0,\ 0.0511)$, where we believe that δ_0 is correct to 4 decimal places. (By a change of variables this equation can be brought into the general form described above.) For the range of parameters we have considered, Shilnikov's homoclinic saddle-node bifurcation occurs on the half-line $\mu = 0$, $\delta > \delta_0$, creating a strange invariant set in $\mu < 0$. The curve of Lorenz bifurcations is locally a half-parabola, $\mu = A(\delta - \delta_0)^2$, for some positive constant A and $\delta < \delta_0$. If $\delta > \delta_0$ then this parabola corresponds to the codimension one pair of heteroclinic connections of Fig 4. In fact, numerical calculations shown in Fig 5 illustrate that these bifurcation curves lie very close to a parabola over a range of parameter values. Plotting $\log \mu$ against $\log |\delta - \delta_0|$ on each of these half-parabolae we find that the points lie on a straight line of slope 2, and that the points on the homoclinic (Lorenz) bifurcation are almost coincident with those from the heteroclinic bifurcations, emphasizing that they lie on the same parabola.

Acknowledgements

We are both grateful for the inspiration provided by the work of Leonid Shil'nikov over many years.

Bibliography

Afraimovich V.S., V.V. Bykov and L.P. Shilnikov (1983). On structurally unstable attracting limit sets of Lorenz Attractor type. Trans. Mosc. Math. Soc. **44**, 153-216.

Afraimovich V.S. and L.P. Shilnikov (1983). Strange attractors and quasi-attractors. In *Nonlinear Dynamics and Turbulence*, eds. G.I.Barenblatt, G.Iooss, & D.D.Joseph, Pitman, London.

Dombre T., U. Frisch, J.M. Greene, M. Hénon, A. Mehr and A.M. Soward (1986). Chaotic streamlines in the ABC flows. J.Fluid Mech. **167**, 353-391.

Glendinning, P. and M.R.E. Proctor (1993). Travelling waves with spatially resonant forcing: bifurcations of a modified Landau equation. Int. J. Bif. and Chaos **3**, 1447-1455.

Guckenheimer, J. and P. Holmes (1983) *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Appl. Math. Sci. 42, Springer, New York.

Kaas-Petersen C. and S.K. Scott (1988). Homoclinic orbits in a simple chemical model of an autocatalytic reaction. Physica D **32**, 461-470.

Lau Y-T and J.M. Finn (1992). Dynamics of a three-dimensional incompressible flow with stagnation points. Physica D **57**, 283-310.

Lorenz E.N. (1963). Deterministic non-periodic flows. J. Atmos. Sci. **20**, 130-141.

Prudnikov A.P., Yu.A. Brychkov and O.I. Marichev (1992) *Integrals and Series Vol. I: Elementary Functions*, 3rd. printing, Gordon and Breach, New York.

Robinson C. (1989). Homoclinic bifurcation to a transitive attractor of Lorenz type. Nonlinearity **2**, 495-518.

Rychlik M.R. (1990). Lorenz attractors through Silnikov-type bifurcation I. Erg. Thy. & Dyn. Syst. **10**, 793-821.

Schecter S. (1987) The saddle-node separatrix-loop bifurcation. SIAM J. Math. Anal. **18**, 1142-1156.

Sparrow C. (1982) *The Lorenz equations: bifurcations, chaos and strange attractors*, Appl. Math. Sci. 41, Springer, New York.

Shilnikov, L.P. (1969). On a new type of bifurcation of multidimensioanl dynamical systems. Sov. Math. Dokl. **10**, 1368-1371.

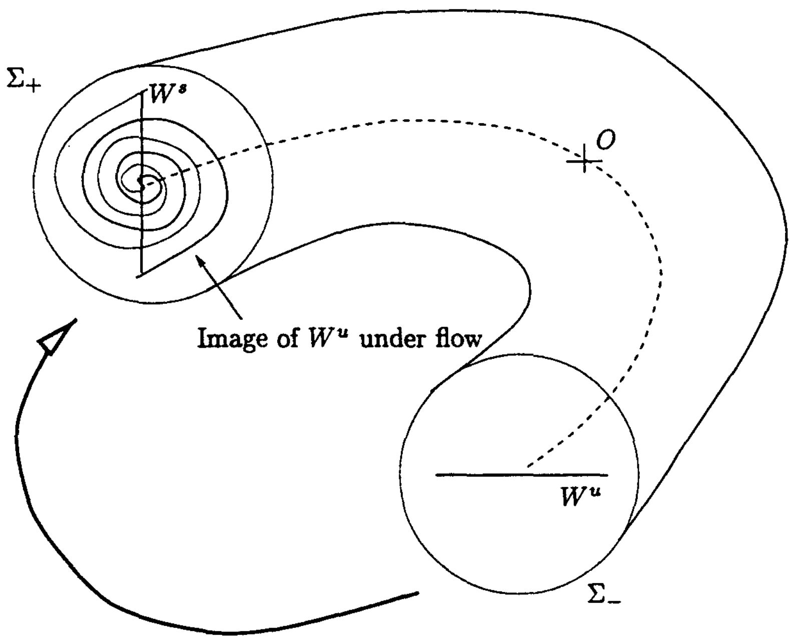

Fig. 1. The return sections $\Sigma_\pm$, and their intersection with the local stable and unstable manifolds of O.

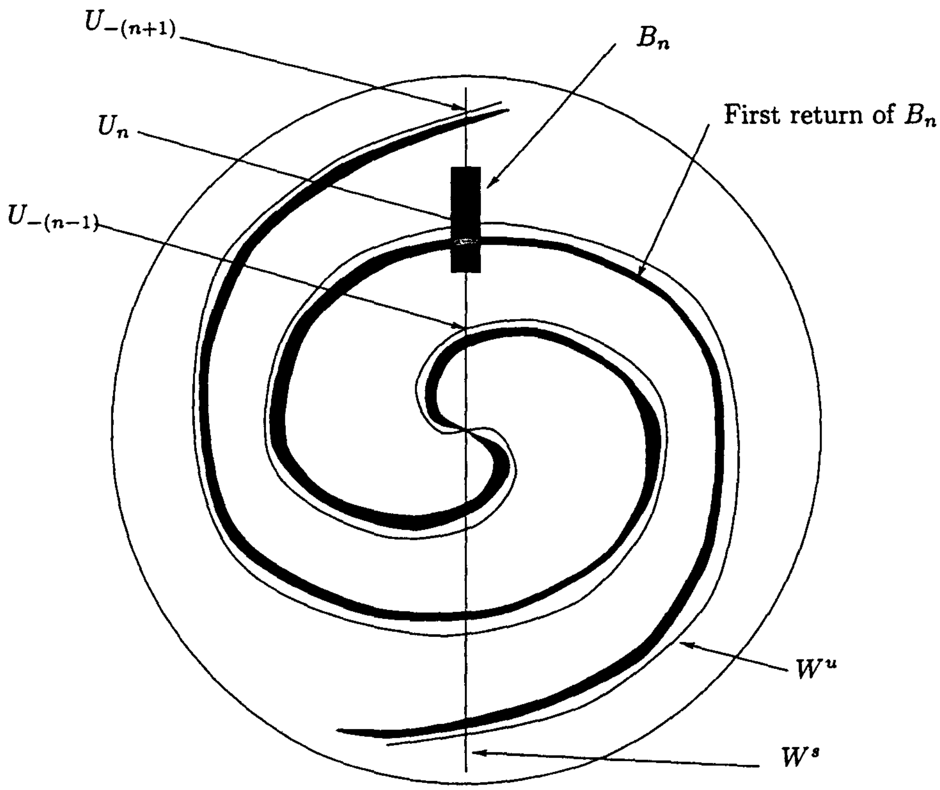

Fig. 2. The box B_n and its image under the first return map on Σ_+.

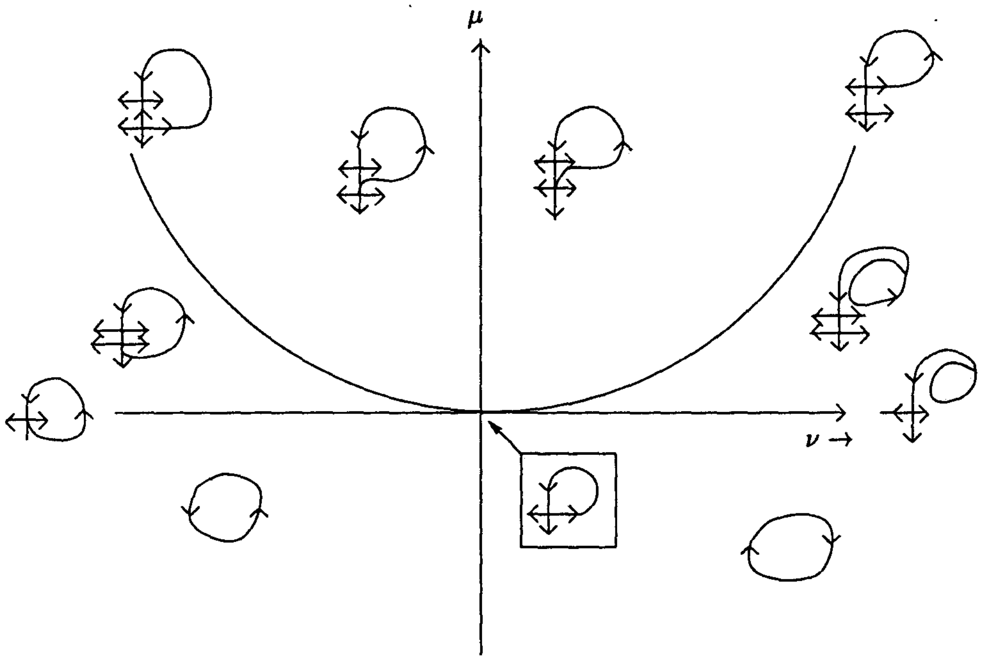

Fig. 3. Two parameter unfolding of the one loop homoclinic saddle-node bifurcation.

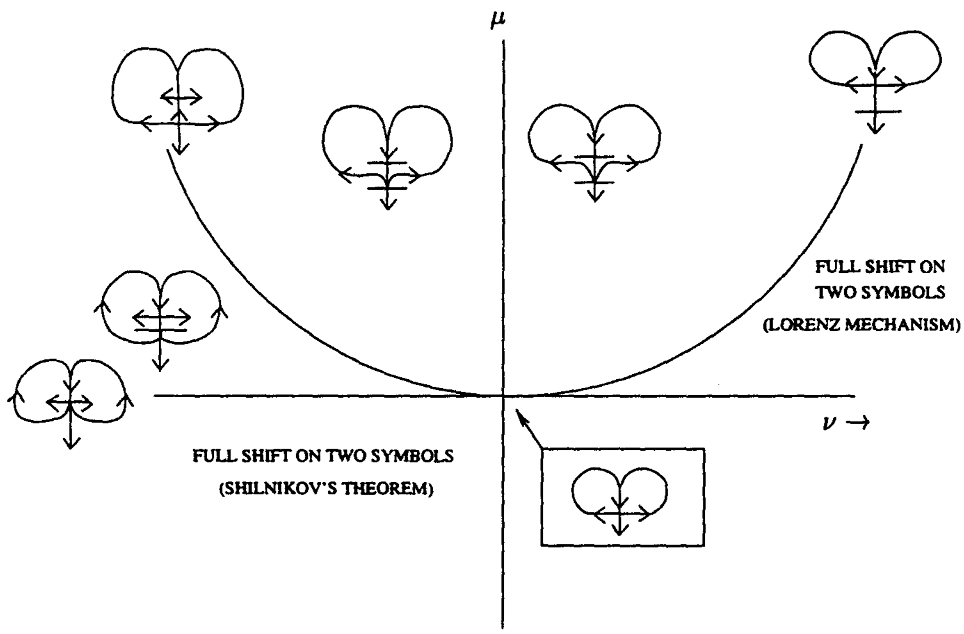

Fig. 4. Two parameter unfolding of the two loop homoclinic saddle-node bifurcation.

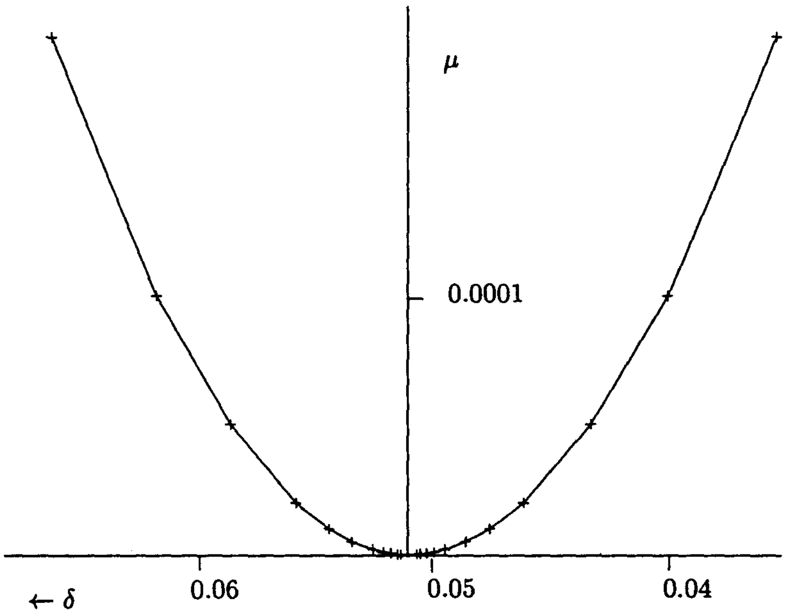

Fig. 5. Numerically calculated bifurcation diagram of equation (3.3), for comparison with Fig 4.