Complex WKB Solutions and Hamiltonian Systems on Riemann Surfaces

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There have been many important developments in which the methods of complex and algebraic geometry have been used to investigate the eigenfunctions of Hill's operator in the context of integrable equations. In this paper we link a new class of Hamiltonian systems on Riemann surfaces to systems of pde's using Bloch eigenfunctions for stationary Schrödinger equations with new types of potentials. In particular, this yields a system of pde's with monodromy.

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1 Introduction

There have been many important developments in which the methods of complex and algebraic geometry have been used to investigate the eigenfunctions of Hill's operator in the context of integrable equations. For example, Bloch eigenfunctions of Hill's operators, which are meromorphic on the associated spectral curves play an important role in the inverse scattering transform method for nonlinear soliton equations. For details see Ablowitz and Segur [1981] and Newell [1985]. Moser [1980,1981] and Knörrer [1982] established a connection between one-dimensional Schrödinger equations with finite-band potentials and the classical C. Neumann and Jacobi problems of mechanics and used Gauss map to find an isomorphism between these classical problems. Baird [1992] investigated harmonic maps associated with the solutions of C. Neumann type Hamiltonian systems on n-dimensional pseudospheres and hyperbolic spaces.

In this paper we build on these results by linking a new class of Hamiltonian systems to systems of pde's using Bloch functions for stationary Shrödinger equations with new types of potentials.

There are different ways of linking quantum mechanical problems to classical ones. One is the semiclassical limit, a second is the inverse scattering method for nonlinear soliton equations and a third is the method of complex geometric asymptotics. One of the goals of this paper is to make some connections between these approaches.

We do this in the following three main steps.

- We link Jacobi geodesic flows on quadrics and the WKB approximation of the eigenfunctions of the associated stationary one-dimensional Schrödinger equation. This is accomplished by using an asymptotic analysis of the Bloch functions. A similar asymptotic analysis can be done for a new two potential Dym system; we discuss this below.

- We introduce a new class of geodesic flows in the presence of potential fields on n-dimensional pseudospheres. A correspondence with the Schrödinger equation is established. The potential depends on two functions u and v and an integrable system for u and v is found.

- Collapsing one of the cuts on the associated Riemann surface, we get a system with monodromy. Then we proceed to construct complex semiclassical solutions on the covering space of the Riemann surface.

With regard to the second item, recall that for the C. Neumann problem, one has the special situation in which the potentials are quadratic and the spectral parameter in the Schrödinger equation appears linearly in the potential (see Moser [1980,1981]). In our cases, the potential also has poles in the spectral parameter. As an example we consider a potential with one pole. Correspodengly, the potential depends on two functions u and v and so we call these potentials *two-potentials*. Using the general approach of Alber, Camassa, Holm and Marsden [1994a,b], we establish a connection between geodesics in the presence of the above two-potentials and an integrable system, namely the following system of two coupled pde's for u and v:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{4}u''' - \frac{3}{2}uu' + v' \\ \frac{\partial v}{\partial t} &= -u'v - \frac{1}{2}uv'.\end{aligned}\right\} \tag{1.1}$$

We apply special limiting procedures (involving the coalescence of roots of the basic polynomial of the spectral curve) to quasiperiodic solutions to obtain billiard and umbilic geodesic flows (in the presence of potentials) on associated limiting Riemann surfaces. In Alber, Camassa, Holm and Marsden [1995] we will uncouple the system of pde's (1.1) and investigate phase space geometry.

Regarding the third point above, our approach to integrable systems with monodromy is demonstrated for classes of solutions of the preceding system of coupled pde's. Thus, this provides an example of a system of pde's with monodromy. Prior to this, such effects were related to mechanical systems. Then we proceed to investigate monodromy of the semiclassical approximation of the spectrum of the corresponding Schrödinger operator, as in Alber and Marsden [1995].

2 The Stationary Schrödinger Equation, Bloch functions and Generating Equations

In this section we set up stationary Schrödinger equations with potentials that, as usual, depend on a complex parameter. However, our potentials will be more general in that we allow poles in the parameter dependence. Using the Bloch function for the stationary Schrödinger equation, and generating equations, one gets a system of Euler-Lagrange equations for an associated classical mechanical system. We illustrate the procedure with the two known cases of the C. Neumann problem and the Jacobi problem of geodesics.

In the next section we consider five new examples of these potentials. Following this, we investigate WKB asymptotics of a variety of Bloch eigenfunctions that are constructed using associated dynamics of particles moving on n-dimensional complex pseudospheres in the presence of certain potentials.

We start with the one-dimensional stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + (V(x) - E)\psi = 0 \tag{2.1}$$

and rewrite it as follows

$$-\frac{\partial^2\psi}{\partial x^2} + W(x,\lambda,E)\psi = 0 \tag{2.2}$$

Where

$$W(x,\lambda,E) = \frac{U(x,E)}{\lambda}, \quad U(x,E) = 2m(V(x) - E) \qquad \text{and} \qquad \lambda = \hbar^2.$$

As usual, the WKB approximation is concerned with small values of the parameter λ.

In what follows we will use the fact that equation (2.2) is associated with the Jacobi problem of geodesics on quadrics and with the Dym equation. We will also investigate asymptotics of the Bloch eigenfunctions used in the inverse scattering transform theory (see Ablowitz and Segur [1981]) for small λ and show that it coincides with the WKB approximation for the usual Schrödinger equation (2.1).

Then we will establish a link between equations of the form (2.2) but determined by more general class of potentials $W(x, \lambda, E)$ and associated mechanical systems of particles moving on n-dimensional complex pseudospheres in the presence of certain potentials.

Then we will consider the asymptotics of the associated Bloch functions. In our notations λ will be the complex parameter, and we will treat it independently from the energy E.

We start by looking for a solution of (2.2) in the form of a Bloch function

$$\psi = \sqrt{\frac{B}{B_0}} \exp\left(\pm \int_{x_0}^{x} \frac{\sqrt{C}}{B}\, dx\right), \tag{2.3}$$

where $B = B(x, \lambda)$ is a function of x and λ and $C = C(\lambda)$ is a function of λ. This gives a solution of (2.2) if and only if

$$-B''B + \frac{B'^2}{2} + 2B^2W = C, \tag{2.4}$$

as can be checked by direct substitution. In particular, it will be interesting for us to choose $B(x, \lambda)$ in the form of a polynomial of λ and $C(\lambda)$ to be a polynomial or rational function with constant coefficients. As we will see, special choices of $W(x, \lambda, E)$, $B(x, \lambda)$ and $C(\lambda)$ related to each other through equation (2.4) will enable us to establish a link with special integrable systems on Riemann surfaces.

A first step in establishing this link is the observation that equation (2.4), remarkably enough, coincides with a generating equation for integrable systems. It is called a generating equation because, for particular choices of W, it generates systems of Euler-Lagrange equations for some well-known mechanical problems and establishes a link between these systems and stationary Hamiltonian flows for classes of solutions of integrable nonlinear pde's. See Alber, Camassa, Holm and Marsden [1994a,b] for an exposition of how this link works.

Following the method of generating equations, we obtain Euler-Lagrange equations by first choosing the functions $B(x, \lambda)$ and $C(\lambda)$ to have the form

$$B(x, \lambda) = \sum_{j=1}^{n+1} \prod_{r=1, r\neq j}^{n+1} (\lambda - l_r) q_j^2(x), \quad C(\lambda) = \frac{\prod_{r=1}^{N}(\lambda - l_r)}{\prod_{j=1}^{M}(\lambda - a_j)} \quad . \tag{2.5}$$

where the integers N and M are related to n, in a way described below. Substituting this into (2.4) and setting $\lambda = l_r$ one by one, we obtain a system of ode's for the functions $q_j(x)$. Notice that for different choices of the function $W(x, \lambda, E)$ one obtains different integrable systems.

In what follows, we are going to use the Lagrange multiplier approach to problems with constraints (see for example, Marsden and Ratiu [1994] for an account). However, for the following integrable systems, something special happens, namely, the Lagrange multiplier turns out to be the negative of the Lagrangian itself. We demonstrate this in the case of some well known systems, and then introduce a new class of potentials in which it is also satisfied.

First we recall (from Moser [1981]) the Lagrangians for the classical C. Neumann and Jacobi problems. We start with the C. Neumann problem for the motion of a particle on the n-sphere in the field of a quadratic potential. Here, $W = u(x) - \lambda$ and the associated Euler-Lagrange equations are

$$q_j'' - q_j(u - l_j) = 0.$$

The Lagrangian and the function u are as follows

$$L = \sum_{j=1}^{n+1} {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) - \sum_{j=1}^{n+1} l_j {q_j}^2, \quad u(x) = \sum_{j=1}^{n+1} l_j {q_j}^2 - \sum_{j=1}^{n+1} {q_j'}^2.$$

For the Jacobi problem of geodesics on n-dimensional quadrics (free motion on quadrics), we take

$$W = \frac{u(x)}{\lambda}$$

and the Euler-Lagrange equations are

$$q_j'' - q_j\frac{u}{l_j} = 0.$$

Here the Lagrangian is as follows

$$L = \sum_{j=1}^{n+1} l_j {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right), \quad u(x) = \frac{\sum_{j=1}^{n+1} {q_j'}^2}{\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}}.$$

Notice that in terms of the q-variables, this problem may be thought of as a problem of geodesics on the n-dimensional pseudosphere in the presence of the potential $u(x)$. All of the systems considered in the next section are of this same type. Notice that they can be linked to harmonic maps in a way similar to an approach used in Baird [1992].

3 New Potentials and WKB Theory for the Associated Hamiltonian Systems

In what follows we will generalize the method described above to a class of "inverse" potentials W, which have poles, of the following form:

$$W(x, \lambda, E) = \sum_{j=-l}^{m} w_j(x, E)\lambda^j. \tag{3.1}$$

As a specific example, we will consider a combination of the potentials for the C. Neumann and Jacobi problems. (Mechanical systems associated with this potential were discussed in Braden [1982]). Here we investigate corresponding spectral problem and establish a connection between this problem and the spatial flow for a particular system of coupled pde's. We will show that this system has umbilic soliton solutions as well as weak billiard solutions. At the same time, an associated spectral problem is a natural development of the approach taken for a family of two-potential systems of coupled KdV equations discussed in Alber [1987] in connection with discrete systems such as the Toda and relativistic Toda lattices. (Discrete systems which correspond to the class of potentials discussed in this paper will be described in forthcoming paper Alber *et al.* [1995]). Two other known examples of two potential systems are the nonlinear Schrödinger equation and the sine-Gordon equation; see, for example, Ablowitz and Segur [1981] and Alber and Alber [1985].

1. (The two-potential Dym system). This example uses a combination of the potentials for the C. Neumann and Jacobi problems. This may also be regarded as a combination of KdV and Dym potentials in view of the association of the C. Neumann and Jacobi problems with the KdV and Dym equations. (For details see Moser [1981] and Alber *et al.* [1994a,b].) We call such systems two-potential systems. In the next section we will use such systems to demonstrate the phenomena of monodromy. Here $W = u(x) + \lambda + \dfrac{v(x)}{\lambda}$ and the Euler-Lagrange equations are

$$q_j'' - q_j\left(u + l_j + \frac{v}{l_j}\right) = 0$$

and the Lagrangian and potential are as follows

$$L = \sum_{j=1}^{n+1} {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) + \sum_{j=1}^{n+1} l_j {q_j}^2 - \frac{1}{\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}},$$

$$u(x) = -\sum_{j=1}^{n+1} {q_j'}^2 - \sum_{j=1}^{n+1} l_j {q_j}^2 + \frac{1}{\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}}.$$

Here we are using an expression for the function $v(x)$, namely

$$v(x) = \frac{\gamma}{\left(\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}\right)^2}$$

which is obtained from a recurrence chain generated by (2.4). Using the dynamical recurrence chain for coefficients b_j obtained from the generating equation

$$-\frac{B'''}{2} + 2B'W + BW' = \frac{\partial W}{\partial t} \tag{3.2}$$

by setting $B(x,\lambda) = b_0(x)\lambda + b_1(x)$ and equating coefficients of the same power of λ, one obtains the system of coupled pde's given in the introduction. Notice that the choice of a polynomial $B(x,\lambda) = b_o(x)\lambda^n + ... + b_n$ of n-th order yields an integrable system of evolution equations from the same hierarchy.

2. (The spatial flow for the two-potential KdV system.) The discrete version of this system corresponds to the Toda and relativistic Toda lattices. (For details see Alber [1987].) Here

$$W = \lambda^2 + v(x)\lambda + u(x)$$

and the Euler-Lagrange equations are

$$q_j'' - q_j\left(u + vl_j + l_j^2\right) = 0.$$

The Lagrangian and potential are as follows

$$L = \sum_{j=1}^{n+1} {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) - \left(\sum_{j=1}^{n+1} l_j {q_j}^2\right)^2 - \sum_{j=1}^{n+1} l_j^2 {q_j}^2,$$

$$u(x) = -\sum_{j=1}^{n+1} {q_j'}^2 + \sum_{j=1}^{n+1} l_j {q_j}^2 + \left(\sum_{j=1}^{n+1} l_j {q_j}^2\right)^2 + \sum_{j=1}^{n+1} l_j^2 {q_j}^2.$$

Here we are using the expression

$$v(x) = 2\gamma \sum_{j=1}^{n+1} l_j q_j^2,$$

which is obtained from the recurrence chain generated by (2.4). Using the recurrence chain generated by the dynamical generating equation (3.2) and and setting $B(x,\lambda) = b_o(x)\lambda + b_1(x)$ one obtains the following integrable system of coupled pde's of KdV type:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{4}v''' - v'u - \frac{1}{2}vu' \\ \frac{\partial v}{\partial t} &= u' - \frac{3}{2}vv'. \end{aligned}\right\} \tag{3.3}$$

3. The spatial flow for the shallow water equation (see Alber *et al.* [1994b]). Here

$$W = 1 + \frac{u(x)}{\lambda},$$

$$q_j'' - q_j\left(\frac{u}{l_j} + 1\right) = 0.$$

and the Lagrangian and potential are as follows

$$L = \sum_{j=1}^{n+1} {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) - \frac{1}{\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}}, \quad u(x) = -\sum_{j=1}^{n+1} {q_j'}^2 + \frac{1}{\sum_{j=1}^{n+1} \frac{q_j^2}{l_j}}.$$

4. (The spatial flow for a generalization of the shallow water equation). Calogero and Degasperis [1982] studied an interesting new class of integrable mechanical systems. We generalize this class by considering a two potential case whose associated system of coupled pde's has the form of a coupled system of equations of shallow water type. Here,

$$W = u(x) + \frac{v(x)}{\lambda}$$

and

$$q_j'' - q_j(u + \frac{v}{l_j}) = 0.$$

The Lagrangian and potential are as follows

$$L = \sum_{j=1}^{n+1} l_j {q_j'}^2 + v\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) - \frac{u}{\sum_{j=1}^{n+1}\frac{q_j^2}{l_j}}, \quad u(x) = -\sum_{j=1}^{n+1} {q_j'}^2 + \frac{1}{\sum_{j=1}^{n+1}\frac{q_j^2}{l_j}}.$$

5. A generalization of the spherical pendulum is described by the following system of Euler-Lagrange equations

$$q_j'' - q_j u - l_j = 0.$$

with Lagrangian and potential of the form

$$L = \sum_{j=1}^{n+1} {q_j'}^2 + u\left(\sum_{j=1}^{n+1} {q_j}^2 - 1\right) - \sum_{j=1}^{n+1} l_j q_j, \quad u(x) = -\sum_{j=1}^{n+1} {q_j'}^2 + \sum_{j=1}^{n+1} l_j q_j.$$

The usual spherical pendulum corresponds to the special case when $l_1, ..., l_n = 0$ and $l_{n+1} = 1$.

To obtain an example of a WKB solution, one starts by considering particular forms of the functions $C(\lambda)$ and $B(x,\lambda)$

$$C(\lambda) = \frac{1}{\lambda}\sum_{k=0}^{2n} c_k \lambda^{2n-k}, \qquad B(x,\lambda) = \sum_{l=0}^{n} b_l(x)\lambda^{n-l} = \prod_{j=1}^{n}\left(\lambda - \mu_j(x)\right). \tag{3.4}$$

The equation (2.4) with potential (3.4) coincides with the generating equation for the geodesic flow on an n-dimensional quadric. It was shown (see Cewen [1990] and Alber *et al.* [1994a,b]) to provide an x-flow for a partial differential equation of Dym type. The equation (2.4) yields a chain of recurrence relations between the coefficients b_l and c_k. (for details see Alber *et al.* [1994a,b]). In particular we have

$$W = \frac{c_{2n}}{b_n^2}$$

and therefore

$$b_n = \sqrt{\frac{c_{2n}}{2m(V-E)}}.$$

As $\lambda \to 0$, the functions $\dfrac{C(\lambda)}{\lambda}$ and $B(\lambda)$ are asymptotic to $\frac{c_{2n}}{\lambda}$ and b_n. This yields the following asymptotics for the Bloch function

$$\psi = \frac{\psi_0}{(V-E)^{\frac{1}{4}}} \exp\left(\pm \frac{i}{\sqrt{\lambda}} \int_{x_0}^{x} \sqrt{2m(V-E)}dx\right) \tag{3.5}$$

which is precisely a well-known WKB solution of the the one-dimensional Schrödinger equation (2.1).

Asymptotics for the Bloch functions which correspond to the "inverse" potentials (3.1) can be found using recurrence relations and geodesic flows on the associated Riemann surfaces. For example, a WKB solution in the case of the 2-potential Dym system described above has the form

$$\psi = \frac{\psi_0}{v^{\frac{1}{4}}} \exp\left(\pm \frac{i}{\sqrt{\lambda}} \int_{x_0}^{x} \sqrt{2mv}\, dx\right). \tag{3.6}$$

A different approach to the semiclassical solutions using a Riemannian metric associated with the complex Hamiltonian system will be described in Section 6.

4 Homoclinic Hamiltonian Flows on Riemann Surfaces

Here we first recall from Alber and Marsden [1994a] a representation of homoclinic flows as flows on noncompact invariant varieties. This representation enables one to treat the homoclinic case in the same manner as the soliton case (see Alber and Marsden [1992,1994b] for the soliton case) and and leads to new exponential Hamiltonians and complex angle representations. Our approach is first elucidated for the C. Neumann problem. Then we describe homoclinic flows for new Hamiltonian systems, which correspond to the class of potentials described above.

Devaney [1978] investigated homoclinic orbits of the C. Neumann problem using a system of first integrals. Moser [1981] studied equilibrium solutions possessing stable and unstable manifolds for this mechanical problem in connection with the spectral theory of Bargmann potentials and soliton x-flows for the KdV equation. In particular, it was shown that equilibrium solutions are associated with the reflectionless potentials which correspond to a set of negative double points of discrete spectrum.

In Alber and Marsden [1994a] we apply a special limiting process to the spectrum associated with the class of quasiperiodic solutions. This yields a change of the Hamiltonian system of equations which describe the dynamics of an auxiliary spectrum μ_j for the finite-zone potentials. A new Hamiltonian on $\mathbf{C}^{2n}$ of exponential type is found for this system, namely

$$H(\mu_j, P_j) = \frac{\sum_{j=1}^{n}(e^{2\sqrt{-\mu_j}P_j} - \bar{C}(\mu_j))}{\prod_{r\neq j}(\mu_j - \mu_r)}, \tag{4.1}$$

where $\bar{C}(\mu) = \prod_{k=1}^{n}(\mu - a_k)$. This system has the following logarithmic first integrals

$$P_j = \frac{\sum_{k=1}^{n} \log(\mu_j - a_k)}{2\sqrt{-\mu_j}} \quad j = 1, ..., n \tag{4.2}$$

and homoclinic angle representations of the form

$$\theta_r = \sum_{j=1}^{n} \frac{1}{2} \int_{\mu_j^0}^{\mu_j} \frac{d\mu_j}{\sqrt{-\mu_j(\mu_j - a_r))}} = x + \theta_r^0, \quad r = 1, ..., n \tag{4.3}$$

defined on a certain noncompact Jacobi variety $\mathcal{J}$. The variety $\mathcal{J}$ is generated by (4.3), which is a generalized Abel Jacobi map, as in Ercolani [1989]. This map is associated with the symmetric product $\Re^n$ of n copies of the Riemann surface

$$\Re: \qquad P = \frac{1}{2\sqrt{-\mu}\prod_{r=1}^{n}(\mu - a_r)}. \tag{4.4}$$

Using the representations (4.3), one can show that as $x \to \infty$ (or $x \to -\infty$), the spectrum which is associated with the homoclinic orbit splits into complex pairs

$$a_j \to (i\alpha_j, -i\alpha_j), \quad a_j = -\alpha_j^2, \quad j = 1, ..., n.$$

Let the double covering of $\Re^n$ be denoted $\widetilde{\Re^n}$. This covering is defined by the following change of variables

$$\xi_j^2 = -\mu_j, \quad j = 1, ..., n. \tag{4.5}$$

The Hamiltonian H defines a dynamical system on $\Re^n$, which lifts to a dynamical system on $\widetilde{\Re^n}$. An analysis of the angle representation shows that the system has a homoclinic point a on $\Re^n$ and, correspondingly, two heteroclinic points α^+ and α^- on $\widetilde{\Re^n}$ associated with the following values of μ_j:

$$\mu_j = a_j, \quad j = 1, ..., n. \tag{4.6}$$

The stable W^s (and unstable W^u) manifolds of the point a are coincident and consist of the orbits in $\Re^n$ that are forward (and backward) asymptotic to the homoclinic point a. On the other hand, the unstable manifold of the point α^+ connects it to the point α^- in $\widetilde{\Re^n}$ and similarly for the stable manifold; these heteroclinic manifolds cover the homoclinic manifold in $\Re^n$.

5 Homoclinic Orbits and Solutions with Monodromy for the Two-potential System

In what follows we will introduce a Hamiltonian system for the set of quasiperiodic solutions of the so-called 2-potential Dym system. Different limiting procedures when applied to this Hamiltonian system yield new umbilic (and in particular, homoclinic) orbits and billiard solutions as well as special solutions with monodromy.

Using the method of Alber, Camassa, Holm and Marsden [1994a,b] one obtains μ-representations for this integrable problem after substituting $E = \mu_j$ into the generating equation (2.4) with the Dym 2-potential:

$$\mu_j' = \frac{1}{\prod_{r\neq j}^{n}(\mu_j - \mu_r)}\sqrt{\frac{C(\mu_j)}{\mu_j}}, \quad j = 1, ..., n. \tag{5.1}$$

Here each of the μ-variables is defined on a copy of the Riemann surface

$$\Re : P^2 = \frac{C(\mu)}{\mu} = -\frac{L_0^2}{\mu} \prod_{k=1}^{2n+2} (\mu - m_k). \tag{5.2}$$

Recall that the μ variables move along cycles on the corresponding Riemann surface (5.2) over the prohibited zones (that is, over the basic cuts between m_{2j} and m_{2j-1} on the Riemann surface).

The system (5.1) is a Hamiltonian system with the Hamiltonian

$$H = \sum_{j=1}^{n} \frac{(P_j^2 - \frac{C(\mu_j)}{\mu_j})}{\prod_{r \neq j}^{n} (\mu_j - \mu_r)}, \quad j = 1, ..., n. \tag{5.3}$$

and the following set of first integrals

$$P_j^2 = \frac{C(\mu_j)}{\mu_j}, \quad j = 1, ..., n. \tag{5.4}$$

Here we are dealing with a degenerate system because the genus of the Riemann surface (5.2) is $(n+1)$ and yet we have only n μ-variables. This produces a degeneracy in the problem of inversion to be described below—one has a similar situation in the case of the focusing nonlinear Schrödinger equation.

This degeneracy can be resolved by introducing an additional μ_{n+1} variable, solving the problem of inversion in terms of Θ functions on the $(n+1)$-dimensional Jacobian and then setting μ_{n+1} equal to a constant m_{2n+2} at the end. This is equivalent to projecting a class of solutions of the enlarged Hamiltonian system onto a subclass of solutions defined on an n-dimensional subspace in phase space.

Therefore, we can describe the quasiperiodic Hamiltonian flow in terms of an angle representation as follows. After adding an equation for the μ_{n+1} variable to (5.1), rearranging the system of equations, summing and using Lagrange-type interpolation formulas, one obtains the following expressions

$$\left. \begin{aligned} \sum_{j=1}^{n+1} \frac{\mu_j^{k+1} \mu_j'}{\sqrt{C(\mu_j)\mu_j}} &= \sum_{j=1}^{n+1} \frac{\mu_j^k}{\prod_{r \neq j}^{n+1} (\mu_j - \mu_r)} = \delta_k^{n-1} \\ \sum_{j=1}^{n+1} \frac{\mu_j^{k+1} \dot{\mu}_j}{\sqrt{C(\mu_j)\mu_j}} &= \sum_{j=1}^{n+1} \frac{\mu_j^k B_2(\mu_j)}{\prod_{r \neq j}^{n+1} (\mu_j - \mu_r)} = \delta_k^{n-2}, \end{aligned} \right\} \tag{5.5}$$

where δ is the Kronecker delta. After integrating (5.5), one obtains an angle representation

$$\alpha_k = \sum_{j=1}^{n+1} \int_{\mu_j^0}^{\mu_j} \frac{\mu_j^{k+1} \, d\mu_j}{\sqrt{C(\mu_j)\mu_j}} = \delta_k^{n-1} x + \delta_k^{n-2} t + \alpha_k^0, \quad k = 0, ..., n, \tag{5.6}$$

where α_k^0 are constants and each μ_j is defined on a copy of the Riemann surface

$$\Re : W^2 = C(\mu)\mu, \tag{5.7}$$

which is a torus of genus $g = n + 1$. The above integrals are taken along cycles a_j over basic cuts on the Riemann surface. This is equivalent to a problem of inversion associated with the Jacobi problem of geodesics on $(n+1)$-dimensional quadrics that can be solved for $\mu_j, j = 1, \ldots, n+1$ as functions of x in terms of Riemann Θ-functions. Finally, we obtain the solution of our Hamiltonian system by setting $\mu_{n+1} = m_{2n+1}$.

After applying a limiting procedure similar to that described in Alber, Camassa, Holm and Marsden [1994b] for the Dym equation and fixing one of the μ-variables, the system (5.6) leads to the so-called umbilic angle representation. Alber, Camassa, Holm and Marsden [1995] shows that these representations generate a class of umbilic solitons and billiard solutions of the coupled system of pde's that were described in the introduction.

On the other hand, setting $m_{2n+2} = m_{2n+1} = b$ in the initial Hamiltonian system (without the additional μ variable) yields a well-defined system of inversion with monodromy. Namely, moving b along a certain closed loop in the space of parameters can lead to a nontrivial shift in action angle variables. This phenomenon is caused by a singularity on the associated Riemann surface and can be demonstrated as follows. In the 1-dimensional case, the limiting process $m_3, m_4 \to b$ yields the angle representation

$$\alpha_1 = \int_{\mu_1^0}^{\mu_1} \frac{\mu_1 d\mu_1}{(\mu_j - b)\sqrt{-\mu_1(\mu_1 - m_1)(\mu_1 - m_2)}} = L_0 x + \alpha_1^0. \tag{5.8}$$

In the case of a genus 3 initial Riemann surface, the limiting angle representation is as follows

$$\left.\begin{aligned} \alpha_1 = -\frac{\partial S}{\partial \beta_1} = \int_{\mu_1^0}^{\mu_1} \frac{\mu_1 d\mu_1}{(\mu_1 - b)\sqrt{C_5(\mu_1)}} + \int_{\mu_2^0}^{\mu_2} \frac{\mu_2 d\mu_2}{(\mu_2 - b)\sqrt{C_5(\mu_2)}} = \alpha_1^0. \\ \alpha_2 = -\frac{\partial S}{\partial \beta_2} = \int_{\mu_1^0}^{\mu_1} \frac{\mu_1 d\mu_1}{\sqrt{C_5(\mu_1)}} + \int_{\mu_2^0}^{\mu_2} \frac{\mu_2 d\mu_2}{\sqrt{C_5(\mu_2)}} = L_0 x + \alpha_2^0. \end{aligned}\right\} \tag{5.9}$$

where

$$S = \int_{\mu_1^0}^{\mu_1} \frac{\sqrt{C_5(\mu_1)}d\mu_1}{(\mu_1 - b)} + \int_{\mu_2^0}^{\mu_2} \frac{\sqrt{C_5(\mu_2)}d\mu_2}{(\mu_2 - b)}$$

is an action function (the generating function of a canonical transformation) and

$$C_5(\mu) = -\mu(\beta_2(\mu - b) + \beta_1 + R_4(\mu)) = -\mu(\mu - m_1)(\mu - m_2)(\mu - m_3)(\mu - m_4).$$

The variables μ_1 and μ_2 move along cycles a_1 and a_2 over the cuts $[m_1, m_2]$ and $[m_3, m_4]$ on the Riemann surface $W^2 = C_5(\mu)$. There is also a singularity at $\mu = b$. Transport of a system of canonical action-angle variables, which linearize the Hamiltonian flow, along a certain loop in the space of parameters (b, m_j) in a way similar to the case of the spherical pendulum and some other integrable systems with monodromy (see, for example, Duistermaat [1980] and Bates and Zou [1993]) will result in a nontrivial shift, which is a manifestation of the monodromy phenomenon. This can be demonstrated as follows. Canonical actions are calculated in the terms of periods of the holomorphic differential

$$I_j = \oint_{a_j} dS = \oint_{a_j} \frac{\sqrt{C_5(\mu)}d\mu}{(\mu - b)}$$

along cycles a_j on the Riemann surface (a torus); for details see Arnold [1978]. Now suppose initially that b does not belong to any of the cycles a_1 and a_2. Then moving b along a closed loop on the Riemann surface around one of the branch points m_j, one continuously transforms one of the a-cycles. At some moment b becomes a branch point itself, which results in the shift of the action variable that is given by the residue of the integrand at $\mu = b$.

Lastly, the method of Alber *et al.* [1994a,b] of associating solutions of nonlinear pde's with finite dimensional Hamiltonian systems on Riemann surfaces leads to the construction of a class of solutions of nonlinear pde's with monodromy.

6 Complex Semiclassical Solutions

In what follows, we recall from Alber [1989, 1991] and Alber and Marsden [1992], a method of complex geometric asymptotics for integrable Hamiltonian flows on Riemann surfaces. We will use geometric asymptotics to describe the quantization conditions of Bohr-Sommerfeld-Keller (BSK) type. Then we will investigate the dependence of these conditions on the parameters (*i.e.,* the first integrals) of the system. This dependence near singularities produces effects caused by the classical and semiclassical monodromy.

Let us consider quadratic complex Hamiltonians of the following form:

$$H = \frac{1}{2}\sum_{j=1}^{n} g^{jj} P_{\mu_j}^2 + V(\mu_1, ..., \mu_n) \tag{6.1}$$

defined on $\mathbf{C}^{2n}$. We think of $\mathbf{C}^{2n}$ as being the cotangent bundle of $\mathbf{C}^n$, with configuration variables $\mu_1, \ldots, \mu_n$ and with canonically conjugate momenta $P_1, \ldots, P_n$.

Notice that the Hamiltonians for quasiperiodic solutions of the systems considered in the previous sections are indeed of this form. We consider the functions g^{jj} as components of a (diagonal) Riemannian metric, construct the associated Laplace-Beltrami operator, and then the stationary Schrödinger equation

$$\nabla^j \nabla_j U + w^2(E - V)U = 0, \tag{6.2}$$

defined on the n-dimensional complex Riemannian manifold $\mathbf{C}^n$. Here ∇^j and ∇_j are covariant and contravariant derivatives defined by the tensor g^{jj} and w (which is the inverse of Planck's constant $\hbar$) and E (the energy eigenvalue) are parameters. Note also that in general, the metric tensor is not constant, and even may have singularities, so that the kinetic term in the expression for H is *not* purely quadratic.

Now we establish a link between equation (6.2) and the Hamiltonian system (6.1) by means of geometric asymptotics; namely, we consider the following function that is similar to the well known Ansatz from WKB theory:

$$\begin{aligned} U(z_1, \ldots, z_n) &= \sum_k A_k(\mu_1, ..., \mu_n) \exp[iwS_k(\mu_1, ..., \mu_n)] \\ &= \sum_k \prod_{j=1}^{n} U_{kj}(\mu_j) = \sum_k \prod_{j=1}^{n} (A_{kj}(\mu_j) \exp[iwS_{kj}(\mu_j)]), \end{aligned} \tag{6.3}$$

which is a multivalued function of several complex variables defined on $\mathbf{C}^n$. If, instead, one considers U to be defined on the covering space of the Jacobi variety of the problem, then U becomes single valued. The functions present in this expression together with r, which denotes a vector of Maslov indices, will be determined below. Note that k also labels the classical paths between initial and current points in the configuration space.

Substituting (6.3) in (6.2), equating coefficients for w and w^2 and integrating, we obtain the amplitude function A, which is a solution of the transport equation, in the form

$$A = \frac{A_0}{\sqrt{(D \det J)}}, \tag{6.4}$$

where $D = \sqrt{\prod_{l=1}^{n} g_{ll}}$ is the volume element of the metric and J is the Jacobian of the change of coordinates from the μ-representation to the angle α-representation. We also find that the phase function S is a solution of the Hamilton-Jacobi equation

$$\Delta^j S \Delta_j S - V = E, \tag{6.5}$$

so that it coincides with the action function.

Now we can apply the above construction to the case of the special class of solutions of the 2-potential Dym system. The Hamiltonian and complex geometric asymptotic solution in this case have the form:

$$H = \frac{1}{2}\frac{(P_1^2 - \frac{C_5(\mu_1)}{\mu_1 - b})}{(\mu_1 - \mu_2)} + \frac{1}{2}\frac{(P_2^2 - \frac{C_5(\mu_2)}{\mu_2 - b})}{(\mu_2 - \mu_1)}. \tag{6.6}$$

and

$$U = \sum_{k=(k_1,k_2)} A_0 \frac{\sqrt{\mu_1 \mu_2}}{((\mu_1 - b)(\mu_2 - b))^{\frac{1}{2}} (C_5(\mu_1)(C_5(\mu_1))^{\frac{1}{4}}} \exp\left[iwS_{k1}(\mu_1) + S_{k2}(\mu_2)\right], \tag{6.7}$$

where

$$S_{k1}(z_1) = \int_{\mu_1^0}^{\mu_1} \frac{\sqrt{C_5(\mu_1)} d\mu_1}{(\mu_1 - b)} + k_1 T_1 + \frac{r_1 \pi}{2}, \tag{6.8}$$

$$S_{k2}(z_2) = \int_{\mu_2^0}^{\mu_2} \frac{\sqrt{C_5(\mu_2)} d\mu_2}{(\mu_2 - b)} + k_2 T_2 + \frac{r_2 \pi}{2}, \tag{6.9}$$

and where $r = (r_1, r_2)$ is a vector of Maslov indices and

$$\left.\begin{aligned} T_1 &= \oint_{a_1} \frac{\sqrt{C_5(\mu_2)} d\mu_2}{(\mu_2 - b)} \\ T_2 &= \oint_{a_2} \frac{\sqrt{C_5(\mu_2)} d\mu_2}{(\mu_2 - b)}. \end{aligned}\right\} \tag{6.10}$$

The amplitude A has singularities at the branch points $\mu_1 = m_1, m_2$, $\mu_2 = m_3, m_4$ and at an additional singular point b on the associated Riemann surface. Each time a trajectory approaches one of these singularities, we continue it in complex x and go around a small circle in the complex plane enclosing the singularity. This results in a phase shift ($\pm i\frac{\pi}{2}$) of the phase function S, which is common in geometric asymptotics. The indices k_1 and k_2 keep track of the number of oriented circuits for μ_1 and μ_2 around a_1 and a_2. The complex mode (6.7) is defined on the covering space of the complex Jacobi variety. Note that in the real case, this complex mode is defined on the covering space of a real subtorus. Keeping this in mind, quantum conditions of BSK type can be imposed as conditions on the number of sheets of the covering space of the corresponding Riemann surface for each coordinate μ_1 and μ_2:

$$\left.\begin{aligned} \frac{\pi}{2}r_1 + wk_1T_1 &= 2\pi N_1 \\ \frac{\pi}{2}r_2 + wk_2T_2 &= 2\pi N_2. \end{aligned}\right\} \tag{6.11}$$

Here N_1, N_2 are integer quantum numbers related to each other and to integer indices k_1, k_2 and r_1, r_2 as follows

$$w = \frac{2\pi N_1}{k_1T_1} = \frac{1}{k_2T_2}\left(2\pi N_2 - \frac{\pi r_2}{2}\right). \tag{6.12}$$

which is an asymptotic formulae for the eigenvalues of the stationary Schrödinger equation. Notice that the quantum conditions (6.11) include a monodromy part after transport along a closed loop in the space of parameters.

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