A General Formulation of Rigid Body Assemblies for Computer Graphics Modeling

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Abstract: In this paper, we present a general approach for the physically-based modeling of rigid-body assemblies. An assembly is defined as a collection of rigid bodies connected to each other through contacts or joints. We present a characterization of contacts between rigid bodies and a methodology to convert the contact into linear constraints on motion parameters such as accelerations of points and vectors on the bodies. We enforce these constraints using an inverse-dynamics approach by computing constraint forces on the rigid bodies. We present a formulation of linear equations that compute the required constraint forces. The contact characterizations can be used as a general primitive in the design of a rigid-body simulator. We present some examples from such a simulator.

1 Introduction

Physically-based modeling is becoming popular to achieve automatic and natural motion of computer graphics models. Many techniques have appeared that solve particular problems using physically-based methods. However, not many general purpose physically-based modeling systems have been presented. This is in contrast to geometric modeling systems, many of which are now available commercially and are very mature. We believe that a similar popularity in physically-based modeling has not come about because of a lack of general and simple primitives or building blocks from which complex and many different simulations can be built easily. Contrast this to geometric building blocks such as polygons or spline surfaces.

In this paper, we attempt to define a set of physical building blocks for simulation of assemblies of rigid bodies. A rigid-body assembly is composed of a collection of rigid bodies connected to each other. The connections or joints are the primitives in our modeling methodology. By selecting the appropriate forms of contacts between the bodies in an assembly, a rich collection of simulations can be performed. To provide a comprehensive collection of contact primitives, we present a characterization of contact between two rigid bodies. This characterization borrows some ideas from the field of applied mechanics.

The concept of contact is a very general one. A contact can not only represent fixed joints such as ball and socket joints but also contacts that permit relative motion such as a shaft rotating in a cylinder. The concept of contacts provides a high-level abstract manner of thinking about assemblies.

We convert such high-level contacts ultimately into forces and torques that can maintain such contacts. To perform dynamic simulations under the motion constraints imposed by the contacts between bodies, we first map each kind of contact into linear constraints on motion parameters of the bodies involved, like accelerations of points and vectors in bodies and angular accelerations of the bodies. To maintain these linear constraints, we apply constraint forces and torques on the rigid bodies involved in a contact. For example, to maintain a constraint on the acceleration of a point on a body, we apply a constraint force. To maintain an acceleration constraint on a vector fixed in a body, we apply a constraint torque. Using Newton's equations of motion, we derive a set of linear equations whose solution provides appropriate constraint forces and torques.

1.1 Previous Work

Dynamics has been the focus of a number of researchers to produce automatic natural motion. Wilhelms ([Wil87]), Armstrong and Green ([AG85]) present a forward-dynamics based system to simulate articulated bodies. Baraff has presented a number of techniques ([Bar89], [Bar90] [Bar91]) for interacting rigid bodies where forces are computed to maintain no inter-penetration of bodies upon contact. Hahn ([Hah88]) has also presented work to simulate colliding rigid bodies. Some other work based on dynamic simulation is [LWB90], [Mil88], [WB85], [PW89].

A number of researchers have reported physically-based inverse techniques for computer graphics. Isaacs and Cohen ([IC87]) describe a method of simulating open loop linkages with a kinematics and inverse-dynamics approach. Barzel and Barr ([BB88]) describe constraint-based simulator for "self-assembly" of rigid objects. Witkin and Kass ([WK88]) describe a *spacetime constraint* based system in which system forces are solved for, over a time interval optimizing criteria such as energy spent over the interval.

Our work is along the lines of and in fact builds upon the work of Barzel and Barr.

1.2 Overview

We start by presenting a characterization of contacts between rigid bodies. In section 3, we present the equations for some basic rigid body motion parameters. In section 4, we consolidate the equations of section 3 to come up with the governing equations for our rigid-body simulator. In section 5, we present some details of implementation and section 6, we present some simulation examples and timings.

2 Characterization of Rigid Body Contacts

Assemblies of rigid bodies are constructed by connecting the bodies through different types of contacts or joints[1]. The contacts are chosen to constrain the motion of the rigid bodies in an assembly to achieve desired motion or to transform one motion into another, such as the transformation of the linear motion of a piston engine into a continuous circular motion. The designing of assemblies consists of finding the proper combination of rigid bodies and connections.

Although it is possible to have multiple bodies connect at a connection, in a majority of cases, two bodies are involved. In this paper, we will restrict ourselves to assemblies where each connection has at most two bodies.

In mechanics literature, contacts between rigid bodies are usually classified into two categories ([DH55]):

- **Lower-Pair Contacts**: If a contact between two bodies is made by two mating surfaces, each sliding with respect to each other, the contact is called a lower-pair contact. An example is a piston sliding in a cylinder.
- **Higher-Pair Contacts**: If a contact between two bodies is confined to a line (as in a roller or cam) or to a point (as in a ball bearing), the contact is called a higher-pair contact.

Each of the contacts restricts the relative motion between the two bodies involved. Therefore, it is possible to map each kind of joint into a system of constraints on the motion parameters of the rigid bodies involved, such as motion of points on the bodies, motion of fixed vectors in bodies or directly the angular acceleration of a body.

[1] In this paper, we use the terms joints and contacts interchangeably.

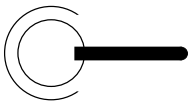
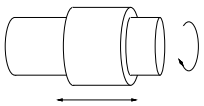
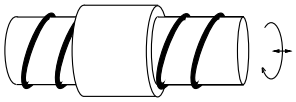
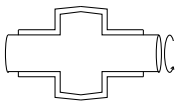
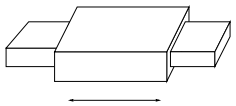
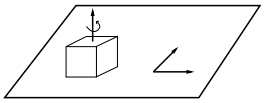
Name (DOFs)	Figure
(a) Spheric Pair Zero Trans Three Rot	
(b) Cylindric Pair One Trans One Rot	
(c) Screw Pair One Trans One(coupled) Rot	
(d) Revolute Pair Zero Trans One Rot	
(e) Prism Pair One Trans Zero Rot	
(f) Plane Pair Two Trans One Rot	

Figure 1: The six Lower-Pair Contacts. The left column shows the degrees of freedoms allowed for each pair. The right column shows an example of each kind of lower-pair.

In the next two subsections, we will discuss lower- and higher-pair contacts and how they can be achieved using contraints on the motion parameters of the bodies. Note that it is possible that more than one mapping from a contact type and motion parameter constraint exists. We only present one of them.

We assume that all bodies start from rest. The connection constraints are met initially in so

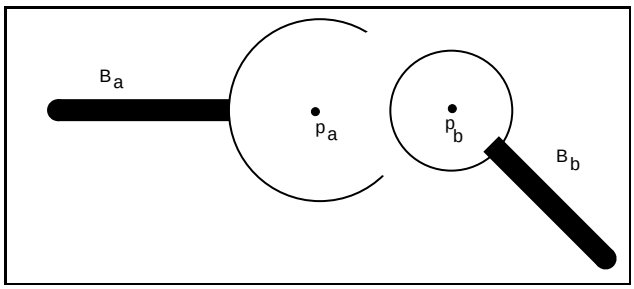


Figure 2: A spheric pair. This pair is constrained by constraining point $\mathbf{p}_a$ and $\mathbf{p}_b$ on bodies $\mathbf{B}_a$ and $\mathbf{B}_b$ respectively to be co-incident.

far as that the positions of the body are consistent with the joint constraint. The subsequent simulation is required to compute the motion of the bodies consistent with these constraints. We make this assumption here because it makes the involved mathematics simpler to explain.

In each of the contacts described, it is assumed that the contact is made between bodies $\mathbf{B}_a$ and $\mathbf{B}_b$. Subscript a will be used for quantities related to body $\mathbf{B}_a$ and subscript b will be used for quantities related to body $\mathbf{B}_b$.

2.1 Lower-Pair Contacts

There are only six possible lower pairs ([DH55]). These six lower-pair contacts are shown in figure 1. To implement each of the lower-pair contacts, we need to constrain the bodies in the contact such that the contact has only the allowed degrees of freedom, as show in the left column of figure 1.

2.1.1 Spheric Pair

The spheric pair, such as a ball-and-socket joint allows a rotation about each of three rectangular axis. The joint does not have any translational degree of freedom (figure 2).

Let us assume that the socket of the spheric joint belongs to $\mathbf{B}_a$, and the ball of the joint belongs to $\mathbf{B}_b$. Let the centers of the socket and the ball be $\mathbf{p}_a$ and $\mathbf{p}_b$ respectively. Since, consistent with our assumptions at the beginning of this section, the bodies start from rest and the bodies are in such a configuration that the ball is in the socket, the constraint to maintain a spheric joint is that points $\mathbf{p}_a$ and $\mathbf{p}_b$ move identically, i.e., the linear accelerations of point $\mathbf{p}_a$ should be equal to the linear acceleration of point $\mathbf{p}_b$.

$$\begin{aligned} \mathbf{a}p_a &= \mathbf{a}p_b \\ \mathbf{a}p_a - \mathbf{a}p_b &= \mathbf{0} \end{aligned} \tag{1}$$

2.1.2 Cylindric Pair

The cylindric pair (figure 3) allows a rotation about the cylinder axis and a translation along the axis. The cylindric pair needs to constrain two degrees of translational freedom and two degrees of rotational degrees.

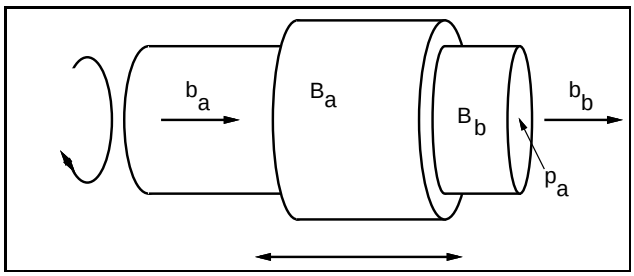


Figure 3: A cylindric pair. This contact is maintained by restricting vector $\mathbf{b}_a$ in body $\mathbf{B}_a$ aligned with the axis vector $\mathbf{a}_b$ in body $\mathbf{B}_b$, and a point $\mathbf{p}_a$ on the axis of cylinder to the axial line.

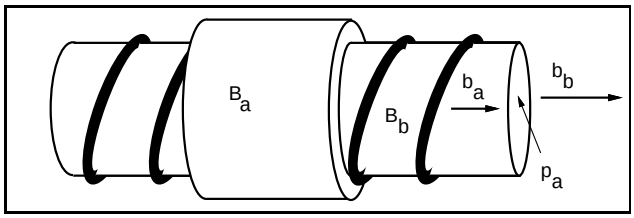


Figure 4: A screw pair. All the constraints of the cylindric pair are needed with the addition of a pitch constraint.

As shown in figure 3, let body $\mathbf{B}_a$ be the sliding cylinder and body $\mathbf{B}_b$ be the body in which the cylinder slides. Let the axis of the cylinder in $\mathbf{B}_b$ be a unit vector $\hat{\mathbf{b}}_b$. Let $\mathbf{p}_a$ be a point on the axis of the cylinder (on body $\mathbf{B}_a$). Let unit vector $\mathbf{b}_a$ be a vector along the axis in body $\mathbf{B}_a$. Then to achieve a cylindric pair, we need to constrain point $\mathbf{p}_a$ to be on axis $\mathbf{b}_b$ and to constraint vector $\mathbf{b}_a$ to always stay aligned with vector $\mathbf{b}_b$. For bodies starting from rest, the equations are:

$$\frac{d^2\hat{\mathbf{b}}_a}{dt^2} - \frac{d^2\hat{\mathbf{b}}_b}{dt^2} = \mathbf{0} \tag{2}$$

$$\mathbf{p}_a \text{ should stay on } \quad \text{axis } \hat{\mathbf{b}}_b \tag{3}$$

The mathematical constraint equivalent to the second textual constraint will be presented in the next subsection.

2.1.3 Screw Pair

The screw pair (figure 4) is similar to the cylindric pair in the degrees of freedom except that the translational and rotational degrees of freedom are coupled by the pitch of the screw[2].

The constraints for a screw pair are the same as the cylinder pair with the additional constraint that the angular acceleration $\underline{\alpha}$ of the screw should be related to the linear acceleration of a point on its axis by the pitch p of the screw. That is:

$$\frac{d^2\hat{\mathbf{b}}_a}{dt^2} - \frac{d^2\hat{\mathbf{b}}_b}{dt^2} = \mathbf{0} \tag{4}$$

$$\mathbf{p}_a \text{ should stay on } \quad \text{axis } \hat{\mathbf{b}}_b \tag{5}$$

$$p\underline{\alpha} - \mathbf{a}p_a = \mathbf{0} \tag{6}$$

[2]The pitch of the screw is the distance traveled by the screw per rotation of the screw

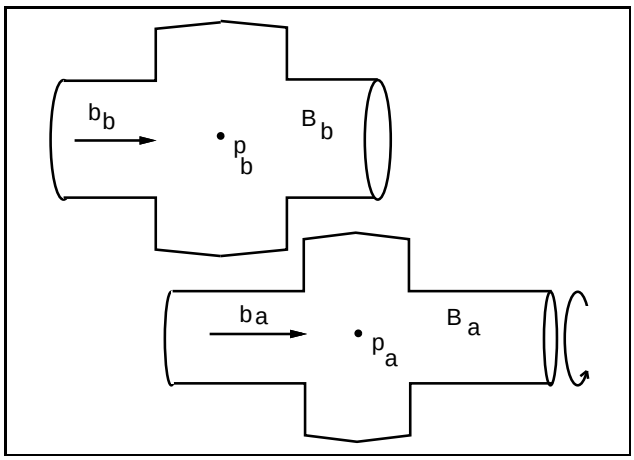


Figure 5: A revolute pair. A point to point constraint and a vector to vector constraint maintains this contact.

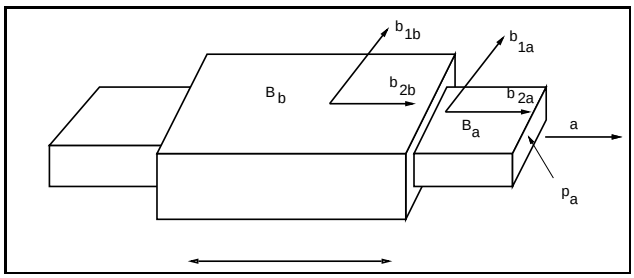


Figure 6: A prism pair. The contact is maintained by keeping two vectors aligned and by keep a point moving on a line parallel to the axisl line.

2.1.4 Revolute Pair

A revolute pair consists of contact surfaces defined as surfaces of revolution. The revolute pair allows only a rotation about its axis and has all of its translational degrees of freedom constrained.

Let points $\mathbf{p}_a$ and $\mathbf{p}_b$ be coincident points on bodies $\mathbf{B}_a$ and $\mathbf{B}_b$ respectively. Further let unitvectors $\mathbf{b}_a$ and $\mathbf{b}_b$ lie along the axes of the two bodies respectively. Then the revolute pair is maintained by constraining the two points to move identically and and the two vectors to stay parallel. That is

$$\mathbf{a}p_a - \mathbf{a}p_b = \mathbf{0} \tag{7}$$

$$\frac{d^2\hat{\mathbf{b}}_a}{dt^2} - \frac{d^2\hat{\mathbf{b}}_b}{dt^2} = \mathbf{0} \tag{8}$$

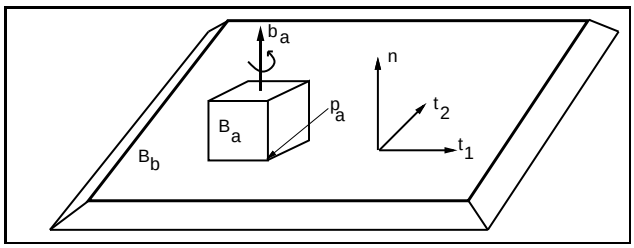


Figure 7: A plane pair. This contact can be maintained by restricting point $\mathbf{p}_a$ on body $\mathbf{B}_a$ on the plane of contact to have no acceleration in the normal direction $\hat{\mathbf{n}}$ and by restricting a fixed vector $\mathbf{b}_a$ in $\mathbf{B}_a$to always be aligned to the normal vector.

2.1.5 Prism Pair

A prism pair has both contact surface cylinders which are other than right cylinders. This type of contact permits only a translation parallel to the generatrix (roughly equivalent to the axis). This contact has no rotational degree of freedom.

Let $\mathbf{p}_a$ be a point on a line parallel to the line of travel of the sliding body $\mathbf{B}_a$. Let unit vector $\hat{\mathbf{b}}_b$ lie on the same line but in body $\mathbf{B}_b$. Let unit vectors $\hat{\mathbf{b}}_{1a}$ and $\hat{\mathbf{b}}_{2a}$ be two vectors in body $\mathbf{B}_a$ and unit vectors $\hat{\mathbf{b}}_{1b}$ and $\hat{\mathbf{b}}_{2b}$ be two vectors in body $\mathbf{B}_b$ parallel to $\hat{\mathbf{b}}_{1a}$ and $\hat{\mathbf{b}}_{2a}$ respectively (see figure 6). Then the prism joint is maintained by keeping point $\mathbf{p}_a$ to be on vector $\hat{\mathbf{b}}_b$, vector $\hat{\mathbf{b}}_{1a}$ parallel to $\hat{\mathbf{b}}_{1b}$ and $\hat{\mathbf{b}}_{2a}$ parallel to $\hat{\mathbf{b}}_{2b}$. That is:

$$\mathbf{p}_a \text{ should stay on } \quad \text{axis } \hat{\mathbf{b}}_b \tag{9}$$

$$\frac{d^2\hat{\mathbf{b}}_{1a}}{dt^2} - \frac{d^2\hat{\mathbf{b}}_{1b}}{dt^2} = \mathbf{0} \tag{10}$$

$$\frac{d^2\hat{\mathbf{b}}_{2a}}{dt^2} - \frac{d^2\hat{\mathbf{b}}_{2b}}{dt^2} = \mathbf{0} \tag{11}$$

2.1.6 Plane Pair

The plane pair consists of one body sliding with respect to the other with a planar contact. The plane pair has one rotational degree of freedom, about the normal to the common plane of contact between the bodies, and two translational degrees of freedom along this plane (figure 7).

Let point $\mathbf{p}_a$ be an arbitrary point on body $\mathbf{B}_a$ on the plane of contact. Let unit vector $\hat{\mathbf{b}}_a$ be vector fixed in body $\mathbf{B}_a$ perpendicular to the plane of contact. Let $\mathbf{n}$ in $\mathbf{B}_b$ be the normal to the plane of contact. Then the plane pair contact can be maintained by requiring that point $\mathbf{p}_a$ have no acceleration normal to the plane of contact and the vector $\mathbf{b}_a$ always stay aligned with the normal. That is:

$$\mathbf{a}p_a \cdot \hat{\mathbf{n}} = \mathbf{0} \tag{12}$$

$$\frac{d^2\hat{\mathbf{b}}_a}{dt^2} - \frac{d^2\hat{\mathbf{n}}}{dt^2} = \mathbf{0} \tag{13}$$

2.2 Higher-Pair Contacts

In this section we discuss examples of some higher-pair contacts between rigid bodies. Some of these constraints were discussed in [BB88] but we present them here as they are implemented in our formulation. Some new constraints are also introduced.

2.2.1 Point on a rigid body connected to a fixed point in space

A fixed point on a body with position vector $\mathbf{p}$ is constrained to be connected to a fixed point $\mathbf{x}$ in space. To maintain this constraint from rest, the point $\mathbf{p}$ should have no acceleration, i.e., point $\mathbf{p}$ should not move. That is:

$$\mathbf{a}_p = 0 \tag{14}$$

2.2.2 Point on a rigid body $\mathbf{B}_a$ fixed to a point on a rigid body $\mathbf{B}_b$

Let the point $\mathbf{p}_a$ on $\mathbf{B}_a$ and the point $\mathbf{p}_b$ on body $\mathbf{B}_b$ be required to stay connected. Then, to maintain the required connectivity, the accelerations of both points, $\mathbf{p}_a$ and $\mathbf{p}_b$ should be same at all times such that the two points move identically. That is

$$\begin{aligned} \mathbf{a}p_a &= \mathbf{a}p_b \\ \mathbf{a}p_a - \mathbf{a}p_b &= 0 \end{aligned} \tag{15}$$

2.2.3 Point on a rigid body $\mathbf{B}_a$ to be on a space curve

Let a space curve parametrized by parameter u be given by

$$\mathbf{r} = \mathbf{r}(u), \tag{16}$$

where $\mathbf{r}(u)$ is a 3-vector valued function of the scalar parameter u.

We wish to keep a point $\mathbf{p}_a$ on body $\mathbf{B}_a$ on the given space curve. Given that the point $\mathbf{p}_a$ is on the space curve to start with and the body is at rest, the point $\mathbf{p}_a$ will continue to stay on the curve provided its acceleration is always tangent to the space curve, which is equivalent to having acceleration of $\mathbf{p}_a$ perpendicular to the curve to be zero.

If unit vector $\hat{\mathbf{t}}$ is the tangent to the curve at the instantaneous point of contact, the equation for this constraint is:

$$\begin{aligned} \mathbf{a}_p - \left(\mathbf{a}_p \cdot \hat{\mathbf{t}}\right) \hat{\mathbf{t}} &= 0 \\ \Leftrightarrow \mathbf{a}_p - \hat{\mathbf{t}}\, \hat{\mathbf{t}}^T\, \mathbf{a}_p &= 0 \\ \Leftrightarrow \left(\mathbf{1} - \hat{\mathbf{t}}\, \hat{\mathbf{t}}^T\right)\, \mathbf{a}_p &= 0 \end{aligned} \tag{17}$$

2.2.4 Point on a rigid body to be on a Surface

Let a space surface parametrized by parameters u and v be given by

$$\mathbf{r} = \mathbf{r}(u, v). \tag{18}$$

where $\mathbf{r}(u)$ is a 3-vector valued function of the scalar parameters u and v.

We wish to keep a point $\mathbf{p}_a$ on body $\mathbf{B}_a$ on the given space surface. Again, given that the point $\mathbf{p}_a$ is on the space surface to start with and the body is at rest, the point $\mathbf{p}_a$ will continue to stay on the surface provided its acceleration perpendicular to the surface is zero, i.e., the point is never permitted to move away from the surface.

If unit vector $\hat{\mathbf{n}}$ is the normal the surface at the point of contact, the equation for this constraint is:

$$\mathbf{a}_p.\hat{\mathbf{n}} = 0 \tag{19}$$

2.2.5 Point on a rigid body to be on a trajectory

We define a trajectory as a parametric curve parametrized by time. That is a curve

$$\mathbf{r} = \mathbf{r}(t), \tag{20}$$

This constraint requires a point $\mathbf{p}$ on a rigid body to be coincident at time t with the point on the trajectory $\mathbf{r}(t)$. The equation of this constraint is:

$$\mathbf{a}_p = \ddot{\mathbf{r}}(t) \tag{21}$$

Note that point on trajectory constraint is different than point on a space curve constraint. If a point moves on a trajectory, it moves monotonically in the positive direction of the curve, i.e., in the direction of the local tangent to the trajectory. For a space curve, no such monotonicity is required. In fact there is no temporalness associated with a (fixed) space curve.

3 Simulating Constrained Rigid Bodies

In the previous section, we formulated the equations for parameters of rigid bodies that maintain high-level contacts or joints between rigid bodies in assemblies. In this section, we relate the body parameters appearing in these equations to forces and torques to perform simulations of rigid-body assemblies.

We use a physically-based approach based on inverse dynamics. For each constraint, we introduce appropriate *constraint* forces and *constraint* torques on the bodies involved in the constraint. During simulation, bodies move in response to the constraint forces and externally applied forces. The constraint forces and torques are computed to be such that the constraints on the rigid bodies are met consistent with the joints and external forces.

We start with a brief review of rigid body dynamics.

3.1 Rigid Body Mechanics

The motion of rigid body under the application of forces and torques is governed by the so called *Newton-Euler equations of rigid body motion*([Gol80]). These equations relate inertial properties of a body to the applied forces and torques.

We will use the following symbols for properties of a rigid body:

$\mathbf{r}_{cm}$	position vector of center of mass
$\mathbf{q}$	quaternion to represent orientation (see note)
$\mathbf{p}$	Linear momentum
$\mathbf{v}$	Linear velocity
$\mathbf{L}$	Angular momentum
$\underline{\omega}$	Angular velocity
m	Mass
$\mathbf{I}$	Moment of Inertia

Note: The orientation of a body is usually represented as a rotation matrix. A quaternion is another representation for the orientation of a rigid body which is numerically superior from the point of view of integrating differential equations of the rigid body. For detailed description of quaternions, please see [Sho85] and [CH53].

The system of differential equations governing the motion of a rigid body is:

$$\begin{aligned} \dot{\mathbf{p}} &= \sum \mathbf{F}_i = \text{Net force on body} \\ \dot{\mathbf{r}}_{CM} &= \mathbf{v} = \frac{1}{m}\mathbf{p} \\ \dot{\mathbf{L}} &= \sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{T}_i = \text{Net Torque} \\ \dot{\mathbf{q}} &= \frac{1}{2}\omega\mathbf{q} = \frac{1}{2}\mathbf{I}^{-1}\mathbf{L}\mathbf{q} \end{aligned} \tag{22}$$

where $\mathbf{r}_i$ is the point of application of the force $\mathbf{F}_i$.

Further the linear and angular momenta are given by:

$$\mathbf{p} = m\mathbf{v} \tag{23}$$

$$\mathbf{L} = \mathbf{I}\omega \tag{24}$$

All moments in the above equations are taken around the center of mass of the body.

The above equations are now used to derive the equations for a acceleration of a point on a rigid body, second order time derivative of a vector fixed in a body and the angular acceleration of a body.

3.2 Equations of Motion of Fixed Point in Body

Let point $\mathbf{p}$ (not necessarily center of mass) be on a body $\mathbf{B}$, with $\mathbf{b}$ being the vector from the center of mass to the point $\mathbf{p}$. Then, as shown in appendix C,

$$\mathbf{a}_p = \frac{1}{m}\sum \mathbf{F}_i$$

$$
\begin{aligned}
&+(\mathbf{b}^{*T}\mathbf{I}^{-1})(\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{T}_i) \\
&+(\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b})
\end{aligned} \tag{25}
$$

Some of the forces on the body will be externally applied forces and some of them will be constraint forces introduced to impose joints. Separating the constraint and external forces and torques, we can rewrite the above equation as:

$$
\begin{aligned}
&\mathbf{a}_p = \\
&\left\{ \frac{1}{m} \sum_i \mathbf{F}_i^c + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{c*}\mathbf{F}_c^i + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{T}_i^c \right\} + \\
&\left\{ \frac{1}{m} \sum_i \mathbf{F}_e^i + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{e*}\mathbf{F}_i^e + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{T}_e^i \right\} + \\
&\left\{ (\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b}) \right\}
\end{aligned} \tag{26}
$$

Note that the all the quantities in the second and third set of braces are known. Also the unknown constraint forces and torques in the first set of braces appear only linearly. Therefore, the above equation can be written as:

$$
\mathbf{a}_p = \sum_i \mathbf{M}_i^{(pF)} \mathbf{F}_i^c + \sum_i \mathbf{M}_i^{(pT)} \mathbf{T}_i^c + \mathbf{v}^{known} \tag{27}
$$

where

$$
\begin{aligned}
\mathbf{M}_i^{(pF)} &= \frac{\mathbf{1}}{m} + \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{c*} \\
\mathbf{M}_i^{(pT)} &= \mathbf{b}^{*T}\mathbf{I}^{-1} \\
\mathbf{v}^{known} &= \text{quantities in second and third braces}
\end{aligned}
$$

Therefore, the grungy equations in expression 26 can be written as the relatively benign looking equation 27.

3.3 Fixed Vector in Body

Let $\mathbf{b}$ be a vector fixed in a body. Then, as derived in appendix D,

$$
\begin{aligned}
\ddot{\mathbf{b}} &= (\mathbf{b}^{*T}\mathbf{I}^{-1})(\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{T}_i) + \\
&\quad (\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b})
\end{aligned} \tag{28}
$$

Again, separating into constraint and external forces,

$$
\begin{aligned}
\ddot{\mathbf{b}} &= \left\{\sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{c*}\mathbf{F}_i^c + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{T}_c^i\right\} + \\
&\quad \left\{\sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{e*}\mathbf{F}_i^e + \sum_i \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{T}_e^i\right\} + \\
&\quad \left\{(\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L}\times\omega)) + \omega\times(\omega\times\mathbf{b})\right\} \qquad (29)\\
\ddot{\mathbf{b}} &= \sum_i \mathbf{M}_i^{bF}F_i^c + \sum_i \mathbf{M}_i^{bT}T_i^c + \mathbf{v}^{known} \qquad (30)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{M}_i^{(bF)} &= \mathbf{b}^{*T}\mathbf{I}^{-1}\mathbf{r}_i^{c*}\\
\mathbf{M}_i^{(bT)} &= \mathbf{b}^{*T}\mathbf{I}^{-1}\\
\mathbf{v}^{known} &= \text{quantities in second and third braces}
\end{aligned}
$$

3.4 Angular Acceleration

As derived in appendix B, the angular acceleration $\underline{\alpha}$ of a rigid body is given as:

$$
\begin{aligned}
\underline{\alpha} &= \sum_i \mathbf{T}_i + \mathbf{I}^{-1}\omega^{*T}\mathbf{L}\\
&= \sum_i \mathbf{T}_i^c + \sum_i \mathbf{T}_i^e + \mathbf{I}^{-1}\omega^{*T}\mathbf{L} \qquad (31)
\end{aligned}
$$

which can be rewritten as:

$$
\begin{aligned}
\underline{\alpha} &= \sum_i \mathbf{T}_i + \mathbf{I}^{-1}\omega^{*T}L \qquad (32)\\
&= \sum_i \mathbf{M}_i^{aF}F_i^c + \sum_i \mathbf{M}_i^{aT}T_i^c + \mathbf{v}^{known} \qquad (33)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{M}_i^{(aF)} &= \mathbf{0}\\
\mathbf{M}_i^{(aT)} &= \mathbf{1}(\text{unit matrix})\\
\mathbf{v}^{known} &= \text{quantities in second and third braces}
\end{aligned}
$$

3.5 Summary of Second Derivative Quantities

As shown in this section, the expressions for acceleration of a point fixed in a body, acceleration of a vector fixed in a body and the angular acceleration of a body all have the same form:

$$\sum_i \mathbf{M}_i^F F_i^c + \sum_i \mathbf{M}_i^T T_i^c + \mathbf{v}^{known} \qquad (34)$$

This is important to note since we use this linearity to simplify the solution for constraint forces.

4 Composing Constraints on Rigid Body Motion Parameters

In section 2, we transformed all the joints into linear combinations of the acceleration of points of bodies, second derivatives of vectors in bodies and angular acceleration of bodies. In section 3, we derived linear expressions for these second derivative quantities. This suggests that we can use more general linear constraints on these quantities.

In this section, let $\mathbf{s}_j$ represent any of acceleration of a point p on a rigid body j, the second derivative of a vector fixed in the rigid body j or the angular acceleration of a body j.

Consider a linear equation of these second derivative quantities as:

$$\sum_j \mathbf{A}_j \mathbf{s}_j = \underline{\beta} \tag{35}$$

where $\mathbf{A}_j$ is an arbitrary matrix not containing terms dependent on the constraint forces, constraint torques and any of $\mathbf{s}_j$ [3].

Since each $\mathbf{s}_j$ can be written as an equation of the form 34, we can rewrite equation 34 as:

$$\begin{aligned} \sum_j \mathbf{A}_j \left\{ \sum_i \left\{ \mathbf{M}_i^{(Fj)} \mathbf{F}_i^c + \mathbf{M}_i^{(Tj)} T_i^c + \mathbf{v}^{(known)j} \right\} \right\} &= \underline{\beta} \\ \sum_i \left\{ \mathbf{A}_j \mathbf{M}_i^{Fj} \right\} \mathbf{F}_i^c + \sum_i \left\{ \mathbf{A}_j \mathbf{M}_i^{Tj} \right\} \mathbf{T}_i^c &= \underline{\gamma} \end{aligned}$$

The above equation is a linear equation in the constraint forces $\mathbf{F}_i^c$ and torques $\mathbf{T}_i^c$ which can be written as:

$$\begin{bmatrix} M_1^{(j)} & M_2^{(j)} & ... & M_{1n}^{(j)} \end{bmatrix} \begin{bmatrix} Fc_1 \\ \vdots \\ Fc_n \\ Tc_1 \\ \vdots \\ Tc_m \end{bmatrix} = \begin{bmatrix} \gamma_1 \end{bmatrix} \tag{36}$$

Each constraint on the second derivative quantities $\mathbf{s}_j$ contributes one such equation. Composing all such equations, we get the matrix equation for constraint forces and torques as

$$\begin{bmatrix} M_{11} & M_{12} & ... & M_{1n} \\ M_{21} & M_{22} & ... & M_{2n} \\ \vdots & \vdots & & \vdots \\ M_{m1} & M_{m2} & ... & M_{mn} \\ \vdots & \vdots & & \vdots \\ M_{n1} & M_{n2} & ... & M_{nn} \end{bmatrix} \begin{bmatrix} Fc_1 \\ \vdots \\ Fc_n \\ Tc_1 \\ \vdots \\ Tc_m \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \\ \vdots \\ \gamma_n \end{bmatrix} \tag{37}$$

[3] Some people would call these constraints strictly pseudo-linear.

Readers who are familiar with the work of Barzel and Barr ([BB88]) will recognize the above equation as the same equation that they derive for their system. Although we use the same form of the equation to compute constraint forces, the terms of the equations are different consistent with the type of constraints (contacts and joints) that we are trying to maintain between bodies.

Summary of Math

Now is a good time to take a breath and to review the mathematical machinery we have developed.

We started with high-level contacts between rigid bodies that allow us to think about assemblies in terms of the relative motion allowed between the bodies involved. An example would be, body **a** can only slide inside a hole in body **b** to from a prismatic contact. We had to eventually relate these contacts to the forces on the bodies to be able to do dynamic simulation. We decided to enforce contacts by introducing constraint forces and torques. We also mapped each high-level contact into a linear equation of accelerations of points or vectors in the rigid bodies.

We then used Newton's equations to come up with a linear set of equations that compute the constraint forces which maintain the desired contacts between rigid bodies in an assembly consistent with external forces.

5 Implementation

The state of a rigid body is uniquely determined by the position of its center of mass and the orientation of the rigid body. The governing equations for these variables are the Newton's equations of motion in equation 22 in section 3.1. We need to integrate these equations to determine the state of each body in an assembly. Since center of mass, linear momentum and angular momentum are all three-vectors and quaternion is a four-component quantity, each rigid body introduces 13 differential equations.

The right hand of these equations depends on known quantities such as external forces and torques, instantaneous velocities and positions etc., and the unknown constraint and torques. These unknown forces and torques are determined by solving the linear system of equations (37).

Therefore to implement the methodology explained in this paper, we need a solver for differential equations and a solver for linear equations. Note that the system of linear equations (37) is very sparse and not necessarily square. Further, it is possible for the system to be singular (or sub-rank) or over-constrained depending on the number and type of constraints applied to the rigid-body assembly. The solver for linear equations need to take care of these situations. Some solvers (such as available in NAG[Lim91]) will solve for a minimal length solution if the system is singular and a least-squares optimal solution if the system is over-constrained.

The simulation of a particular rigid-body assembly required two steps, a set-up step and the solution step.

5.1 Simulation Setup

For each rigid body in the assembly, we add the 13 first order differential equations of the body into the differential equation solving system.

For each constraint in the system, we determine the number of constraint forces or torques required to impose that constraint. To impose a point constraint, i.e., to control the acceleration of a point, we add a constraint force at that point. To impose a vector constraint, we add a constraint torque. To impose an angular acceleration, we also add a constraint torque. Note that when two bodies are involved in a constraint, equal and opposite force or torques are added to each body.

The number of "columns" in linear system of equations for the constraint forces and torques is augmented by the number of constraint force and torque added by a constraint. The number of "rows" in the linear system is equal to the number of constraints. We write rows and columns in quotes, since each element M as shown in equation 37 is actually a matrix.

5.2 Simulation Solution

As mentioned above, the solution of the simulation is just the solution of the differential equations of each of the rigid bodies. The system of equations can be written in a general form as:

$$y_i{}' = y_i(t, y_1, y_2, \ldots, y_n, \mathcal{X}) \tag{38}$$

where the right hand sides, y_i are a function of the constraint forces and torques, external forces and torques and the state of each body.

Every time the differential equation solver needs to compute the right hand side, we set up the linear set of equations 37, solve for the constraint forces and torques, and compute each y_i.

Once the position of center of mass and orientation is solved for at any desired time, the quaternion is converted to a rotation matrix, the rigid body is transformed by the rotation matrix obtained, and translated to its center of mass. The body is then rendered.

Of course, to provide a rich modeling environment, we should also provide interesting rigid body shapes and external forces. For example, in our simulation system, we have friction forces, piecewise linear forces (as a function of time), sinusoid forces, forces as a function of position of bodies etc.

6 Simulation Examples

Using the above methodology, we have implemented a rigid-body simulator. In this section we present some simulation examples. In all of these examples, all the motion results due to the simulation of external forces and constraint forces (and torques) according to Newton's equation. No part of the motion is specified kinematically. The constraints applied to each mechanism cause the appropriate constraint forces to generate the correct motion.

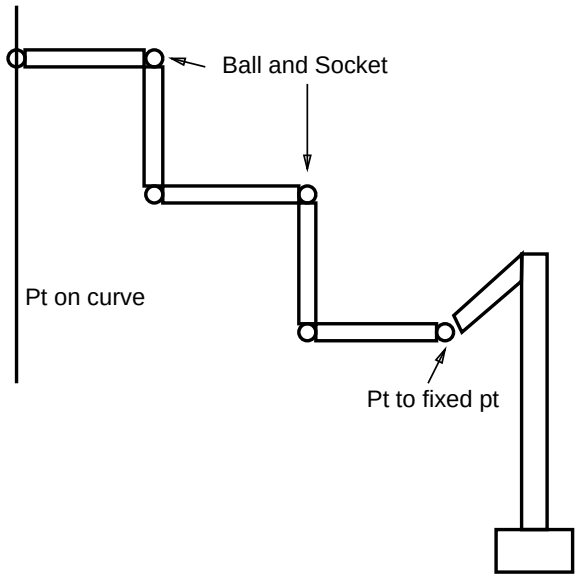


Figure 8: A chain constructed of rigid links. The links are connected by ball and socket joints. One end of the chain is restricted to stay on a space curve. The other end is fixed in space.

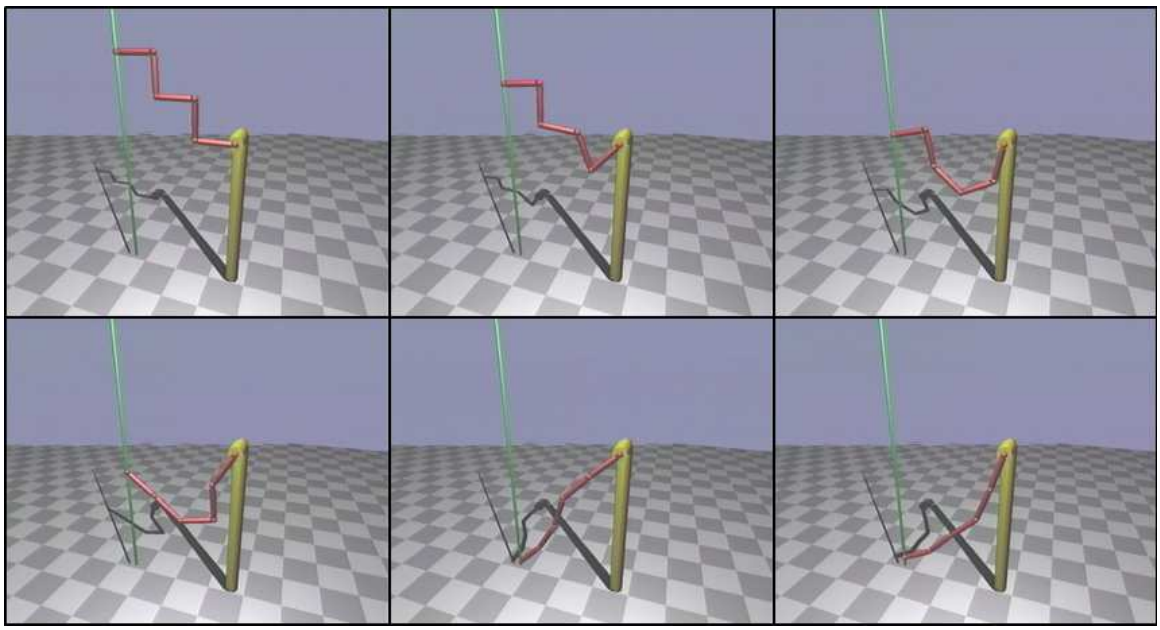


Figure 9: Frames from the animation of the chain set-up in figure 8.

6.1 Chain of Rigid Links

The first example show a rather contrived example of an assembly. The assembly is a number of rigid cylinders connected end to end to form a chain. The connections between the cylinders are formed by ball and socket joints as explained in section 2.1.1. One end of

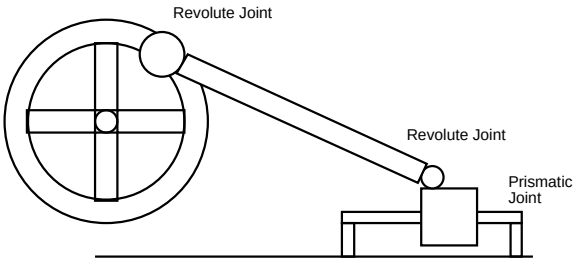


Figure 10: A slider mechanism that converts the rotary motion of a wheel to an oscillatory translational motion of a slider. There are three revolute joints and one prismatic joint.

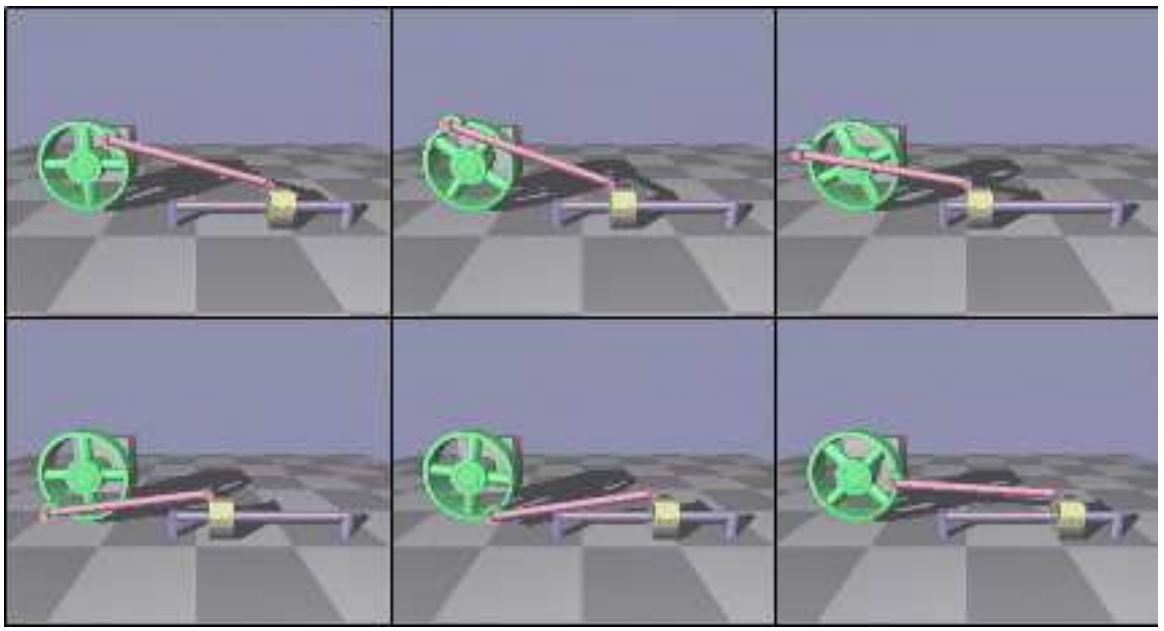


Figure 11: Frames from the animation of the slider set-up in figure 10.

the chain is nailed to a fixed point in space (section 2.2.1). The other end is constrained to move on a straight line with a point to space curve constraint (section 2.2.3). Figure 8 shows the starting position of the chain. Figure 9 shows some frames from the simulation.

External force of gravity is applied to each link. In addition a velocity-proportional damping force is also applied to each link.

6.2 Slider Mechanism

The slider mechanism shown in figures 10 and 11 converts the rotary motion of the wheel to the oscillatory linear motion of a slider. The slider and the fixed post form a prismatic joint. The connecting rod is connected to the piston and the wheel with revolute joints. Further, the wheel is connected to the engine shaft with another revolute joint.

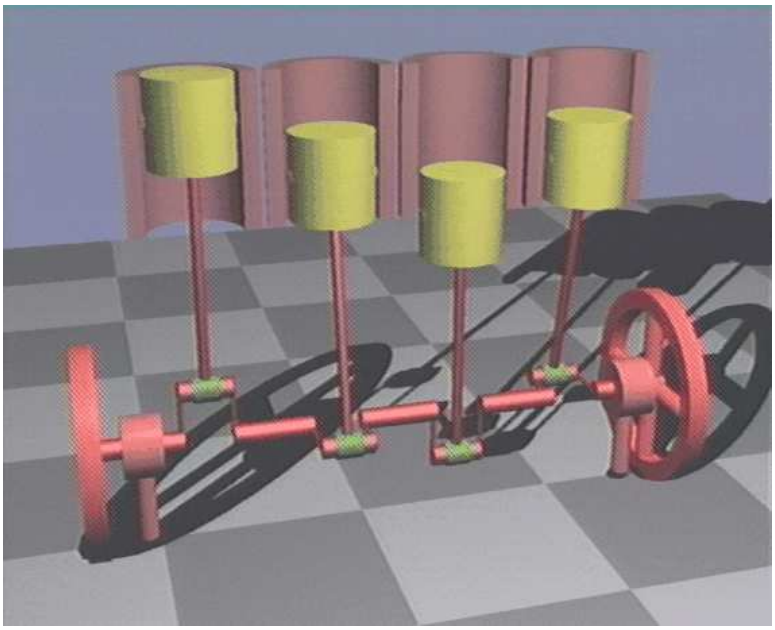


Figure 12: Crank shaft setup. The mechanism consists of 10 revolute joints and four cylindric joints. A position dependent force is applied to each piston

The external forces applied is a driving torque to the wheel and an angular-velocity dependent friction torque also to the wheel. The prismatic joint is friction-less.

6.3 Crank Shaft Mechanism

In this example, we simulate a crank shaft mechanism driven by four pistons (Figure 12). The crank shaft is fixed to the ground through two bearings implementing revolute joints. Each of the pistons is connected to the crank shaft by a rod through revolute joints (at the green bearings) and the rod is connected to the piston through revolute joints.

External forces are applied to the pistons. The piston forces are position and velocity dependent forces. When a piston is moving up, there no force is applied to it. When the piston reaches the top, a force is turned on (to simulate the firing of a spark plug as in a car engine). The force falls linearly to zero as the piston moves down to its bottom extreme.

A frictional torque is applied to the crank to prevent the speed of the system to increase unbounded.

6.4 Performance Figures

The following table gives the timing of the example simulations on a Hewlett-Packard Model 730 workstation (76 MIPS, 23 MFlops).

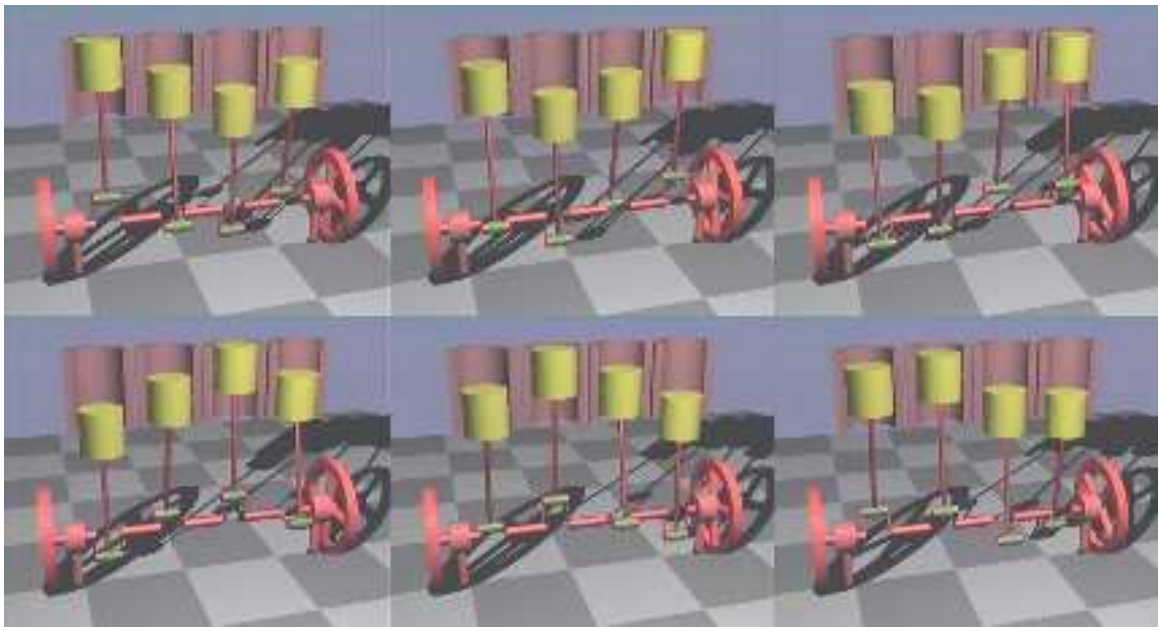


Figure 13: Frames from the animation of the crank set-up in figure 12.

Animation	Chain	Slider	Crank
No of Differential Eqns	65	52	117
No of Linear Eqns	18	18	54
Simulation Time	30s	30s	100s
Computer User Time	166.7s	63.6s	234.6s

The times shown in the table do not include times required to do the rendering which obviously depends on the quality of rendering.

7 Conclusion

Our goal in this research was to come up with primitives or building blocks for a rigid-body simulator. We have developed two concepts towards this goal.

First, we have presented a characterization of rigid-body contacts. We have discussed lower-pair contacts in which two bodies contact over an extended area. We have presented all possible such contacts. We have also discussed a number of higher-pair contacts where bodies contact along points and lines. This characterization and its further treatment provide some basic and general building primitives to form complex rigid body assemblies as shown in our examples.

Secondly, we have presented a methodology to enforce the above contacts between bodies using inverse-dynamics. We impose constraint forces and torques to maintain desired contacts. We have mapped each type of contact into a linear constraint between a number of second order rigid body quantities such as acceleration of points, angular velocity etc. We

further showed that an arbitrary linear combination of such second order quantities (with matrix coefficients) can be converted using Newton's equations of motion into a linear set of equations. The solution to these equations provides constraint forces and torques that produce configurations of rigid bodies consistent with the constraint. The constraints for different contacts discussed in this paper are just a subset of the constraints allowed by this arbitrary linear constraint.

These two concepts can be used to design a general rigid-body simulator, an instance of which we have implemented and we described some examples.

We also believe that the arbitrary linear combination of second order quantities is more powerful than just for implementing rigid-body contacts described in this paper. This is subject of further study.

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A Dual of a vector

The dual of a vector $\mathbf{b} = [b_1, b_2, b_3]^t$ is the matrix $\mathbf{b}^*$:

$$\mathbf{b}^* = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \tag{39}$$

B Angular Acceleration

We have

$$\begin{aligned} \mathbf{L} &= \mathbf{I}\underline{\omega} \\ \underline{\omega} &= \mathbf{I}^{-1}\mathbf{L} \\ \dot{\underline{\omega}} &= \frac{d\mathbf{I}^{-1}}{dt}\mathbf{L} + \mathbf{I}^{-1}\frac{d\mathbf{L}}{dt} \\ &= \underline{\omega}^*\mathbf{I}^{-1}\mathbf{L} + \mathbf{I}^{-1}\underline{\omega}^{*t}\mathbf{L} + \mathbf{I}^{-1}\mathbf{T} \\ &= \mathbf{I}^{-1}(\mathbf{L} \times \underline{\omega} + \mathbf{T}) \end{aligned} \tag{40}$$

C Fixed Point on Body

Let point $\mathbf{p}$ (not necessarily CM) be on a body B, with $\mathbf{b}$ being the vector from the CM to the point $\mathbf{p}$. Then,

$$\begin{aligned}
\mathbf{r}_p &= \mathbf{r}_{CM} + \mathbf{b} \\
\mathbf{v}_p &= \dot{\mathbf{r}}_{CM} + \omega \times \mathbf{b} \\
\mathbf{a}_p &= \ddot{\mathbf{r}}_{CM} + \dot{\omega} \times \mathbf{b} + \omega \times (\omega \times \mathbf{b}) \\
&= \ddot{\mathbf{r}}_{CM} + (\mathbf{b}^{*T}\mathbf{I}^{-1})\mathbf{T} \\
&\quad +(\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b}) \\
&= \frac{1}{m}\sum \mathbf{F}_i \\
&\quad +(\mathbf{b}^{*T}\mathbf{I}^{-1})(\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{T}_i) \\
&\quad +(\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b})
\end{aligned} \tag{41}$$

D Fixed Vector

$$\begin{aligned}
\dot{\mathbf{b}} &= \omega \times \mathbf{b} \\
\ddot{\mathbf{b}} &= \dot{\omega} \times \mathbf{b} + \omega \times (\omega \times \mathbf{b}) \\
&= (\mathbf{b}^{*T}\mathbf{I}^{-1})\mathbf{T} + (\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \\
&\quad \omega \times (\omega \times \mathbf{b}) \\
&= (\mathbf{b}^{*T}\mathbf{I}^{-1})(\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{T}_i) + \\
&\quad (\mathbf{b}^{*T}\mathbf{I}^{-1}(\mathbf{L} \times \omega)) + \omega \times (\omega \times \mathbf{b})
\end{aligned} \tag{42}$$