Constant Affine Velocity Predicts the 1/3 Power Law of Drawing and Planar Motion Perception

Frank E. Pollick*, Guillermo Sapiro
Computer Peripherals Laboratory
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*ATR Human Information Processing Research Laboratories, 2-2 Hikaridai, Seika-cho Soraku-gun Kyoto 619-02, Japan.
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Frank E. Pollick
ATR Human Information Processing Research Laboratories
2-2 Hikaridai, Seika-cho Soraku-gun
Kyoto 619-02, Japan

Guillermo Sapiro
Hewlett-Packard Laboratories
1501 Page Mill Rd.
Palo Alto, CA 94304, USA

Abstract

Numerous studies have shown that the power of 1/3 is important in relating Euclidean velocity to radius of curvature (R) in the production and perception of planar movement. Although the relation between velocity and curvature is clear and very intuitive, no valid explanation for the specific 1/3 value was found yet. We show that if instead of computing the Euclidean velocity we compute the affine one, a velocity which is invariant to affine transformations, then we obtain that the unique function of R which will give an affine invariant velocity is precisely $R^{1/3}$. This means that the 1/3 power law, experimentally found in the studies of hand-drawing and planar motion perception, implies motion at constant affine velocity. Since drawing at constant affine velocity implies that curves of equal affine length will be drawn in equal time, we performed an experiment to further support this result. Results showed that drawing was performed at constant affine velocity. Possible reasons for the appearance of affine transformations in the production and perception of planar movement are discussed.

1 Introduction

When humans draw planar curves, the instantaneous tangential velocity of the hand decreases as the curvature increases [1, 2, 3]. This relationship is best described as a power law where velocity is proportional to the 1/3 power of the radius of curvature [3]. An identical power law has been observed in planar motion perception [4, 5], where it was shown that a point-light moving on a plane is perceived as moving with constant velocity when its real velocity holds this 1/3 power law. Although aspects of drawing [6, 7, 8, 9], its development [10], as well as the visual perception of motion [4, 5], show that the power law influences the organization of both perception and action, no adequate explanation for the specific 1/3 value has been offered.

When considering how to account for this 1/3 power we note that although the physical world which we see and manipulate can be described by Euclidean geometry, there is reason to doubt that properties such as Euclidean distance and angles are faithfully reproduced in our internal representations. For example, judgments of static form show that the structure of human visual space [11, 12] as well as motor space

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[13] deviate from Euclidean geometry. In addition to these deviations stands the fact that regularities and invariances of the relations between internal representations and the physical world need not be expressed in Euclidean geometry. And in this paper we show that the power law relating figural and kinematic aspects of movement - that Euclidean tangential velocity V_e is proportional to the radius of curvature R to the 1/3 power - can be explained by examination of the affine space rather than the Euclidean one.

Why affine? In vision, affine transformations are obtained when a planar object is rotated and translated in space, and then projected into the eye (camera) via a parallel projection. This is a good model of the human visual system when the object is flat enough, and away from the eye, as in the case of drawing. Accordingly, affine concepts have been applied to the analysis of image motion and the perception of three-dimensional structure from motion [14, 15, 16, 17, 18] as well as the recognition of planar form [24]. Another way that affine invariance could arise is that the transforms from visual input to motor output could approximate the true Euclidean transformations [19] and do so with affine approximations. Although in this work we do not attempt to isolate the stage in visuo-motor processing at which the affine geometry enters, the essential explanation of the 1/3 power remains the same.

2 Affine velocity and constant planar motion

We proceed now to explain the 1/3 power law experimental findings based on differential geometry. A planar curve may be regarded as the trajectory of a point $p \in [0, a]$ on the plane. For each value of p, a point $\mathcal{C}(p) = [x(p), y(p)] \in \mathbf{R}^2$ on the curve is obtained, Figure 1. The velocity of the trajectory is given by the tangent vector $\frac{\partial \mathcal{C}}{\partial p}$. Different parametrizations p give different velocities, but define the same trace or geometric curve. That means that given an increasing function $q(p) : \mathbf{R}^+ \to \mathbf{R}^+$, although the traveling velocities are different since $\frac{\partial \mathcal{C}}{\partial p} \neq \frac{\partial \mathcal{C}}{\partial q}$, the curve $\mathcal{C}(q)$ defines the same trace as $\mathcal{C}(p)$. Figure 1 presents a picture explaining these concepts.

An important parametrization is the *Euclidean arc-length* v [20], which means that the curve is traveled with constant velocity, that is $\| \frac{\partial \mathcal{C}}{\partial v} \| \equiv 1$, see Figure 1. In this case the Euclidean length of a curve between v_0 and v_1 is

$$l_e(v_0, v_1) := \int_{v_0}^{v_1} dv.$$

This *Euclidean arc-length* parametrization is invariant with respect to rotations and translations (Euclidean transformations). This means the following: Assume $\tilde{\mathcal{C}}$ is obtained from $\mathcal{C}$ via a rotation and a translation, i.e.,

$$\tilde{\mathcal{C}} = \mathcal{R}\mathcal{C} + T,$$

where $\mathcal{R}$ is a 2×2 rotation matrix and T is a 2×1 translation vector. Let v_0 and v_1 be two points in $\mathcal{C}$ and $\tilde{v_0}$ and $\tilde{v_1}$ their corresponding points after the transformation $(\mathcal{R}, T)$, see Figure 2. Then, the Euclidean invariance of the arc-length v gives that $dv = d\tilde{v}$, meaning that distances measured via dv are preserved; $l_e(v_0, v_1) = l_e(\tilde{v_0}, \tilde{v_1})$.

Having the definition of Euclidean arc-length and length, we can define the *Euclidean velocity* via

$$V_e := \frac{dv}{dt},$$

where t stands for time. This is the classical definition of velocity, which relates the (Euclidean) distance l_e traveled with the time it takes to travel it. Since l_e is invariant to rotations and translations, so is V_e. This velocity V_e is the one measured in the experiments of hand-drawing and planar point motion. As we saw in previous section it was found that in this case

$$V_e = cR^{1/3}, \tag{1}$$

where c is a constant and R is the radius of curvature. Recall that the radius of curvature $R(p)$ is defined as the radius of the circle that best approximates the curve $\mathcal{C}$ at the point p. This radius R is also the inverse of the curvature κ, defined as the rate of change of the unit tangent vector $\vec{\mathcal{T}}$, that is

$$\kappa := \| \frac{\partial \vec{\mathcal{T}}}{\partial v} \| .$$

Suppose now that instead of only rotations and translations, we allow *affine* transformations, which means that the curve can be stretched with different values in the horizontal and vertical directions. An affine transformation of a curve $\mathcal{C}$ is formally defined as

$$\tilde{\mathcal{C}} = \mathcal{A}\mathcal{C} + T,$$

where $\mathcal{A}$ is a 2×2 non-singular matrix [1] and T is a translation vector as before. For the affine group, the Euclidean arc-length v is not invariant any more, $dv \neq d\tilde{v}$ and $l_e(v_0, v_1) \neq l_e(\tilde{v_0}, \tilde{v_1})$. We can define a new notion of ***affine arc-length*** (s), and based on it an ***affine length*** (l_a), which are affine invariant [21, 22]. The affine arc-length is given by the requirement $\left|\frac{\partial \mathcal{C}}{\partial s} \times \frac{\partial^2 \mathcal{C}}{\partial s^2}\right| = 1$, which means that the area [2] of the parallelogram determined by the vectors $\frac{\partial \mathcal{C}}{\partial s}$ and $\frac{\partial^2 \mathcal{C}}{\partial s^2}$ is constant. This gives the simplest affine invariant parametrization. [3] Based on this, we define the affine invariant distance as

$$l_a(v_0, v_1) := \int_{v_0}^{v_1} ds,$$

and the affine velocity as

$$V_a := \frac{ds}{dt}.$$

The affine velocity relates the affine distance l_a with the time it takes to travel it, and both l_a and V_a are affine invariant (see Figure 2).

As we pointed out before, parametrizations only describe the velocity the curves are traveled, and define the same geometric curve or trace. It is possible in general to transform a curve $\mathcal{C}(p)$ parametrized by p into another one parametrized by q, with q being a function of p. This process is called re-parametrization. Assume that the curve is originally parametrized via Euclidean arc-length v, and we want to re-parametrize it by affine arc-length s. Then, using the relation between an arbitrary parametrization and s [21], we have

$$\frac{ds}{dv} = \left|\frac{\partial \mathcal{C}}{\partial v} \times \frac{\partial^2 \mathcal{C}}{\partial v^2}\right|^{1/3} = |\vec{\mathcal{T}} \times \kappa \vec{\mathcal{N}}|^{1/3} = \kappa^{1/3},$$

where $\vec{\mathcal{T}}$, $\vec{\mathcal{N}}$, and $\kappa = 1/R$ are the unit tangent, unit normal, and the Euclidean curvature respectively. In the expression above we used classical relations of differential geometry. Therefore

$$V_a = \frac{ds}{dt} = \frac{ds}{dv}\frac{dv}{dt} = \kappa^{1/3} V_e = \frac{1}{R^{1/3}} V_e.$$

This is the general formula that relates Euclidean velocity with affine one. For the case of hand-drawing velocity (1) we have that

$$V_a \propto c, \tag{2}$$

[1] We assume that the determinant of $\mathcal{A}$ is equal to 1.

[2] Length is non affine invariant, but area is.

[3] Affine differential geometry is not defined at inflection points ($R = \infty$) and thus the definitions are correct for only non-inflection points. However, since inflection points are affine invariant, that is, preserved via an affine transformation, this causes no problems.

which means that the curve is traveled with constant affine velocity. This means for example that a circle and an ellipse will be traveled at times proportional to k, since they are related by an affine transformation. Looking at Figure 2, the 1/3 power law predicts that the drawing times from p_1 and p_2 in Fig. 2b and Fig. 2c are the same, since both curves are related by an affine transformation.

From (2) we conclude that traveling with velocity proportional to the 1/3 power of the radius of curvature means that the affine velocity is constant. Moreover, it is easy to prove that the unique function of R that will give constant affine velocity is this 1/3 power. This means that the 1/3 power is the unique function of the curvature giving that two curves related by an affine transformation are drawn in the same time. The same is true for a point-light moving on two planar trajectories related by affine transformations.

3 Experiment

We performed an experiment to determine if, as predicted, curves were drawn at constant affine velocity and that drawing time remained constant for shapes of equal affine length.

3.1 Methods

Six volunteers from the lab staff participated in a single session where they twice traced each of 16 planar figures for a period of 45 seconds. The 16 figures which subjects traced were 4 hippopedes [23], and three affine transformed curves with equal affine length (see Figure 3).

Position data was sampled at 205 HZ from a digitizing pad with 0.02 mm accuracy and was digitally filtered with a fifth order butterworth filter with a cutoff frequency of 10 Hz. Subjects reproduced the Euclidean perimeter with an average error of 0.6 mm (SD 1.3 mm) which showed no statistically significant variation with the amount of stretch or affine length.

3.2 Results

Results showed that curves were drawn with constant affine velocity (Figure 4a,b) , but that shapes with equal affine lengths did not have equal drawing times (Figure 4c). This apparent contradiction is consistent with the scenario where drawing occurs at constant affine velocity but the form of the figure is incorrectly reproduced. Errors in reproducing local shape will result in the total affine length of the drawn-figure being unequal to the affine length of the presented figure. For example, drawing movements which underestimate the local radius of curvature will result in overestimates of total affine length and thus longer total drawing times.

We examined the drawing movements for errors which were consistent with the increased drawing times. This was done by plotting the drawing times versus the cumulative error in the reproduction of local radius of curvature (Figure 5). It was found that drawing time increased with total underestimation of the local radius of curvature. These results are suggestive that subjects could not reproduce the radius of curvature past a threshold value and that errors in drawing time were a result of the presented radius of curvature passing this threshold.

The idea of a threshold of radius of curvature is consistent with neurophysiological data obtained from monkeys. These studies, exploring cortical mechanisms of the population coding [26] of movement direction, indicated that as the radius of curvature increased past a threshold value, the population coding of movement was no longer predictive of the actual movement [25].

4 Concluding remarks

The 1/3 power law of human hand-drawing and planar motion perception has been an intriguing issue since it was first experimentally discovered. In this work we proved that it is the unique function of curvature that gives a constant affine motion. Considering that affine transformations are good approximations of real object-visual system transformations in the visual world when the observed object is flat enough, this theoretical finding suggests that the 1/3 power law results from approximations in visuo-motor transformations involving affine rather than Euclidean distances. It also suggests common affine mechanisms in the production and perception of form. The large amount of experimental data supporting the 1/3 power law, as well as the new data provided here, supports this theory. We are currently working on further investigations of affine representations for planar motion and their potential role in cortical mechanisms of movement control.

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The authors thank Prof. Peter Giblin for introducing us and asking the following important question: "Do you think there is any relation between your works?" The answer to this question is the topic of this paper.

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Figure Legends

- **Figure 1.** Geometry of a planar curve. In Figure 1a a curve parametrized by p is given. Note that the tangent vectors have different lengths, since the parametrization p is arbitrary. This tangent vectors represent the curve traveling velocity. In Figure 1b, the same geometric curve is presented, with a different parametrization. In this case, the parametrization is given by the Euclidean arc-length v, which means that the curve is traveled with constant velocity. This makes the tangent vectors equal in length. Although tangent vectors are different in both curves, since the trajectories have different velocities, both curves have the same trace.

- **Figure 2.** Curves related by Euclidean (Fig. 2a and 2b.) and affine transformations (fig. 2a and 2c). While the Euclidean distance between corresponding points in a and b is preserved, it is not so between b and c. In this case, the affine distance is the preserved one. The 1/3 power predicts that the traveling time from p_1 to p_2 is the same since the figures are related by affine transformations.

- **Figure 3.** The sixteen shapes used in the drawing experiment. Each column contains four figures with equal affine length and corresponds to an affine-transformed hippopede. The polar equation of a hippoped is $r^2 = 4b(a - b\sin^2\theta))$ and the 4 hippopedes were obtained with the values $a = 4.3$mm and $b = \frac{a}{5}, \frac{a}{4}, \frac{a}{3.25}, \frac{a}{2}$ for the columns left to right (with each rotated so that its long axis was vertically aligned). The area-preserving affine transformation used in obtaining the 4 rows was to stretch by an amount α in the vertical direction while compressing by an amount $\frac{1}{\alpha}$ in the horizontal direction. The four rows, from top to bottom, correspond to values of α=1.2, 1.85, 2.5 and 3.25.

- **Figure 4.** a) An example of corresponding instantaneous Euclidean and affine velocities (filtered at 1 Hz cutoff). Euclidean velocity is periodic with the drawing motion while affine velocity is roughly constant (units of velocity: Euclidean ($\frac{mm}{sec}$), affine ($\frac{mm^{2/3}}{sec}$)). b) Averages of subjects' instantaneous affine and Euclidean velocities. Average instantaneous affine velocity (open marks) was constant for all shapes while average instantaneous Euclidean velocity (filled marks) increased with the Euclidean perimeter (units of velocity: Euclidean ($\frac{mm}{sec}$), affine ($\frac{mm^{2/3}}{sec}$)). c) Average drawing time did not remain constant for shapes of equal affine length, but increased for shapes with greater Euclidean perimeter.

- **Figure 5.** Subjects' errors in reproducing the local form of the presented shape were related to their increase in drawing time. This can be seen by plotting the drawing times versus the average error in the total radius of curvature. This error was defined as the sum of the radius of curvature of the drawn shape minus the approximate numerical integral of the radius of curvature of the presented shape.

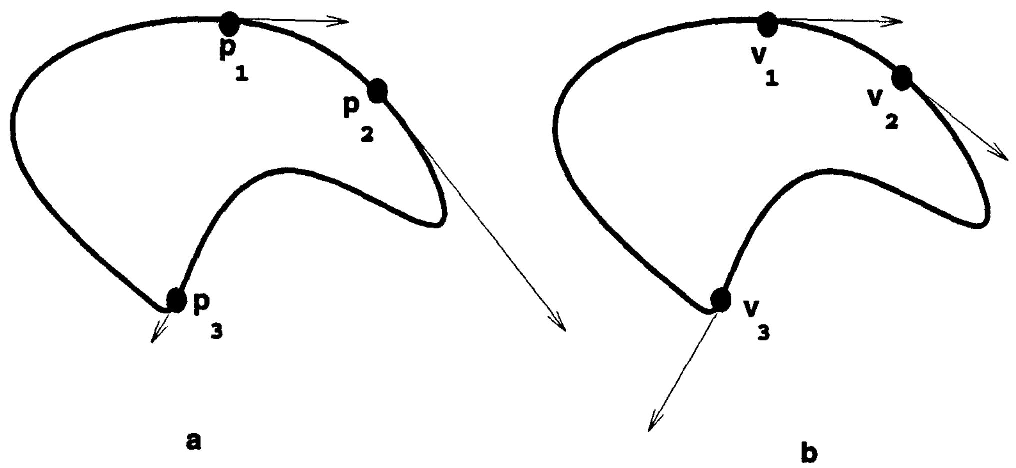


Figure 1

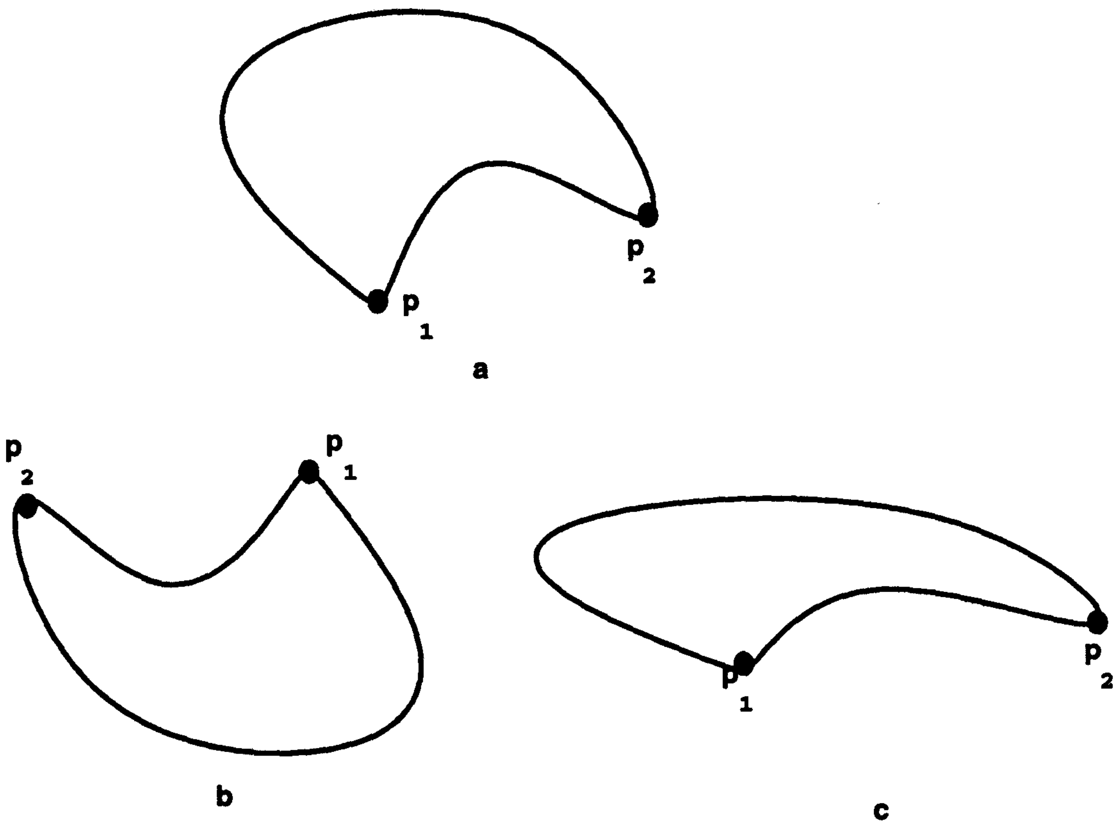


Figure 2

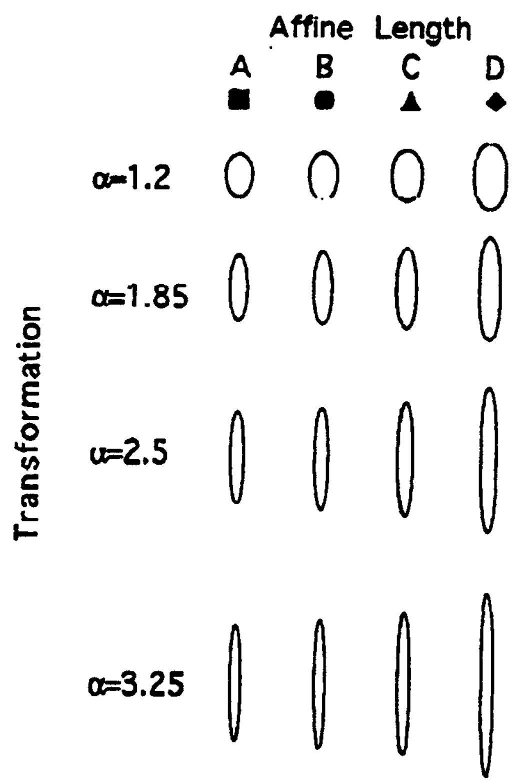


Figure 3

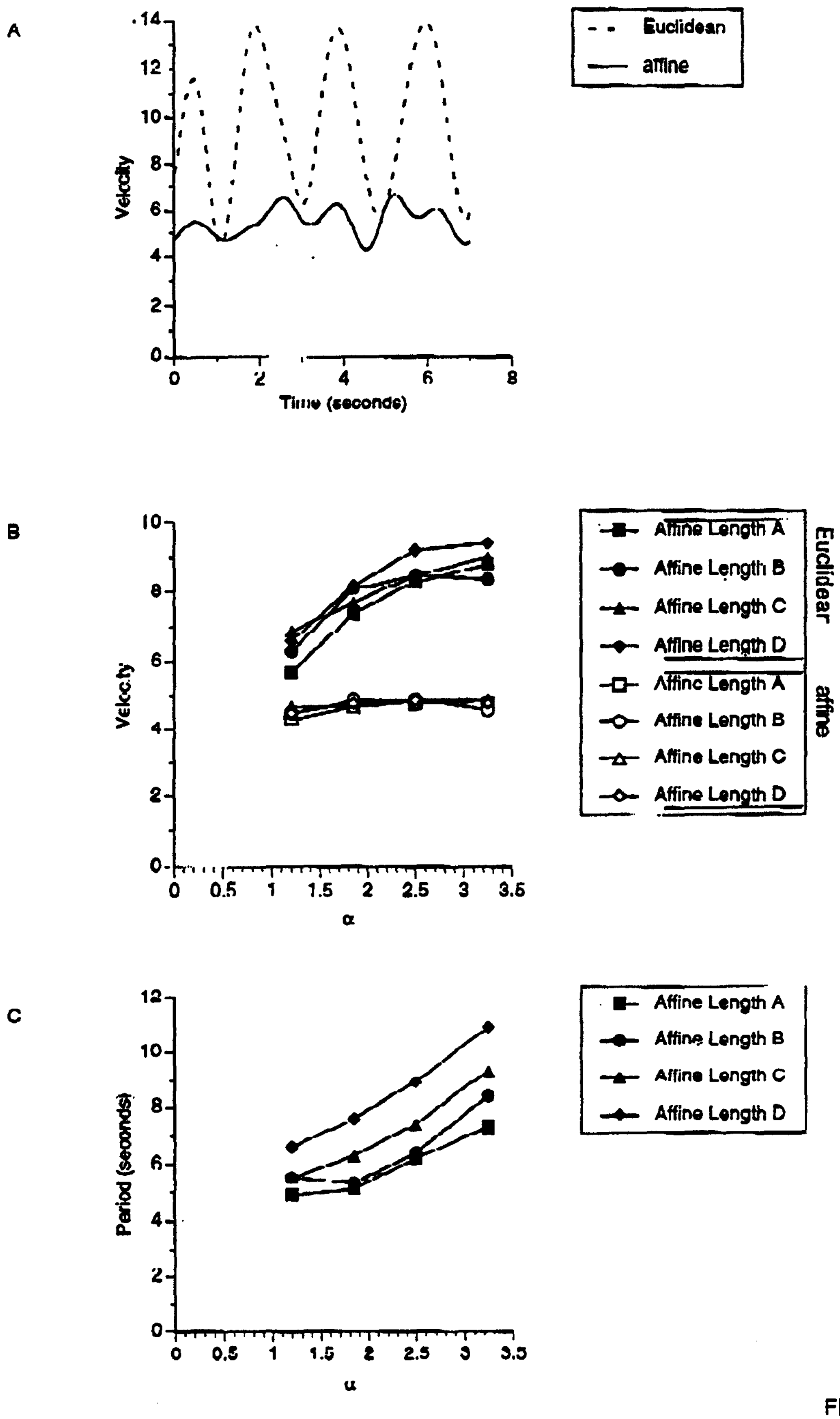
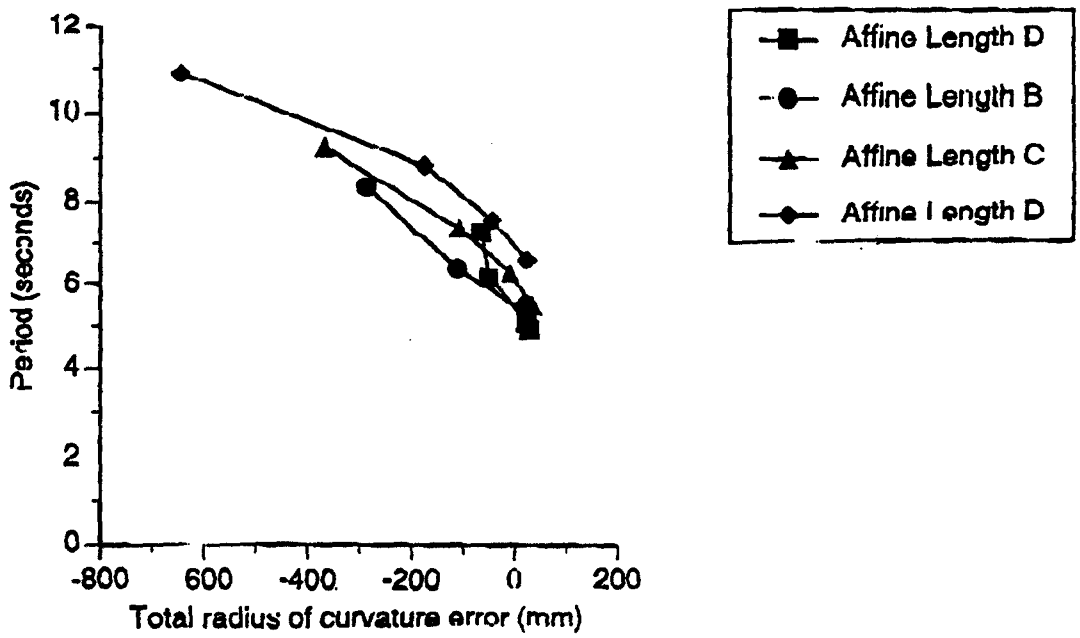

A
Euclidean
affine
Velocity
Time (seconds)
B
Affine Length A
Affine Length B
Affine Length C
Affine Length D
Euclidean
affine
Velocity
α
C
Period (seconds)
α

Figure 4



12
10
8
6
4
2
0
Period (seconds)
-800
600
-400
-200
0
200
Total radius of curvature error (mm)
Affine Length D
Affine Length B
Affine Length C
Affine Length D

Figure 5