Histogram Modification via Partial Differential Equations

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Abstract

The explicit use of partial differential equations (PDE's) in image processing became a major topic of study in the last years. In this work we present an algorithm for histogram modification via PDE's. We show that the histogram can be modified to achieve any given distribution. The modification can be performed while simultaneously reducing noise. This avoids the noise sharpening effect in classical algorithms. The approach is extended to local contrast enhancement as well. A variational interpretation of the flow is presented and theoretical results on the existence of solutions are given.

Key words: Histogram modification, partial differential equations, denoising, variational formulation.

1 Introduction

The use of partial differential equations (PDE's) for image processing became a major research topic in the past years. The idea is not to think of image processing in the discrete domain but in the continuous one, combined with efficient numerical implementations. In general, let $\Phi_0 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ represent a gray-level image, where $\Phi_0(x, y)$ is the gray-level value. The algorithms that we describe are based on the formulation of partial differential equations of the form

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}[\Phi(x, y, t)],$$

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where $\Phi(x,y,t) : \mathbf{R}^2 \times [0,\tau) \to \mathbf{R}$ is the evolving image, $\mathcal{F} : \mathbf{R} \to \mathbf{R}$ is a given function which depends on the algorithm, and the image Φ_0 is the initial condition. The solution $\Phi(x,y,t)$ of the differential equation gives the processed image.

Most of the use of PDE's for image processing was done for image debluring or denoising [1, 2, 8, 13, 14, 19, 23, 24]. PDE's where recently used as well for a number of problems in computer vision as shape analysis [9], shape from shading [10, 18], segmentation [5, 6, 12], invariant shape smoothing [22], and mathematical morphology [1, 20]. See [17] for a recent book on the topic.

One of the advantages of the use of PDE's for image processing is the possibility to combine algorithms. If two procedures are given by

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}_1[\Phi(x,y,t)] \quad , \quad \frac{\partial \Phi}{\partial t} = \mathcal{F}_2[\Phi(x,y,t)],$$

then they can be combined as

$$\frac{\partial \Phi}{\partial t} = \alpha \mathcal{F}_1[\Phi(x,y,t)] + \mathcal{F}_2[\Phi(x,y,t)],$$

where $\alpha \in \mathbf{R}^+$. This was successfully used for example in [3], where $\mathcal{F}_1$ was the smoothing operator in [2] and $\mathcal{F}_2$ the debluring one in [13].

Other advantage of this methodology is the accuracy achieved when efficient numerical implementations are used. This makes the algorithms very appropriate for example for medical applications.

In this work we present a novel PDE for histogram modification. We show how to obtain any grey-level distribution, and present examples. Then, as an example, we combine it with the smoothing operator proposed in [23], obtaining contrast normalization and denoising at the same time. A variational interpretation of the histogram modification flow and theoretical results regarding existence of solutions to the proposed PDE's are presented as well.

Before proceeding with the algorithm we should point out that in [15] the authors recently presented a diffusion network for image normalization. In their work, the image $\Phi(x,y)$ is normalized via $\frac{\Phi - \Phi_a}{\Phi_M - \Phi_m}$, where Φ_a, Φ_M, and Φ_m are the average, maximum, and minimum of Φ over local areas. These values are computed using a diffusion flow, which minimizes a cost functional. The method was generalized computing a full local frame of reference for the gray level values of the image. This is achieved changing the variables in the flow. A number of properties, including existence of the solution of the diffusion flow, were presented as well. In contrast with their work, in our case we have full control of the final distribution of the gray-levels, that means while their work is on contrast normalization, our is on histogram modification. Also, the modified image is obtained in this work as the steady state solution of the PDE, without any extra operations as required in [15]. This allows straightforward combination with other PDE based algorithms as explained above.

2 Histogram modification

We start from the PDE for histogram equalization, and then we extend it for any distribution. Assume that he image $\Phi(x, y, t) : [0, N]^2 \to [0, M]$ evolves according to

$$\frac{\partial \Phi(x, y, t)}{\partial t} = (N^2 - N^2/M \ \Phi(x, y, t)) - \mathcal{A}[(v, w) : \Phi(v, w, t) \geq \Phi(x, y, t)], \tag{1}$$

where $\mathcal{A}[\cdot]$ represents area (or number of pixels in the discrete case). For the steady state solution ($\Phi_t = 0$) we have

$$\mathcal{A}[(v, w) : \Phi(v, w) \geq \Phi(x, y)] = (N^2 - N^2/M \ \Phi(x, y)).$$

Then, for $a, b \in [0, M]$, $b > a$, we have

$$\mathcal{A}[(v, w) : b \geq \Phi(v, w) \geq a] = (N^2/M) * (b - a),$$

which means that he histogram is constant. Therefore, the steady state solution of (1) gives the image after normalization via histogram equalization. Note than in spite of the fact that $\mathcal{A}[\cdot]$ is a global operator, special data structures can be used which make its computation very fast. The values of $\mathcal{A}[\cdot]$ need to be updated after each iteration, and not re-computed. This makes the algorithm very fast. As we will see in sections 3 and 4, we can also just compute $\mathcal{A}[\cdot]$ in the neighborhood of the pixel that is being updated, or just perform local histogram equalization dividing the image into regions as in classical histogram modification algorithms.

From (1) we can extend the algorithm to obtain any given gray-value distribution $h : [0, M] \to \mathbf{R}^+$. Let $H(s) := \int_0^s h(\xi) d\xi$. That is, $H(s)$ gives the density of points between 0 and s. Then, if the image evolves according to

$$\frac{\partial \Phi(x, y, t)}{\partial t} = (N^2 - H[\Phi(x, y, t)]) - \mathcal{A}[(v, w) : \Phi(v, w, t) \geq \Phi(x, y, t)], \tag{2}$$

the steady state solution is given by

$$\mathcal{A}[(v, w) : \Phi(v, w) \geq \Phi(x, y)] = (N^2 - H[\Phi(x, y)]).$$

Therefore,

$$\mathcal{A}[(v, w) : \Phi(x, y) \leq \Phi(v, w) \leq \Phi(x, y) + \delta] = H[\Phi(x, y) + \delta] - H[\Phi(x, y)]),$$

and tacking Taylor expansion when $\delta \to 0$ we obtain the desired result. Note that of course (1) is a particular case of (2), with $h = $ constant.

2.1 Existence and uniqueness of the flow

We present now results related to the existence and uniqueness of the proposed flow for histogram modification. Results on existence of the flow obtained when combined with a smoothing operator will be presented in Section 3.1. Results on the smoothing operator itself can be found in the mentioned references.

Let Φ_0 be an image defined in $[0,N]^2$ with values in the range $[a,b]$, $0 \le a < b \le M$. We assume that the distribution function of Φ_0 is continuous, that is

$$\mathcal{A}[X : \Phi_0(X) = \lambda] = 0 \tag{3}$$

for all $X \in [0,N]^2$ and all $\lambda \in [a,b]$. To equalize the histogram of Φ_0 we look for solutions of

$$\Phi_t(t,X) = \mathcal{A}[Z : \Phi(t,Z) < \Phi(t,X)] - \frac{N^2}{b-a}(\Phi(t,X) - a) \tag{4}$$

which also satisfy

$$\mathcal{A}[X : \Phi(t,X) = \lambda] = 0. \tag{5}$$

Hence the distribution function of $\Phi(t,X)$ is also continuous. This requirement, mainly technical, avoids the possible ambiguity of changing the sign " $<$ " by " $\le$ " in the computation of $\mathcal{A}$. Let's recall the definition of $\mathrm{sign}^-(\cdot)$:

$$\mathrm{sign}^-(r) = \begin{cases} 1 & \text{if } r < 0 \\ [0,1] & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}$$

With this notation, Φ satisfying (4) and (5) can be written as

$$\Phi_t(t,X) = \int_{[0,N]^2} \mathrm{sign}^-(\Phi(t,Z) - \Phi(t,X))dZ - \frac{N^2}{b-a}(\Phi(t,X) - a). \tag{6}$$

Observe that as a consequence of (5), the real value of sign^- at zero is unimportant, avoiding possible ambiguities. In order to simplify the notation, let us normalize Φ such that it is defined on $[0,1]^2$ and takes values in the range $[0,1]$. This is done just by the change of variables given by $\Phi(t,X) \leftarrow \frac{\Phi(\lambda t, NX) - a}{b-a}$, where $\lambda = \frac{b-a}{N^2}$. Then, Φ satisfies the equation

$$\Phi_t(t,X) = \int_{[0,1]^2} \mathrm{sign}^-(\Phi(t,Z) - \Phi(t,X))dZ - \Phi(t,X). \tag{7}$$

Therefore, without loss of generality we can assume $N = 1$, $a = 0$, and $b = 1$, and analyze (7). For this flow we have the following result:

Theorem 1 *For any continuous function $\Phi_0 : [0,1]^2 \to [0,1]$ such that $\mathcal{A}[Z : \Phi_0(Z) = \lambda] = 0$ for all $\lambda \in [0,1]$, there exists a unique continuous solution $\Phi(t,X)$ in $[0,1]^2$ with range $[0,1]$ satisfying the flow (7) with initial condition given by Φ_0, and such that $\mathcal{A}[Z : \Phi(t,Z) = \lambda] = 0$ for all $\lambda \in [0,1]$. Moreover, as $t \to \infty$, $\Phi(t,X)$ tends to the histogram equalization of $\Phi_0(X)$.*

Proof: We start with the existence. We look for a solution $\Phi(t,X)$ such that for all $t>0$, $X,X'\in[0,1]^2$, $\Phi(t,X)<\Phi(t,X')$ if and only if $\Phi_0(t,X)<\Phi_0(t,X')$. This assumption is enough to prove existence. Later we will proof that it holds for any solution of (7). In that case

$$\mathcal{A}[Z:\Phi(t,Z)<\Phi(t,X)]=\mathcal{A}[Z:\Phi(0,Z)<\Phi(0,X)], \tag{8}$$

that is, is independent of t. Let us denote this function by $\mathcal{F}_0(X)$. Then, (7) can be re-written as

$$\Phi_t=\mathcal{F}_0(X)-\Phi(t,X), \tag{9}$$

whose explicit solution is

$$\Phi(t,X)=\exp\{-t\}\Phi_0+(1-\exp\{-t\})\mathcal{F}_0(X). \tag{10}$$

Observe that the solution has range $[0,1]$ and satisfies

$$\mathcal{A}[Z:\Phi(t,Z)=\lambda]=0\ ,\ t\geq 0\ ,\ \lambda\in[0,1]. \tag{11}$$

Since (10) also satisfies (8), it is a solution to the histogram equalization flow.

In order to proof uniqueness, we need to following Lemma:

Lemma 1 *Let $\Phi(t,X)$ be a continuous solution of (7) satisfying (11). Let $X,X'\in[0,1]^2$ be such that $\Phi(0,X)<\Phi(0,X')$. Then*

$$\Phi(t,X')-\Phi(t,X)\geq\exp\{-t\}(\Phi(0,X')-\Phi(0,X)). \tag{12}$$

Proof: Let $\delta>0$. Then

$$\frac{d}{dt}\left(-\frac{1}{2}\ln(\delta^2+\Phi(t,X')-\Phi(t,X))^2\right)$$
$$=\frac{\Phi(t,X')-\Phi(t,X)}{\delta^2+(\Phi(t,X')-\Phi(t,X))^2}\left(\int\mathrm{sign}^-(\Phi(t,Z)-\Phi(t,X'))dZ-\int\mathrm{sign}^-(\Phi(t,Z)-\Phi(t,X))dZ\right)$$
$$+\frac{(\Phi(t,X')-\Phi(t,X))^2}{\delta^2+(\Phi(t,X')-\Phi(t,X))^2}.$$

One easily verifies that the first term above is negative. Therefore

$$\frac{d}{dt}\left(-\frac{1}{2}\ln(\delta^2+\Phi(t,X')-\Phi(t,X))^2\right)\leq\frac{(\Phi(t,X')-\Phi(t,X))^2}{\delta^2+(\Phi(t,X')-\Phi(t,X))^2}\leq 1.$$

After integration we observe

$$\sqrt{\delta^2+(\Phi(t,X')-\Phi(t,X))^2}\geq\exp\{-t\}\sqrt{\delta^2+(\Phi(0,X')-\Phi(0,X))^2}.$$

Letting $\delta\to 0$,

$$|\Phi(t,X')-\Phi(t,X)|\geq\exp\{-t\}|\Phi(0,X')-\Phi(0,X)|>0.$$

Since $\Phi(t,X)$ is continuous, this implies (12). $\Box$

Together with (3), this Lemma implies that if $X, X' \in [0,1]^2$ are such that $\Phi(0,X) = \Phi(0,X')$, then $\Phi(t,X) = \Phi(t,X')$. Indeed since $\mathcal{A}[Z : \Phi(0,Z) = 0] = 0$, one can find sequences $X_n, X'_n \in [0,1]^2$ such that

$$\begin{array}{ll} X_n \to X & , \quad X'_n \to X' \\ \Phi(0,X_n) = \lambda_n > 0 & , \quad \lambda_n \to \Phi(0,X) = 0 \\ \Phi(0,X'_n) = \lambda'_n > 0 & , \quad \lambda'_n \to \Phi(0,X') = 0 \end{array}$$

and $\lambda_n > \lambda'_n$. By the previous Lemma, $\Phi(t,X_n) > \Phi(t,X'_n)$. Letting $n \to \infty$ we have $\Phi(t,X) \geq \Phi(t,X')$. The other inequality follows in the same way. Hence $\Phi(t,X) = \Phi(t,X')$ $\forall\, t \geq 0$. Using the lemma above and the last observation we see again that

$$\begin{array}{l} [\quad Z : \Phi(t,Z) < \Phi(t,X)] = [Z : \Phi(0,Z) < \Phi(0,X)] \\ [\quad Z : \Phi(t,Z) = \Phi(t,X)] = [Z : \Phi(0,Z) = \Phi(0,X)] \ (= 0). \end{array}$$

Therefore,

$$\int \operatorname{sign}^-(\Phi(t,Z) - \Phi(t,X))dZ = \int \operatorname{sign}^-(\Phi(0,Z) - \Phi(0,X))dZ = \mathcal{F}_0(X).$$

The flow can be re-written as (9), and (10) gives the solution. Letting $t \to \infty$, $\Phi(t,X)$ tends to $\mathcal{F}_0(X)$, which corresponds to the equalized histogram for $\Phi_0(X)$. $\Box$

Remark. The above proof can be adapted to any required gray-value distribution h, proving the results for the flow

$$\Phi_t(t,X) = \int \operatorname{sign}^-(\Phi(t,Z) - \Phi(t,X))dZ - \Psi(\Phi(t,X)), \qquad (13)$$

where Ψ is any strictly increasing Lipschitz continuous function.

2.2 Variational interpretation of the histogram flow

The formulation given by equations (6) and (7) not only helps to prove the theorem above, also gives a variational interpretation of the histogram modification flow. Variational approaches are frequently used in image processing. They not only give explicit solutions for a number of problems, also help very often to give an intuitive interpretation of the solution, interpretation which is many times not so easy to achieve from the corresponding Euler-Lagrange or PDE. Variational formulations help to derive new approaches as well.

Let us consider the following functional

$$\mathcal{U}(\Phi) = \frac{1}{2}\int \left(\Phi(X) - \frac{1}{2}\right)^2 dX - \frac{1}{4}\int\int |\Phi(X) - \Phi(Z)| dX dZ, \qquad (14)$$

where $\Phi \in \mathrm{L}^2[0,1]^2$, $0 \leq \Phi(X) \leq 1$. $\mathcal{U}$ is a Lyapounov functional for the equation (7).

Lemma 2 *Let Φ be the solution of (7) with initial data Φ_0 as in Theorem 1. Then*

$$\frac{d\mathcal{U}(\Phi)}{dt} \leq 0.$$

Proof:

$$\frac{d\mathcal{U}(\Phi)}{dt} = \int \left(\Phi(X) - \frac{1}{2}\right) \Phi_t(X) dX - \frac{1}{4} \int \int \operatorname{sign}(\Phi(Z) - \Phi(X))(\Phi_t(Z) - \Phi_t(X)) dX dZ.$$

Let's denote the first integrand in the equation above by A and the second by B. Observe that due to (11), part B of the integral above is well defined. Let us re-write B as

$$B = \frac{1}{4} \int \int \operatorname{sign}(\Phi(Z) - \Phi(X)) \Phi_t(Z) dX dZ - \frac{1}{4} \int \int \operatorname{sign}(\Phi(Z) - \Phi(X)) \Phi_t(X) dX dZ.$$

Interchanging the variables X and Z in the first part of the expression above we obtain

$$B = -\frac{1}{2} \int \int \operatorname{sign}(\Phi(Z) - \Phi(X)) \Phi_t(X) dX dZ.$$

Fixing X we have

$$\int \operatorname{sign}(\Phi(t, Z) - \Phi(t, X)) dZ = 1 - 2 \int \operatorname{sign}^-(\Phi(t, Z) - \Phi(t, X)) dZ,$$

and we may write

$$B = -\frac{1}{2} \int \Phi_t(X) dX + \int \int \operatorname{sign}^-(\Phi(Z) - \Phi(X)) \Phi_t(X) dZ dX.$$

Hence

$$\begin{aligned}
\frac{d\mathcal{U}(\Phi)}{dt} &= \int \Phi(t, X) \Phi_t(t, X) dX - \int \int \operatorname{sign}^-(\Phi(t, Z) - \Phi(t, X)) \Phi_t(t, X) dX dZ \\
&= \int \left\{ \Phi(t, X) - \int \operatorname{sign}^-(\Phi(t, Z) - \Phi(t, X)) dZ \right\} \Phi_t(t, X) dX \\
&= -\int \Phi_t(t, X)^2 dX \leq 0.
\end{aligned}$$

This concludes the proof. $\Box$.

Therefore, when solving (7) we are indeed minimizing, by the steepest descent method, the functional $\mathcal{U}$ given by (14) restricted to the condition that the minimizer satisfies (11). That means, (7) is

$$\Phi_t = \mathcal{U}'.$$

The first term in $\mathcal{U}$ stands for the variance of the signal, while the second one gives the correlation between values at different positions. Having this in mind, other functionals might be proposed to achieve contrast modification.

3 Simultaneous anisotropic diffusion and histogram modification

We present now a flow for simultaneous de-noising and histogram modification. This is just an example of the possibility of combination of different algorithms in the same PDE.

In [23], we presented a geometric flow for edge preserving anisotropic diffusion, based on the results in [1, 2] and [21, 22]. The idea is to smooth the image only in the direction parallel to the edges, achieving this via curvature flows. The flow is given by

$$\frac{\partial \Phi}{\partial t} = \frac{1}{1+ \parallel \nabla(G * \Phi) \parallel} \kappa^{1/3} \parallel \nabla \Phi \parallel, \tag{15}$$

which is equivalent to

$$\frac{\partial \Phi}{\partial t} = \frac{1}{1+ \parallel \nabla(G * \Phi) \parallel} (\Phi_x^2 \Phi_{yy} - 2\Phi_x \Phi_y \Phi_{xy} + \Phi_y^2 \Phi_{xx})^{1/3}, \tag{16}$$

where κ is the Euclidean curvature of the level-sets of Φ, G is a Gaussian, and (16) is obtained from (15) via explicit computation of this curvature. The above equation means that each one of the level sets of Φ is evolving according to the affine heat flow developed in [21, 22] for planar shape smoothing, with the velocity "altered" by the function $\frac{1}{1+\|\nabla(G*\Phi)\|}$ as in [2] in order to reduce smoothing in the edges. This flow, and its Euclidean version (with κ instead of $\kappa^{1/3}$ in (15)), were tested in [1, 2, 23], and proved to give very satisfactory results. See the mentioned references for more details.

As we pointed out in the Introduction, the flows (2) and (15) can be combined to obtain a new flow which performs anisotropic diffusion (denoising) while simultaneously modifies the histogram. The flow is given by

$$\frac{\partial \Phi}{\partial t} = \frac{\alpha \kappa^{1/3} \parallel \nabla \Phi \parallel}{1+ \parallel \nabla(G * \Phi) \parallel} + (N^2 - H[\Phi(x,y,t)]) - \mathcal{A}[(v,w) : \Phi(v,w,t) \geq \Phi(x,y,t)] \tag{17}$$

where $\alpha \in \mathbf{R}^+$ is a parameter which controls the trade-off between smoothing and histogram modification. This flow is tested in Section 4.

Note that other smoothing operator can be used as well. For instance, in [19] (see also [11] for theoretical results), the authors proposed to minimize the total variation of the image, given by

$$\int \parallel \nabla \Phi(X) \parallel dX.$$

It is easy to show that the Euler-Lagrange of this functional is given by the curvature κ of the level-sets, that is

$$\mathrm{div}\left(\frac{\nabla \Phi}{\parallel \nabla \Phi \parallel}\right) = \kappa,$$

which leads to the flow

$$\Phi_t = \kappa.$$

Using this smoothing operator, together with the histogram modification part, gives very similar results as those obtained with the affine based flow. If this smoothing operator is combined with the histogram flow, the total flow

$$\frac{\partial \Phi}{\partial t} = \alpha\kappa + (N^2 - H[\Phi(x,y,t)]) - \mathcal{A}[(v,w) : \Phi(v,w,t) \geq \Phi(x,y,t)] \tag{18}$$

will therefore be such that it minimizes

$$\alpha \int \| \nabla\Phi(X) \| \, dX + \mathcal{U}, \tag{19}$$

where $\mathcal{U}$ is given by (14), obtaining a complete variational formulation [1] of the combined histogram-equalization/smoothing approach. This is precisely the formulation we analyze below.

Before going into the proof let us mention that the PDE formulation above permits also to consider simultaneous denoising and *local* histogram modification. Indeed, we may consider the model

$$\frac{\partial \Phi}{\partial t} = \alpha\kappa + (N^2 - H[\Phi(x,y,t)]) - \mathcal{A}[(v,w) \in \mathrm{B}(v,w,\delta) : \Phi(v,w,t) \geq \Phi(x,y,t)], \tag{20}$$

where $\mathrm{B}(v,w,\delta)$ is a ball of center (v,w) and radius δ. $\mathrm{B}(v,w)$ can also be any other surrounding neighborhood, obtained from example from previously performed segmentation. Experiments with this model are presented in Section 4 as well.

3.1 Existence of the flow

We present now a theoretical result related to the simultaneous smoothing and contrast modification flow (18).

Before proceeding with the existence proof of the variational problem (19), let's recall the following standard notation:

1. $\mathcal{C}([0,T],\mathcal{H}) := \{\Phi : [0,T] \to \mathcal{H}$ continuous $\}$, where $T > 0$ and $\mathcal{H}$ is a Banach space (and in particular for a Hilbert space).
2. $\mathrm{L}^p([0,T],\mathcal{H}) := \{\Phi : [0,T] \to \mathcal{H}$ such that $\int_0^T \| \Phi \|^p < \infty\}$, with $1 \leq p < \infty$.
3. $\mathrm{L}^\infty([0,T],\mathcal{H}) := \{\Phi : [0,T] \to \mathcal{H}$ such that $\text{ess sup}_{t\in[0,T]} \| \Phi \| < \infty\}$.
4. $\Phi \in \mathrm{L}^p_{loc}([0,\infty),\mathcal{H})$ means that $\Phi \in \mathrm{L}^p([0,T],\mathcal{H})$ for all $T > 0$.

[1] The affine flow has a variational interpretation as well, but much more complicated.

5. $W^{1,2}([0,T],\mathcal{H}) := \{\Phi : [0,T] \to \mathcal{H} \text{ such that } \Phi, \Phi_t \in \mathrm{L}^2([0,T],\mathcal{H})\}$.

In order to simplify notations, later we will assume $\Omega =]0,1[^2$ and $\mathcal{H} = \mathrm{L}^2(\Omega)$.

We proceed now to prove existence of the solution to the Euler-Lagrange equation corresponding to the variational problem (19), given by ($\alpha = 1$)

$$\Phi_t = \operatorname{div}\left(\frac{\nabla\Phi}{\| \nabla\Phi \|}\right) + \int_{[0,1]^2} \operatorname{sign}^-(\Phi(t,Z) - \Phi(t,X)) - \Phi(t,X), \tag{21}$$

together with the initial and boundary conditions

$$\Phi(0,X) = \Phi_0(X)\ ,\ X \in [0,1]^2, \qquad \frac{\partial\Phi}{\partial\vec{n}}(t,X) = 0\ ,\ t > 0,\quad X \in \partial[0,1]^2,$$

where $\vec{n}$ stands for the normal direction. We shall use results from the theory on non-linear semigroups on Hilbert space [4]. Before proceeding, we need a number of additional definitions. A function $\Phi \in \mathrm{L}^1(\Omega)$ whose derivatives in the sense of distributions are measures with finite total variation in Ω, is called a function of bounded variation. The class of such functions will be denoted by $\mathrm{BV}(\Omega)$. Thus, $\Phi \in \mathrm{BV}(\Omega)$ if there are Radon measures μ_1, ..., μ_n defined in $\Omega \subset \mathbf{R}^n$ such that its total mass $|D\mu_i|(\Omega)$ is finite and

$$\int_\Omega \Phi(X) D_i\phi(X) dX = -\int_\Omega \phi(X) d\mu_i(X)$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega)$. The gradient of Φ will therefore be a vector valued measure with finite total variation

$$\| \nabla\Phi \| = \sup\left\{\int_\Omega \Phi \operatorname{div} v\, dX\ :\ v = (v_1,\ ...,\ v_n) \in \mathcal{C}_0^\infty(\Omega, \mathbf{R}^n),\ |v(X)| \leq 1,\ X \in \Omega\right\}.$$

The space $\mathrm{BV}(\Omega)$ will have the norm

$$\| \Phi \|_{\mathrm{BV}} = \| \Phi \|_1 + \| \nabla\Phi \| .$$

The space $\mathrm{BV}(\Omega)$ is continuously embedded in $\mathrm{L}^p(\Omega)$ for all $p \leq \frac{n}{n-1}$. The inmersion is compact if $p < \frac{n}{n-1}$ ([25], Theorem 2.5.1). If Φ_i is a sequence of functions in $\mathrm{BV}(\Omega)$ converging to the function Φ in $\mathrm{L}^1(\Omega)$, then $\| \nabla\Phi \| \leq \lim_i \inf \| \nabla\Phi_i \|$ ([25], Theorem 5.2.1). Moreover, given a function $\Phi \in \mathrm{BV}(\Omega)$, there exists a sequence of functions $\Phi \in \mathrm{BV}(\Omega)$ such that $\Phi_i \to \Phi$ in $\mathrm{L}^1(\Omega)$ and such that $\| \nabla\Phi \| = \lim_i \| \nabla\Phi_i \|$ ([25], Theorem 5.2.3).

Let $\mathcal{H}$ be a Hilbert space and let $\phi : \mathcal{H} \to]-\infty, +\infty]$ be convex and proper. Given $X \in \mathcal{H}$, the subdifferential of ϕ at X, $\partial\phi(X)$, is given by

$$\partial\phi(X) = \{Y \in \mathcal{H} :\ \forall\xi \in \mathcal{H}\ \ \phi(\xi) - \phi(X) \geq \langle Y, \xi - X \rangle\}.$$

We write $\mathrm{dom}(\phi) := \{X \in \mathcal{H} : \phi(X) < +\infty\}$, $\mathrm{dom}(\partial\phi) := \{X \in \mathcal{H} : \partial\phi(X) \neq \emptyset\}$. From now on we shall write, as we mentioned before, $\Omega =]0,1[^2$ and $\mathcal{H} = \mathrm{L}^2(\Omega)$. We also define the functionals $\phi, \psi : \mathcal{H} \to]-\infty, +\infty]$ by

$$\phi(\Phi) := \begin{cases} \| \nabla\Phi \| + \frac{1}{2}\int_\Omega (\Phi(X) - \frac{1}{2})^2 dX & \Phi \in \mathrm{BV}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\psi(\Phi) := \frac{1}{4}\int_\Omega \int_\Omega |\Phi(Z) - \Phi(X) dX dZ.$$

Note that both functionals are convex, lower semicontinuous, and proper on $\mathcal{H}$. We introduced both functionals since formally (21) is associated with the following abstract problem:

$$\Phi_t + \partial\phi(\Phi) \ni \partial\psi(\Phi). \tag{22}$$

To make such formulation precise, let us recall the following ([4], Definition 3.1): Let $T > 0$, $f \in \mathrm{L}^1([0,T], \mathcal{H})$. We call $\Phi \in \mathcal{C}([0,T], \mathcal{H})$ a strong solution of

$$\Phi_t + \partial\phi(\Phi) \ni f(t) \tag{23}$$

is Φ is differentiable almost everywhere on $]0,T[$, $\Phi \in \mathrm{dom}(\partial\phi)$ a.e. in t and

$$-\Phi_t + f(t) \in \partial\psi(\Phi(t)) \tag{24}$$

almost everywhere (a.e.) on $]0,T[$. In particular, if $\Phi \in W^{1,2}([0,T],\mathcal{H})$, $\Phi(t) \in \mathrm{dom}(\partial\phi)$ a.e. and (24) holds a.e. on $]0,T[$, then Φ is a strong solution of (23). We say that $\Phi \in \mathcal{C}([0,T],\mathcal{H})$ is a strong solution of (22) if there exists $\omega(t) \in \mathrm{L}^1([0,T],\mathcal{H})$, $\omega(t) \in \partial\psi(\Phi(t))$ a.e. in $]0,T[$, such that Φ is a strong solution of

$$\Phi_t + \partial\psi(\Phi) \ni \omega. \tag{25}$$

With these preliminaries, we reformulate (21) as an abstract evolution problem of the form (22) and use the machinery of non-linear semigroups on Hilbert spaces to prove existence of solutions of (22).

Theorem 2 *For any $\Phi_0 \in \mathrm{BV}(\Omega)$, $0 \leq \Phi_0 \leq 1$, there exists a strong solution $\Phi \in W^{1,2}([0,T],\mathcal{H})$, $\forall T > 0$, of (22) with initial condition $\Phi(0) = \Phi_0$, and such that $0 \leq \Phi \leq 1$, $\forall t > 0$. Moreover, the functional $\mathcal{V}(\Phi) = \phi(\Phi) - \psi(\Phi)$ is a Lyapounov functional for (22).*

Proof: The proof is divided in a number of steps.

1st Step. Regularization:

Define for each $\epsilon > 0$, the following functionals

$$\phi_\epsilon(\Phi) := \begin{cases} \frac{\epsilon}{2}\int_\Omega \| \nabla\Phi \|^2 dX + \frac{\epsilon}{2}\int_\Omega \sqrt{\epsilon^2 + \| \nabla\Phi \|^2} dX + \frac{1}{2}\int_\Omega \left(\Phi(X) - \frac{1}{2}\right)^2 dX & \Phi \in \mathrm{BV}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\psi_\epsilon(\Phi) := \frac{1}{4}\int_\Omega\int_\Omega \beta_\epsilon(\Phi(Z) - \Phi(X))dXdZ,$$

where $\beta_\epsilon(r) := \sqrt{\epsilon^2 + r^2}$, $r \in \mathbf{R}$. Observe that β_ϵ is smooth, β_ϵ' is an odd function, and $|\beta_\epsilon'| \leq 1$. Both ϕ_ϵ and ψ_ϵ are convex, lower semicontinuous, and proper on $\mathcal{H}$. Let $\Phi_{\epsilon 0} \in W^{1,2}(\Omega)$ be such that $0 \leq \Phi_{\epsilon 0} \leq 1$, $\Phi_{\epsilon 0} \to \Phi_0$ in $\mathcal{H}$, $\int_\Omega \| \nabla\Phi_{\epsilon 0} \| \to \| \nabla\Phi_0 \|$, and $\epsilon\int_\Omega \| \nabla\Phi_{\epsilon 0} \|^2 \to 0$ as $\epsilon \to 0$. Hence $\phi_\epsilon(\Phi_{\epsilon 0}) \to \phi(\Phi_0)$ as $\epsilon \to 0$ (such sequence exists from Lemma 3.1 in [7]). Since ϕ_ϵ is convex, lower semicontinuous, and proper, $\partial\phi_\epsilon$ is a maximal monotone operator on $\mathcal{H}$ ([4], 2.3.4). On the other hand, letting

$$B_\epsilon := \psi_\epsilon'(\Phi) = -\frac{1}{2}\int_\Omega \beta_\epsilon'(\Phi(Z) - \Phi(X))dZ, \quad \Phi \in \mathcal{H}$$

$B_\epsilon : \mathcal{H} \to \mathcal{H}$ satisfies

$$\| B_\epsilon(\Phi) - B_\epsilon(\hat{\Phi}) \|_2 \leq \| \beta_\epsilon'' \|_\infty \| \Phi - \hat{\Phi} \|_2, \quad \Phi, \hat{\Phi} \in \mathcal{H}.$$

That is B_ϵ is a Lipschitz operator on $\mathcal{H}$. Hence, $\partial\psi_\epsilon + B_\epsilon$ generates a strongly continuous semigroup on $\mathcal{H}$ ([4], Proposition 3.12). Therefore, there is a strong solution $\Phi_\epsilon \in \mathcal{C}([0,T],\mathcal{H})$ of

$$\Phi_t + \partial\phi_\epsilon(\Phi) + B_\epsilon(\Phi) = 0$$

such that $\Phi_\epsilon(0) = \Phi_{\epsilon 0}$. Writing $f_\epsilon(t) = -B_\epsilon(\Phi_\epsilon(t)) = \psi_\epsilon'(\Phi_\epsilon(t))$, $|f_\epsilon| \leq \frac{1}{2}$. Therefore $f_\epsilon \in \mathrm{L}^1([0,T),\mathcal{H})$ and we see that Φ_ϵ is a strong solution of

$$\Phi_t + \partial\phi_\epsilon(\Phi) = f_\epsilon(t) \tag{26}$$

with $\Phi_\epsilon(0) = \Psi_{\epsilon 0}$. Since $\Phi_{\epsilon 0} \in \mathrm{dom}(\phi_\epsilon)$, using the regularization properties of semigroups generated by subdifferentials ([4], Theorem 3.6), we have the following estimates: $\Phi_\epsilon \in W^{1,2}([0,T],\mathcal{H})$, $t \to \phi_\epsilon(\Phi_\epsilon(t))$ is absolutely continuous on $[0,T]$ and

$$\int_\Omega |\Phi_{\epsilon t}|^2 + \frac{d}{dt}\phi_\epsilon(\Phi_\epsilon) = < f_\epsilon, \Phi_{\epsilon t} > = \frac{d}{dt}\psi_\epsilon(\Phi_\epsilon). \tag{27}$$

In particular, from (27)

$$\int_\Omega |\Phi_{\epsilon t}|^2 + \frac{d}{dt}\phi_\epsilon(\Phi_\epsilon) \leq \| f_\epsilon \|_2 \| \Phi_{\epsilon t} \|_2 \leq \frac{(\| f_\epsilon \|_2)^2 + (\| \Phi_{\epsilon t} \|_2)^2}{2}. \tag{28}$$

Integrating the equation above from 0 to T we obtain

$$\frac{1}{2}\int_0^T \int_\Omega |\Phi_{\epsilon t}|^2 dX\, dt + \phi_\epsilon(\Phi_\epsilon) \leq \phi_\epsilon(\Phi_{\epsilon 0}) + \frac{T}{8}. \tag{29}$$

We have the following bounds independent of ϵ:

$$\int_\Omega \| \nabla \Phi_\epsilon \| \leq M \ , \ \int_\Omega \| \Phi_\epsilon \|^2 \leq M \ , \ \int_\Omega \| \Phi_{\epsilon t} \|^2 \leq M, \tag{30}$$

$\forall t > 0$, $\forall \epsilon > 0$, and some $M > 0$. The bounds above mean that $\Phi_\epsilon \in W^{1,2}([0,T], \mathcal{H}) \cap \mathrm{L}^\infty([0,T], \mathrm{BV}(\Omega))$ for any $T > 0$.

2nd Step. Uniform bounds on Φ_ϵ:

To get the bounds $0 \leq \Phi_\epsilon \leq 1$, let us observe that since ϕ_ϵ is strictly convex and ψ_ϵ is smooth, Φ_ϵ is the classical solution of the corresponding PDE

$$\Phi_t = \mathrm{div}\left(\epsilon \nabla \Phi + \frac{\nabla \Phi}{\sqrt{\epsilon^2 + \| \nabla \Phi \|^2}}\right) - \left(\Phi - \frac{1}{2}\right) - \frac{1}{2}\int_\Omega \beta'_\epsilon(\Phi(t,Z) - \Phi(t,X))dZ. \tag{31}$$

It will be important to re-write (31) as

$$\Phi_t = \mathrm{div}\left(\epsilon \nabla \Phi + \frac{\nabla \Phi}{\sqrt{\epsilon^2 + \| \nabla \Phi \|^2}}\right) - \frac{1}{2}\int_\Omega (1 - \beta'_\epsilon(\Phi(t,Z) - \Phi(t,X))dZ - \Phi. \tag{32}$$

To prove that $\Phi_\epsilon \geq 0$, let $j \in \mathcal{C}^\infty(\mathbf{R})$, $j \geq 0$, $j' \leq 0$, $j'' \geq 0$, $j(X) = 0\ \forall X \geq 0$, and $0 < j(X) \leq |X|\ \forall X < 0$. Then

$$\frac{d}{dt}\int_\Omega j(\Phi_\epsilon) = \int_\Omega j'(\Phi_\epsilon)\Phi_{\epsilon t}. \tag{33}$$

Based on (32), and integrating by parts, we obtain that the expression above is equal to

$$\begin{aligned} - & \int_\Omega j''(\Phi_\epsilon)\nabla\Phi_\epsilon \left[\epsilon \nabla \Phi_\epsilon + \frac{\nabla \Phi_\epsilon}{\sqrt{\epsilon^2 + \| \nabla \Phi_\epsilon \|^2}}\right] dX \\ + & \frac{1}{2}\int_\Omega j'(\Phi_\epsilon(t,X)) \int_\Omega (1 - \beta'_\epsilon(\Phi_\epsilon(t,Z) - \Phi_\epsilon(t,X))dZ dX - \int_\Omega j'(\Phi_\epsilon)\Phi_\epsilon. \end{aligned}$$

The first term is negative by definition. Since $|\beta'_\epsilon| \leq 1$, and $j' \leq 0$, so is the second term. Since $j' \leq 0$ and $j(X) = 0$ for $x \geq 0$, the third term is negative as well. Hence, integrating (33) from 0 to T

$$\int_\Omega j(\Phi_\epsilon(T,X)dX \leq \int_\Omega j(\Phi_\epsilon(0,X))dX.$$

In particular, if $\Phi_\epsilon(0) \geq 0$, $j(\Phi_\epsilon) = 0$. Hence $\Phi_\epsilon \geq 0$.

To prove that $\Phi_\epsilon \leq 1$, let us compute for $p > 2$

$$\frac{d}{dt}\int_\Omega \Phi_\epsilon(t,X)^p dX = p\int_\Omega \Phi_\epsilon^{p-1}\Phi_{\epsilon t} dX.$$

Using (32) and integrating by parts, this expression is equal to

$$\begin{aligned} - \quad & p(p-1)\int_\Omega \Phi_\epsilon(t,X)^{p-2}\nabla\Phi_\epsilon \left[\epsilon\nabla\Phi_\epsilon + \frac{\nabla\Phi_\epsilon}{\sqrt{\epsilon^2 + \| \nabla\Phi_\epsilon \|^2}}\right] dX \\ + \quad & \frac{1}{2}p\int_\Omega \Phi_\epsilon(t,X)^{p-1}\int_\Omega (1 - b'_\epsilon(\Phi_\epsilon(t,Z) - \Phi_\epsilon(t,X)) dZ dX - p\int_\Omega \Phi_\epsilon(t,X)^p dX. \end{aligned}$$

It is clear that the first term is negative. The second term can be majorized using $(1-\beta'_\epsilon) \leq 2$. Hence

$$\begin{aligned} \frac{d}{dt}\int_\Omega \Phi_\epsilon(t,X)^p dX &\leq p\int_\Omega \Phi_\epsilon(t,X)^{p-1} dX - p\int_\Omega \Phi_\epsilon(t,X)^p dX \\ &\leq p\left(\int_\Omega \Phi_\epsilon(t,X)^p dX\right)^{\frac{p-1}{p}} - p\int_\Omega \Phi_\epsilon(t,X)^p dX, \end{aligned}$$

where Hölder inequality has been used. Defining $\mathcal{A}_{\epsilon p}(t) := (\int_\Omega \Phi_\epsilon(t,X)^p dX)^{1/p}$, we have

$$\frac{d}{dt}\mathcal{A}_{\epsilon p}(t)^p \leq p\mathcal{A}_{\epsilon p}(t)^{p-1} - p\mathcal{A}_{\epsilon p}(t)^p.$$

It follows that, where $\mathcal{A}_{\epsilon p}(t) > 0$,

$$\frac{d}{dt}\mathcal{A}_{\epsilon p}(t) \leq 1 - \mathcal{A}_{\epsilon p}(t),$$

$$\mathcal{A}_{\epsilon p}(t) \leq 1 + \exp\{-t\}(\mathcal{A}_{\epsilon p}(0) - 1).$$

If $p \to \infty$, $\mathcal{A}_{\epsilon p}(t) \to \| \Phi_\epsilon(t) \|_\infty$, $\mathcal{A}_{\epsilon p}(0) \to \| \Phi_\epsilon(0) \|_\infty$, we have

$$\| \Phi_\epsilon(t) \|_\infty \leq 1 + \exp\{-t\}(\| \Phi_\epsilon(0) \|_\infty - 1) \leq 1.$$

Hence

$$0 \leq \Phi_\epsilon \leq 1, \quad \text{a.e. } \epsilon > 0 \tag{34}$$

3rd Step. Letting $\epsilon \to 0$:

From (30), we know that there exists a sequence Φ_ϵ such that $\Phi_\epsilon \to \Phi$ in $\mathrm{L}^1_{loc}([0,\infty[\times\bar{\Omega})$. From (34) we also have that $\Phi_\epsilon \to \Phi$ in $\mathrm{L}^p_{loc}([0,\infty[\times\bar{\Omega})$, $\forall p < \infty$, and $\Phi_\epsilon \to \Phi$ weak-* in $\mathrm{L}^\infty_{loc}([0,\infty[\times\bar{\Omega})$. Therefore,

$$\Phi \in W^{1,2}_{loc}([0,\infty[,\mathcal{H}) \cap \mathrm{L}^\infty_{loc}([0,\infty[,\mathrm{BV}(\Omega)).$$

(The subindex "loc" could be omitted as we will see below). We also have $0 \leq \Phi \leq 1$ almost everywhere. Now, since Φ_ϵ is a strong solution of (27)

$$\phi_\epsilon(\hat{\Phi}) - \phi_\epsilon(\Phi_\epsilon) \geq < \Phi_{\epsilon t} - \psi'_\epsilon(\Phi_\epsilon), \Phi_\epsilon - \hat{\Phi} >, \quad \forall \hat{\Phi} \in \mathcal{H}. \tag{35}$$

Let $\mu_\epsilon(t, Z, X) := \Phi_\epsilon(t, Z) - \Phi_\epsilon(t, X)$. Since $\mu_\epsilon(t, Z, X) \to \Phi(t, Z) - \Phi(t, X)$ in $\mathrm{L}^1_{loc}([0, \infty[\times \bar{\Omega})$, there exists a subsequence (call it again $\mu_\epsilon(t, Z, X)$) such that $\beta'_\epsilon(\mu_\epsilon)$ converges to $s(t, X, Z) \in \mathrm{sign}(\Phi(t, Z) - \Phi(t, X))$ weakly-$*$ in $\mathrm{L}^\infty([0, T] \times \bar{\Omega})$ for all $T > 0$. Let $\omega := \int_\Omega s(t, Z, X) dZ$. We have

$$< \psi'_\epsilon(\Phi_\epsilon), \Phi_\epsilon > \to < \omega, \Phi > \quad , \quad < \psi'_\epsilon(\Phi_\epsilon), \hat{\Phi} > \to < \omega, \hat{\Phi} >$$

weakly-$*$ in $\mathrm{L}^\infty([0, T])$ for all $T > 0$. Therefore, converges also weakly in $\mathrm{L}^1([0, T])$ for all $T > 0$. On the other hand, since $\Phi_\epsilon \to \Phi$ in $\mathrm{L}^2([0, T], \mathcal{H})$,

$$< \Phi_{\epsilon t}, \Phi_\epsilon > = -\frac{1}{2}\frac{d}{dt} < \Phi_\epsilon, \Phi_\epsilon > \to \frac{1}{2}\frac{d}{dt} < \Phi, \Phi >$$

weakly in $\mathrm{L}^1([0, T])$ for all $T > 0$. Similarly, $< \Phi_{\epsilon t}, \hat{\Phi} > \to < \Phi_t, \hat{\Phi} >$ weakly in $\mathrm{L}^1([0, T])$ for all $T > 0$. Finally, since $\phi(\Phi(t)) \leq \lim_i \inf \phi_\epsilon(\Phi_\epsilon)$, $\phi(\hat{\Phi}) = \lim_\epsilon \phi_\epsilon(\hat{\Phi})$ for all $\hat{\Phi} \in W^{1,2}(\Omega)$, letting $\epsilon \to 0$ in (35) we obtain

$$\phi(\hat{\Phi}) - \phi(\Phi(t)) \geq < \Phi_t - \omega, \Phi - \hat{\Phi} >, \quad \forall \hat{\Phi} \in W^{1.2}(\Omega)$$

Now it is straightforward to see that

$$\phi(\hat{\Phi}(t)) - \phi(\Phi) \geq < \Phi_t - \omega, \Phi - \hat{\Phi} >, \quad \forall \hat{\Phi} \in \mathcal{H}.$$

That is $\Phi(t) \in \mathrm{dom}(\partial\phi)$ and

$$-\Phi_t + \omega(t) \in \partial\phi(\Phi(t)) \quad a.e.$$

To justify the last assertion of Theorem 1, let us observe from (27)

$$\frac{d}{dt}(\phi_\epsilon(\Phi_\epsilon) - \psi_\epsilon(\Phi_\epsilon)) \leq 0,$$

that is

$$\phi_\epsilon(\Phi_\epsilon(t)) - \psi_\epsilon(\Phi_\epsilon(t)) \leq \phi_\epsilon(\Phi_\epsilon(0)) - \psi_\epsilon(\Phi_\epsilon(0)). \tag{36}$$

Since $\psi_\epsilon(\Phi_\epsilon(t)) \to \psi(\Phi(t))$, $\psi_\epsilon(\Phi_\epsilon(0)) \to \psi(\Phi(0))$, $\phi_\epsilon(\Phi_\epsilon(0)) \to \phi(\Phi(0))$, and ϕ is lower semicontinuous, we get ($\epsilon \to 0$)

$$\phi(\Phi(t)) - \psi(\Phi(t)) \leq \phi(\Phi(0)) - \psi(\Phi(0)) \;\; a.e.. \tag{37}$$

Since Φ is a continuous function of t, we may assume that (37) holds for all t. Hence, $\phi - \psi$ is a Lyapounov functional for the problem (22). $\Box$.

Note that the Theorem above proofs existence of the solution. There is no result so far related to uniqueness.

Before concluding this sections, let's make some remarks on the asymptotic behavior of Φ as $t \to \infty$. Integrating (10) we have

$$\int_0^T \int_\Omega |\Phi_{\epsilon t}|^2 dX dt = \phi_\epsilon(\Phi_\epsilon(t)) - \phi_\epsilon(\Phi_\epsilon(0)) + \psi_\epsilon(\Phi_\epsilon(t)) - \psi_\epsilon(\Phi_\epsilon(0)) \le \phi_\epsilon(\Phi_\epsilon(0) + \frac{1}{4}.$$

If $\epsilon \to 0$,

$$\int_0^T \int_\Omega |\Phi_t|^2 dX dt + \phi(\Phi(t)) \le \psi(\Phi(0) + \frac{1}{4}.$$

In particular, if $T \to \infty$

$$\int_0^\infty \int_\Omega |\Phi_t|^2 dX dt \le \psi(\Phi(0) + \frac{1}{4}.$$

Therefore, for a subsequence $t = t_n$ we have $\Phi_t(t_n) \to 0$ in $\mathcal{H}$ as $n \to \infty$. Since, on the other hand, $\phi(\Phi)$ is bounded, we may assume that $\Phi(t_n) \to \bar{\Phi}$ in L^1. Since $0 \le \Phi \le 1$, also $\Phi(t_n) \to \bar{\Phi}$ in L^2. Now, since $-\Phi_t + \omega \in \partial\phi(\Phi)$ a.e., we may assume that $-\Phi_t(t_n) + \omega(t_n) \in \partial\phi(\Phi(t_n))$ for all n. Hence

$$\phi(\hat{\Phi}) - \phi(\Phi(t_n)) \ge < -\Phi_t(t_n) + \omega(t_n), \Phi(t_n) - \hat{\Phi} >, \quad \forall \hat{\Phi} \in \mathcal{H}. \tag{38}$$

Moreover, we may assume that $\omega(t_n) \to \hat{\omega} \in \partial\psi(\hat{\Phi})$ weakly in $\mathcal{H}$. Letting $n \to \infty$ in the expression above, we get

$$\phi(\hat{\Phi}) - \phi(\bar{\Phi}) \ge < \bar{\omega}, \Phi - \hat{\Phi} > \quad \forall \hat{\Phi} \in \mathcal{H}.$$

That is $\hat{\omega} \in \partial\phi(\hat{\Phi})$ where $\hat{\omega} \in \partial\psi(\hat{\Phi})$ We may say that essentially all limit points of $\Phi(t)$ as $t \to \infty$ are critical points of $\phi(\Phi) - \psi(\Phi)$.

4 Experimental results

Before presenting experimental results, let's make some remarks on the complexity of the algorithm. Each iteration of (2) (or (17)) has $O(N^2)$ operations. In our examples we observed that no more than 5 iterations are usually required to converge. Therefore, the complexity of the proposed algorithm is $O(N^2)$, which is the minimal expected for any image processing procedure that operates on the whole image.

The first example is given in Figure 1. The original image is presented on the top. On the middle we present the image after histogram equalization performed using the popular software *xv*, [2] and in the bottom the one obtained from the steady state solution of (1). This is repeated in Figure 2. Note that since the original image in Figure 2 is just the one in Figure 1 but with the pixel values scaled by a constant factor, the result of running the

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PDE for this image is the same of for the one in Figure 1, as expected from an histogram modification algorithm.

Figure 3 presents an example of equation (2) for h being a linear function of the form in the top. The original image is the same as in Figure 1. In the bottom the modified image is presented.

Figure 4 gives an example of the result of tacking the original image in Figure 1 and performing histogram equalization for each block independently. In Figure 5 we compute $\mathcal{A}$ only in a 32×32 surrounding area of the pixel. Note that this kind of histogram modification is usually used in order to detect details in the image, and not for visual presentation. We present them here just as examples of the use of the PDE approach to compute variations of global histogram equalization. Other variations, as adaptive histograms [16], can be performed as well.

An example of the simultaneous denoising and histogram equalization is given in Figure 6 for an ultrasound image of the heart. On the top left the original image is given, and on the bottom the one obtained from our histogram equalization approach without smoothing. The result of the combined histogram modification and smoothing is given in the right column for two different values of α. This is repeated in Figure 7 for MRI. Note how the contrast is improved, while at the same time noise is removed without mayor edge distortion. This result is of great importance for segmentation algorithms as those proposed in [5, 6, 12]. Another example of this procedure is given in Figure 8 for a fingerprint image. [3]

Finally, in Figure 9 we present an example of combining local and global histogram modification. The original image is given in the top. The image in the bottom is obtained from 4 iterations of local histogram as in Figure 5, followed by 3 iterations of global histogram equalization.

5 Concluding remarks

In this paper, a novel partial differential equation for histogram modification was presented. The modified image is obtained as the steady state solution of a PDE, which uses the original image as initial condition. Existence and uniqueness results, and a variational interpretation of the approach, were presented as well. The histogram modification flow can be combined with previously developed flows for anisotropic diffusion, in order to obtain a single PDE which simultaneously performs contrast normalization and denoising. The algorithm was tested on a number of images, and proved to converge very fast.

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[3] The image is from the database of the National Institute of Standards and Technology.

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