Cycle Times and Fixed Points of Min-Max Functions

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The dynamic behavior of discrete event systems with only maximum timing constraints (or, dually, only minimum timing constraints) can be studied by linear methods based on max-plus algebra, as discussed in the recent book *"Synchronization and Linearity"* by Baccelli, Cohen, Quadrat and Olsder. Systems with both maximum and minimum constraints are non-linear from this perspective and only limited results about their behavior have so far been obtained (Olsder 1991, 1993; Gunawardena, 1993). In this paper we describe some new results and conjectures about such systems which generalize the initial earlier work and shed a new light on aspects of the linear theory. Our main result is a necessary and sufficient condition for the existence of a fixed point (eigenvector) for any min-max function. The work described here was done as part of project STETSON, a joint project between HP Labs and Stanford University on asynchronous hardware design.

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1 Introduction

It will be convenient to use the infix operators $a \vee b$ and $a \wedge b$ to stand for maximum (least upper bound) and minimum (greatest lower bound) respectively: $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Note that addition distributes over both maximum and minimum:

$$h + (a \vee b) = h + a \vee h + b, \qquad h + (a \wedge b) = h + a \wedge h + b. \tag{1}$$

Definition 1.1 *A min-max expression, f, is a term in the grammar:*

$$f := x_1, x_2, \cdots \mid f + a \mid f \wedge f \mid f \vee f$$

where $x_1, x_2, \cdots$ are variables and $a \in \mathbf{R}$.

For example, $x_1 + x_2 \wedge x_3 + 2$ and $x_1 \vee 2$ are forbidden but $x_1 - 1 \vee x_2 + 1$ is allowed. (In expressions such as these $+$ has higher binding than $\wedge$ or $\vee$.)

Definition 1.2 *A min-max function of dimension n is any function, $F : \mathbf{R}^n \to \mathbf{R}^n$, each of whose components, $F_i : \mathbf{R}^n \to \mathbf{R}$, is a min-max expression of n variables $x_1, \cdots, x_n$.*

The theory of min-max functions is concerned with studying F as a dynamical system: with studying the behaviour of the sequence $\vec{x}, F(\vec{x}), F^2(\vec{x}), \cdots$, for varying $\vec{x} \in \mathbf{R}^n$. For a discussion of the origins of this problem see [2, §1] and for information on applications see [3].

There are three existing sources of information about the behaviour of such functions. The earliest is the work on the linear theory of max-plus algebra, [1, Chapter 3]. This work applies to max-only functions: those in which $\wedge$ does not appear in any F_i. (Dually for min-plus algebra and min-only functions. Because $-(a \wedge b) = (-a \vee -b)$, dual results do not require a separate proof and we leave it to the reader to formulate them where necessary.) It is easy to see that any max-only function can be written in canonical form:

$$F_i(x_1, \cdots, x_n) = (a_{i1} + x_1 \vee \cdots \vee a_{in} + x_n),$$

where $a_{ij} \in \mathbf{R} \cup \{-\infty\}$, and that this expression is unique, [2, Lemma 2.1]. If $A = (a_{ij})$ is the corresponding matrix in max-plus algebra then, using max-plus matrix notation, $F(\vec{x}) = A\vec{x}$ (considering vectors as column vectors). Hence we can expect the theory of min-max functions to include the linear theory as a special case.

Let $\vec{c}(h) = (h, h, \cdots, h)$ denote the vector each of whose components has the same value h.

Definition 1.3 *F has a fixed point, $\vec{x} \in \mathbf{R}^n$, if, and only if, $F(\vec{x}) = \vec{x} + \vec{c}(h)$, for some $h \in \mathbf{R}$.*

If F is a max-only function and A is the corresponding matrix then, for a fixed point $\vec{x}$ of F, $A\vec{x} = h\vec{x}$. That is, $\vec{x}$ is an eigenvector of A with eigenvalue h. This number, h, will turn out to be part of the cycle time vector of F which we shall study in §2.

The second source of work on min-max functions is due to Olsder, who undertook in [6] the first investigation of a system with mixed constraints. He considered separated functions

$F : \mathbf{R}^n \to \mathbf{R}^n$ for which F_i is max-only for $1 \leq i \leq k$ and F_i is min-only for $k+1 \leq i \leq n$. Olsder gave a necessary and sufficient condition for the existence of a fixed point, provided that F satisfied certain reasonable conditions, [6, Theorem 2.1], [1, Theorem 9.25]. A more recent analysis of the same class of functions, [7], is less relevant to our purposes here.

The third and final work is [2]. This paper has two main results: a proof of eventual periodicity for min-max functions of dimension 2, [2, Theorem 3.3], and a formula for the cycle time of an arbitrary min-max function with a fixed point, [2, Theorem 5.1] (see (11) below).

The purpose of the present paper is to describe some new results and conjectures which provide a coherent explanation of this previous work. We believe that they capture some of the essential properties of min-max functions and provide a foundation for a deeper study of some of the remaining unsolved problems. They also shed a new light on aspects of the linear theory.

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2 The cycle time vector

We shall frequently use numerical operations and relations and apply them to vectors. These should always be assumed to be applied to each component separately. Hence $\vec{u} \leq \vec{v}$ means $u_i \leq v_i$ for each i. Similarly, $(\bigwedge_l \vec{a}_l)_i = \bigwedge_l (\vec{a}_l)_i$. We begin by recalling some simple properties of a min-max function F. First, F is continuous. Second, F is monotone:

$$\vec{u} \leq \vec{v} \implies F(\vec{u}) \leq F(\vec{v}), \tag{2}$$

Third, F is homogeneous, in the sense that, for any $h \in \mathbf{R}$,

$$F(\vec{u} + \vec{c}(h)) = F(\vec{u}) + \vec{c}(h), \tag{3}$$

This follows easily from (1), [2, Lemma 2.3]. The next property is not quite so obvious. Let $|\vec{u}|$ denote the maximum norm on vectors in $\mathbf{R}^n$: $|\vec{u}| = \bigvee_{1\leq i\leq n} |u_i|$, where $|u_i|$ is the usual absolute value on real numbers. We require a simple preliminary observation.

Lemma 2.1 *For any real numbers $a_i, b_i \in \mathbf{R}$ with $1 \leq i \leq n$,*

$$|(\bigvee_{1\leq i\leq n} a_i) - (\bigvee_{1\leq i\leq n} b_i)| \;\leq\; \bigvee_{1\leq i\leq n} |a_i - b_i|, \qquad |(\bigwedge_{1\leq i\leq n} a_i) - (\bigwedge_{1\leq i\leq n} b_i)| \;\leq\; \bigvee_{1\leq i\leq n} |a_i - b_i|.$$

Proof: Suppose that $a_k = \bigvee_{1\leq i\leq n} a_i$ and $b_j = \bigvee_{1\leq i\leq n} b_i$. If $a_k \leq b_j$ then $|a_k - b_j| \leq |a_j - b_j|$. If $b_j \leq a_k$ then $|a_k - b_j| \leq |a_k - b_k|$. Similarly for the second inequality.

QED

Lemma 2.2 *(Non-expansive property.) Let F be a min-max function of dimension n. If $\vec{u}, \vec{v} \in \mathbf{R}^n$ then $|F(\vec{u}) - F(\vec{v})| \leq |\vec{u} - \vec{v}|$.*

Proof: Assume that $\vec{u}$ and $\vec{v}$ are fixed and let $h = |\vec{u} - \vec{v}|$. It is sufficient to show that if f is any min-max expression of n variables, then $|f(\vec{u}) - f(\vec{v})| \leq h$. The proof of this is by induction on the structure of f. If $f \equiv x_k$, then $|f(\vec{u}) - f(\vec{v})| = |u_k - v_k| \leq h$ as required. Now suppose as an inductive hypothesis that the required result holds for f and g. It is obvious that it must then hold for $f + a$. By the preceding Lemma for $n = 2$ and the inductive hypothesis,

$$|(f \wedge g)(\vec{u}) - (f \wedge g)(\vec{v})| \leq |f(\vec{u}) - f(\vec{v})| \vee |g(\vec{u}) - g(\vec{v})| \leq h.$$

Similarly for $f \vee g$. The result follows by structural induction.

QED

F is not contractive. If it were, the Contraction Mapping Theorem, [4, Theorem 3.1.2], would imply that the dynamic behaviour of F was trivial. But suppose that at some point $\vec{x} \in \mathbf{R}^n$

$$\lim_{s \to \infty} \frac{F^s(\vec{x})}{s} \tag{4}$$

exists. (Note that this is a vector quantity in $\mathbf{R}^n$.) It then follows from Lemma 2.2 that this limit must exist at all points of $\mathbf{R}^n$ and must have the same value. In the applications of min-max functions, the state vector $\vec{x}$ is often interpreted as a vector of occurrence times of certain events and the vector $F(\vec{x})$ as the times of next occurrence. Hence the limit in (4) can be thought of as the vector of asymptotic average times to the next occurrence of the events:

$$\frac{F^s(\vec{x}) - F^{s-1}(\vec{x}) + \cdots + F(\vec{x}) - \vec{x}}{s} = \frac{F^s(\vec{x}) - \vec{x}}{s},$$

which tends to (4) as $s \to \infty$, [2, §1]. This motivates the following definition.

Definition 2.1 *Let F be a min-max function. If the limit (4) exists somewhere, it is called the cycle time vector of F and denoted by $\vec{\chi}(F) \in \mathbf{R}^n$.*

This brings us to our first problem. When does the cycle time vector exist? Suppose first that F is a max-only function of dimension n and that A is the associated $n \times n$ matrix in max-plus algebra. We recall that the precedence graph of A, [1, Definition 2.8], denoted $\mathcal{G}(A)$, is the directed graph with annotated edges which has nodes $\{1, 2, \cdots, n\}$ and an edge from j to i if, and only if, $A_{ij} \neq -\infty$. The annotation on this edge is then the real number A_{ij}. A path in this graph has the usual meaning of a sequence of directed edges and a circuit is a path which starts and ends at the same node, [1, §2.2]. The weight of a path p, $|p|_{\mathsf{w}}$, is the sum of the annotations on the edges in the path. The length of a path, $|p|_\ell$, is the number of edges in the path. If g is a circuit, the ratio $|g|_{\mathsf{w}}/|g|_\ell$ is the cycle mean of the circuit, [1, Definition 2.18].

Definition 2.2 *If A is an $n \times n$ matrix in max-plus algebra, let $\vec{\mu}(A) \in (\mathbf{R} \cup \{-\infty\})^n$ be the vector such that $\mu_i(A) = \bigvee\{\, |g|_{\mathsf{w}}/|g|_\ell \mid g$ a circuit in $\mathcal{G}(A)$ upstream from node $i\}$. Dually, if B is a matrix in min-plus algebra, then $\vec{\eta}(B)$ will denote the vector of minimum cycle means.*

A circuit is upstream from node i if there is a path in $\mathcal{G}(A)$ from some node on the circuit to node i. Because of the conventions of canonical form, [2, §2], if A is a matrix associated to a max-only function then any node in $\mathcal{G}(A)$ has at least one upstream circuit and so $\vec{\mu}(A) \in \mathbf{R}^n$.

Proposition 2.1 *Let F be a max-only function and A the associated matrix in max-plus algebra. The limit (4) always exists and the cycle time vector is given by $\vec{\chi}(F) = \vec{\mu}(A)$.*

Proof: Let $\mu_1(A) = h$. Suppose initially that $h = 0$ and consider the sequence of numbers $\alpha(s) = (A^s)_{1*}(0, \cdots, 0)$. We can interpret $\alpha(s)$ as the maximum weight among paths in $\mathcal{G}(A)$ which are of length s and which terminate at node 1, [1, §2.3.1]. If we consider any path terminating at node 1 then the only positive contribution to the weight of the path can come from those edges which are not repeated on the path: a repeated edge would be contained in a circuit, whose contribution to the path weight is at most 0. Since there are only finitely many edges, the weight of any path must be bounded above by $\sum_{A_{ij}>0} A_{ij}$. Hence $\alpha(s)$ is bounded above. We also know that there is some circuit upstream from node 1 whose weight is 0. Call this circuit g. For s sufficiently large, we can construct a path, $p(s)$, leading to node 1 whose starting point cycles round the circuit g. The weight of this path can only assume a finite set of values because $|g|_{\mathsf{w}} = 0$. Since $\alpha(s)$ is the path of maximum weight of length s, it follows that $\alpha(s) \geq |p(s)|_{\mathsf{w}}$ and so $\alpha(s)$ is also bounded below. We have shown that there exist $m, M \in \mathbf{R}$ such that, for all $s \geq 0$, $m \leq \alpha(s) \leq M$. It follows immediately that $\lim_{s\to\infty} \alpha(s)/s = 0$. Hence,

$$\lim_{s\to\infty} \frac{(F^s(0,\cdots,0))_1}{s} = \lim_{s\to\infty} \frac{(A^s)_{1*}(0,\cdots,0)}{s} = \lim_{s\to\infty} \frac{\alpha(s)}{s} = 0 = \mu_1(A).$$

If $h \neq 0$ then replace F by $G(\vec{x}) = F(\vec{x}) - \vec{c}(h)$. G is also a min-max function and it follows from (3) that B, the associated matrix, satisfies $B_{ij} = A_{ij} - h$. Hence $\mu_1(B) = 0$ and we can apply the argument above to show that $\lim_{s\to\infty}(G^s(0,\cdots,0))_1/s = 0$. But since $F = G + \vec{c}(h)$, it follows from (3) that $\lim_{s\to\infty} F^s(0,\cdots,0))_1/s = h = \mu_1(A)$. The same argument can be applied to any component of F and the result follows.

QED

This result associates certain real numbers to any max-only function. Examples show that these are not the same as the eigenvalues of A and the relationship between cycle times and eigenvalues has yet to be determined. In §3 we shall see that when F has a fixed point, $\vec{v}$, then there is a unique cycle time which coincides with the eigenvalue of the eigenvector $\vec{v}$.

We now return to the case of an arbitrary min-max function of dimension n. Each component of F can be placed in conjunctive normal form:

$$F_k(\vec{x}) = (A^k_{11} + x_1 \vee \cdots \vee A^k_{1n} + x_n) \wedge \cdots \wedge (A^k_{\ell(k)1} + x_1 \vee \cdots \vee A^k_{\ell(k)n} + x_n), \qquad (5)$$

where $A^k_{ij} \in \mathbf{R} \cup \{-\infty\}$, [2, §2]. Here $\ell(k)$ is the number of conjunctions in the component F_k. It is an important property of min-max functions that conjunctive normal form is unique up to re-ordering of the conjunctions, [2, Theorem 2.1]. We can now associate a max-plus matrix A to F by choosing, for the k-th row of the matrix, one of the $\ell(k)$ conjunctions in (5): $A_{kj} = A^k_{i_k j}$ where $1 \leq i_k \leq \ell(k)$ specifies which conjunction is chosen in row k.

Definition 2.3 *The matrix A constructed in this way is called a max-only projection of F. The set of all max-only projections is denoted* $\mathsf{P}(F)$. *Dually, the set of min-only projections of F,* $\mathsf{Q}(F)$, *is constructed from the disjunctive normal form of F.*

If A is any max-only projection of F, it is clear from the construction above that $F(\vec{x}) \leq A\vec{x}$ for any $\vec{x} \in \mathbf{R}^n$. It follows from (2) that $F^s(\vec{x}) \leq A^s\vec{x}$ for all $s \geq 0$. Now choose $\epsilon > 0$.

It then follows from Proposition 2.1 that, for all sufficiently large s, $F^s(\vec{x})/s \leq \vec{\mu}(A) + \vec{c}(\epsilon)$. Since this holds for any max-only projection, and there are only finitely many such, we see that $F^s(\vec{x})/s \leq (\bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A)) + \vec{c}(\epsilon)$ for all sufficiently large s. By a dual argument applied to the min-only projections of F, we can conclude that

$$(\bigvee_{B \in \mathsf{Q}(F)} \vec{\eta}(B)) - \vec{c}(\epsilon) \leq \frac{F^s(\vec{x})}{s} \leq (\bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A)) + \vec{c}(\epsilon). \tag{6}$$

for all sufficiently large s.

Conjecture 2.1 *(The Duality Conjecture.) Let F be any min-max function. Then,*

$$\bigvee_{B \in \mathsf{Q}(F)} \vec{\eta}(B) = \bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A).$$

The significance of this should be clear. It implies that any min-max function has a cycle time and gives us a formula for computing it. Of course (4) could still exist without the Conjecture being true. We should note in passing that since ϵ was arbitrary, (6) already tells us that

$$\bigvee_{B \in \mathsf{Q}(F)} \vec{\eta}(B) \leq \bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A). \tag{7}$$

What evidence is there to support the Duality Conjecture? It is easy to show that it holds for any max-only (dually, any min-only) function. In dimension 2 it follows from the geometric methods of [2] that (4) always exists. It can also be shown that the Duality Conjecture holds but this requires a complicated combinatorial argument (unpublished). Finally, we shall see in the next section that the Conjecture holds for any function with a fixed point.

3 The existence of fixed points

Lemma 3.1 *Let F be any min-max function of dimension n and let $\mathsf{P}(F) = \{A_1, \cdots, A_N\}$. The function $\{1, \cdots, n\} \times \{1, \cdots, N\} \to \mathbf{R}$ taking $(i, j) \to \mu_i(A_j)$ has a saddle point:*

$$\bigvee_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq N} \mu_i(A_j) = \bigwedge_{1 \leq j \leq N} \bigvee_{1 \leq i \leq n} \mu_i(A_j).$$

Proof: Let $\bigwedge_{1 \leq j \leq N} \mu_i(A_j) = h_i$ and suppose that the h_i are ordered so that $h_{i1} \geq h_{i2} \geq \cdots \geq h_{in}$ where $\{i1, \cdots, in\} = \{1, \cdots, n\}$. We shall exhibit a matrix $X \in \mathsf{P}(F)$ such that $\bigvee_{1 \leq i \leq n} \mu_i(X) = h_{i1}$. It then follows that

$$\bigwedge_{1 \leq j \leq N} \bigvee_{1 \leq i \leq n} \mu_i(A_j) \;\leq\; h_{i1} \;=\; \bigvee_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq N} \mu_i(A_j)$$

and since the reverse inequality always holds, [5, Lemma 1.3.1], the result will be established. We shall need some extra notation for sets A and B: $A \backslash B = \{i \in A \mid i \notin B\}$.

By construction, there is some matrix $A \in \mathsf{P}(F)$ such that $\mu_{i1}(A) = h_{i1}$. Let $S_1 \subseteq \{1, \cdots, n\}$ be those nodes in $\mathcal{G}(A)$ for which some path exists to $i1$. Note that $\{i1\} \subseteq S_1$. For each index $i \in S$, let $X_{i*} = A_{i*}$. Note that if X_{i*} is defined and $X_{ij} \neq -\infty$, the X_{j*} is also defined. Hence, no matter how we fill in the remaining rows of X, there can be no path in $\mathcal{G}(X)$ which starts at some node $j \notin S_1$ and terminates at some node $i \in S_1$. Furthermore any circuit in $\mathcal{G}(X)$ which lies entirely in S_1 must have cycle mean at most h_{i1}.

Now consider h_{i2}. As before, there is some matrix $A' \in \mathsf{P}(F)$ such that $\mu_{i2}(A') = h_{i2}$. Let $S' \subseteq \{1, \cdots, n\}$ be those nodes in $\mathcal{G}(A')$ for which some path exists to $i2$. Let $X_{i*} = (A')_{i*}$ for any index $i \in S' \backslash S_1$ and let $S_2 = S' \cup S$ be those indices for which X_{i*} is now defined. Note that $\{i1, i2\} \subseteq S_2$. S_2 has the same properties as S_1. If $i \in S_2$ and $X_{ij} \neq -\infty$ then $j \in S_2$. Furthermore, because of this same property for S_1, if we complete the remaining rows of X in any manner whatsoever, then any circuit in $\mathcal{G}(X)$ whose nodes lie entirely in S_2 must in fact lie entirely in S_1 or entirely in $S' \backslash S_1$. If the former, then the cycle mean is at most h_{i1}; if the latter, then the cycle mean is at most h_{i2}. In either case the cycle mean is at most h_{i1}.

We can proceed in this way to inductively build up a complete matrix $X \in \mathsf{P}(F)$. By construction, $\bigvee_{1 \leq i \leq n} \mu_i(X) = h_{i1}$. This completes the proof.

QED

If A is an $n \times n$ matrix in max-plus algebra let $\mu(A)$ denote the maximum cycle mean of A: $\mu(A) = \bigvee_{1 \leq i \leq n} \mu_i(A)$. Similarly, let $\eta(B)$ denote the minimum cycle mean of a min-plus matrix B. It follows from Lemma 3.1 and (7) that

$$\vec{c}\Big(\bigvee_{B \in \mathsf{Q}(F)} \eta(B) \Big) \leq \bigvee_{B \in \mathsf{Q}(F)} \vec{\eta}(B) \leq \bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A) \leq \vec{c}\Big(\bigwedge_{A \in \mathsf{P}(F)} \mu(A) \Big). \tag{8}$$

Now let F be any min-max function and suppose that $\vec{x}$ is a fixed point of F: $F(\vec{x}) = \vec{x} + \vec{c}(h)$. It follows from (3) that $F^s(\vec{x}) = \vec{x} + s\vec{c}(h)$. Hence the cycle time vector exists and is given by $\vec{\chi}(F) = \vec{c}(h)$. All the components of the cycle time vector are equal.

What can we say about the cycle means? It was shown in [2, Theorem 5.1] that when F has a fixed point, $\bigvee_{B \in \mathsf{Q}(F)} \eta(B) = \bigwedge_{A \in \mathsf{P}(F)} \mu(A)$ and if h denotes this common value then $\vec{\chi}(F) = \vec{c}(h)$. This should be considered as a generalization to min-max functions of the classical "Perron-Frobenius" theorem on the eigenvalues of an irreducible max-plus matrix, [1, Theorem 3.23]. It follows from (8) that in this case the Duality Conjecture is true for F. In fact we can distinguish three conditions which are implied by the existence of a fixed point:

$$\bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A) = \vec{c}(h), \tag{9}$$

$$\bigvee_{B \in \mathsf{Q}(F)} \vec{\eta}(B) = \vec{c}(h), \tag{10}$$

$$\bigvee_{B \in \mathsf{Q}(F)} \eta(B) = h = \bigwedge_{A \in \mathsf{P}(F)} \mu(A). \tag{11}$$

Because of (8), (11) implies both (9) and (10). If the Duality Conjecture holds then Lemma 3.1 shows that either (9) or (10) separately imply (11) and all the conditions are equivalent.

The remainder of this section is concerned with determining when the necessary conditions above are also sufficient. As usual, we first consider the simple case.

Proposition 3.1 *Let F be a max-only function and A the associated matrix in max-plus algebra. F has a fixed point if, and only if, $\vec{\mu}(A) = \vec{c}(h)$, for some $h \in \mathbf{R}$.*

Proof: If F has a fixed point then the result is just (9). So suppose that $\vec{\mu}(A) = \vec{c}(h)$. By using the same trick as in Proposition 2.1 we may assume that $h = 0$. Let i be any node in $\mathcal{G}(A)$. There is some circuit upstream from i which has the maximum weight of 0. Let k be a node on this circuit. The vector $v(i) = A^+_{*k} = \bigvee_{s \geq 1} A^s_{*k}$ can be shown to be an eigenvector of A with eigenvalue 0, [2, Lemma 4.3] (see also [1, §3.7.2]). However, $v(i) \notin \mathbf{R}^n$ in general. For instance, if there is no path from k to j, then $A^+_{jk} = -\infty$. But note that $A^+_{ik} \neq -\infty$ since, by construction, there is a path from k to i. We can carry out this construction for each node i because $\mu_i(A) = 0$ by hypothesis. Hence we can find vectors $v(1), \cdots, v(n)$ such that $v(i)_i \neq -\infty$ and $Av(i) = v(i)$. But then, by max-plus linearity, $v = \bigvee_i v(i)$ is clearly an eigenvector: $Av = v$. Moreover, $v_i \neq -\infty$ for any i. Hence v is a fixed point of F. This completes the proof.

QED

This result has not appeared before in the max-plus literature, to the best of our knowledge.

Theorem 3.1 *Let F be any min-max function. F has a fixed point if, and only if,*

$$\bigvee_{B \in \mathsf{Q}(F)} \eta(B) = \bigwedge_{A \in \mathsf{P}(F)} \mu(A).$$

Proof: If F has a fixed point then the result is just (11) above. So suppose that (11) holds. As usual, we may assume that $h = 0$. Let $X \in \mathsf{P}(F)$ be any max-only projection for which $\mu(X) = h$. Then, by (8),

$$\vec{c}(h) \geq \vec{\mu}(X) \geq \bigwedge_{A \in \mathsf{P}(F)} \vec{\mu}(A) = \vec{c}(h).$$

It follows that $\vec{\mu}(X) = \vec{c}(h)$. Hence, by Proposition 3.1, X has a real eigenvector $\vec{v}$. By construction of the max-only projections, it follows that $F(\vec{v}) \leq X\vec{v} = \vec{v}$. (Recall that h was assumed to be 0.) Hence, by (2), $F^s(\vec{v})$ is a decreasing sequence. We can apply a similar argument to the min-only projections of B and find a vector $\vec{u}$ such that $\vec{u} \leq F(\vec{u})$. Again using (2), $F^s(\vec{u})$ is an increasing sequence. However, neither sequence can respectively decrease or increase too far because F is non-expansive. It is easy to see from Lemma 2.2 that $F^s(\vec{v})$ must be bounded below and $F^s(\vec{u})$ must be bounded above. But a bounded monotonic sequence must converge. Hence there exist vectors $\vec{u}_\infty$ and $\vec{v}_\infty$, not necessarily distinct, such that $\lim_{s \to \infty} F^s(\vec{v}) = \vec{v}_\infty$ and $\lim_{s \to \infty} F^s(\vec{u}) = \vec{u}_\infty$. Since F is continuous it follows that $F(\vec{v}_\infty) = \vec{v}_\infty$ and $F(\vec{u}_\infty) = \vec{u}_\infty$. This completes the proof.

QED

Theorem 3.1 is relatively weaker than Proposition 3.1 because the former uses (11) while the latter uses (9). (11) is inconvenient in practice because it requires information on both $\mathsf{P}(F)$ and $\mathsf{Q}(F)$. For example, if the Duality Conjecture were true then it could be shown that the conditions of Olsder's fixed point theorem for separated functions, [6, Theorem 2.1], imply (9) and we could deduce Olsder's theorem as a corollary of Theorem 3.1. In the absence of the Conjecture, this appears difficult. As another example suppose that F has a periodic point, $\vec{x}$, where $F^k(\vec{x}) = \vec{x} + \vec{c}(h)$, [2, Definition 2.3]. It follows easily from (3) that the cycle time vector exists and $\vec{\chi}(F) = \vec{c}(h/k)$. If the Duality Conjecture were true we could deduce the strong result that a min-max function has a periodic point if, and only if, it has a fixed point. This, and stronger results, are already known to hold in dimension 2, [2, Theorem 3.3].

4 Conclusion

The main contribution of this paper is the identification of the Duality Conjecture and the demonstration of its significance for the deeper study of min-max functions.

References

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Note to the reader

Copies of [2, 3] are available by ftp from hplose.hpl.hp.com, IP address 15.254.100.100. Use "anonymous" as user name and refer to "/pub/jhcg/README" for more information.