A Non-Markovian version of Pitman's 2M – X Theorem

B.M. Hambly[1], James Martin[2], Neil O'Connell
Basic Research Institute in the Mathematical Sciences
HP Laboratories Bristol
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Pitman's representation theorem, telegrapher's process

Let (ξ_k, $k \geq 0$) be a Markov chain on {–1, +1} with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 \mid \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 \mid \xi_k = -1) = b < a$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. We prove that the process $2M - X$ has the same law as that of X conditioned to stay non-negative.

Pitman's representation theorem [18] states that, if (X_t, $t \geq 0$) is a standard Brownian motion and $M_t = \max_{s \leq t} X_s$, then $2M - X$ has the same law as the 3-dimensional Bessell process. This was extended in [19] to the case of non-zero drift, where it is shown that, if X_t is a standard Brownian motion with drift, then $2M - X$ is a certain diffusion process. This diffusion has the significant property that it can be interpreted as the law of X conditioned to stay positive (in an appropriate sense). Pitman's theorem has the following discrete analogue [18, 15] : if X is a simple random walk with non-negative drift (in continuous or discrete time) then $2M - X$ has the same law as X conditioned to stay non-negative (for the symmetric random walk this conditioning is in the sense of Doob). Here we present a non-Markovian version of Pitman's theorem. Let (ξ_k, $k \geq 0$) be a Markov chain on {–1, +1} with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 \mid \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 \mid \xi_k = -1) = b$. We will assume that $1 > a > b > 0$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$.

* Internal Accession Date Only Approved for External Publication
[1] St. Annes College, University of Oxford, Oxford, UK
[2] Statistical Laboratory, Cambridge University, Cambridge, UK

A NON-MARKOVIAN VERSION OF PITMAN'S $2M - X$ THEOREM

B.M. Hambly[1], James Martin[2] and Neil O'Connell[3]

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Abstract

Let $(\xi_k,\ k \geq 0)$ be a Markov chain on $\{-1,+1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 |\ \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 |\ \xi_k = -1) = b < a$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$. We prove that the process $2M - X$ has the same law as that of X conditioned to stay non-negative.

Pitman's representation theorem [18] states that, if $(X_t,\ t \geq 0)$ is a standard Brownian motion and $M_t = \max_{s \leq t} X_s$, then $2M - X$ has the same law as the 3-dimensional Bessel process. This was extended in [19] to the case of non-zero drift, where it is shown that, if X_t is a standard Brownian motion with drift, then $2M - X$ is a certain diffusion process. This diffusion has the significant property that it can be interpreted as the law of X conditioned to stay positive (in an appropriate sense). Pitman's theorem has the following discrete analogue [18, 15]: if X is a simple random walk with non-negative drift (in continuous or discrete time) then $2M - X$ has the same law as X conditioned to stay non-negative (for the symmetric random walk this conditioning is in the sense of Doob).

Here we present a non-Markovian version of Pitman's theorem. Let $(\xi_k,\ k \geq 0)$ be a Markov chain on $\{-1,+1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1 |\ \xi_k = 1) = a$ and $P(\xi_{k+1} = -1 |\ \xi_k = -1) = b$. We will assume that $1 > a > b > 0$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0 \leq k \leq n} X_k$.

Theorem 1 *The process $2M - X$ has the same law as that of X conditioned to stay non-negative.*

Note that, if $b = 1 - a$, then X is a simple random walk with drift and we recover the original statement of Pitman's theorem in discrete time.

To prove Theorem 1, we first consider a two-sided stationary version of ξ, which we denote by $(\eta_k,\ k \in \mathbb{Z})$, and define a stationary process $\{Q_n, n \in \mathbb{Z}\}$ by

$$Q_n = \max_{m \leq n} \left(-\sum_{j=m}^{n} \eta_j \right)^+ .$$

[1]Oxford University
[2]Cambridge University
[3]BRIMS, HP Labs, Bristol BS34 8QZ, UK.

Note that Q satisfies the Lindley recursion $Q_{n+1} = (Q_n - \eta_{n+1})^+$, and we have the following queueing interpretation. The number of customers in the queue at time n is Q_n; if $\eta_{n+1} = -1$ a new customer arrives at queue and $Q_{n+1} = Q_n + 1$; if $\eta_{n+1} = 1$ and $Q_n > 0$, a customer departs from the queue and $Q_{n+1} = Q_n - 1$; otherwise $Q_{n+1} = Q_n$.

Note that the process η can be recovered from Q, as follows:

$$\eta_n = \begin{cases} -1 & \text{if } Q_n > Q_{n-1} \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

For $n \in \mathbb{Z}$, set $\bar{Q}_n = Q_{-n}$.

Theorem 2 *The processes Q and $\bar{Q}$ have the same law.*

Proof: We first note that it suffices to consider a single excursion of the process Q from zero. This follows from the fact that, at the beginning and end of a single excursion, the values of η are determined, and so these act as regeneration points for the process. To see that the law of a single excursion is reversible, note that the probability of a particular excursion path depends only on the numbers of transitions (in the underlying Markov chain η) of each type which occur within that excursion path, and these numbers are invariant under time-reversal. □

Thus, if we define, for $n \in \mathbb{Z}$,

$$\hat{\eta}_n = \begin{cases} -1 & \text{if } Q_n > Q_{n+1} \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

we have the following corollary of Theorem 2.

Corollary 3 *The process $\hat{\eta}$ has the same law as η.*

Proof of Theorem 1: Note that we can write $\hat{\eta}_n = \eta_{n+1} + 2(Q_{n+1} - Q_n)$. Summing this, we obtain, for $n \geq 1$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = \tilde{X}_n + 2(Q_n - Q_0). \tag{3}$$

where $\tilde{X}_n = \sum_{j=1}^{n} \eta_j$. If we adopt the convention that empty sums are zero, and set $\tilde{X}_0 = 0$, then this formula remains valid for $n = 0$. It follows that, on $\{Q_0 = 0\}$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = 2\tilde{M}_n - \tilde{X}_n, \tag{4}$$

where $\tilde{M}_n = \max_{0 \le m \le n} \tilde{X}_m$.

Note also that, for $m \in \mathbb{Z}$,

$$Q_m = (Q_{m+1} - \hat{\eta}_m)^+ = \max_{n \ge m} \left(- \sum_{j=m}^{n} \hat{\eta}_j \right)^+ . \tag{5}$$

The law of X conditioned to stay non-negative is the same as the law of $\tilde{X}$ conditioned to stay non-negative, since the events $X_1 \ge 0$ and $\tilde{X}_1 \ge 0$ respectively require that $\xi_1 = 1$ and $\eta_1 = 1$, and so the difference in law between ξ and η becomes irrelevant. By Corollary 3, the law of $\tilde{X}$ conditioned to stay non-negative is the same as the law of the process

$$\left(\sum_{j=0}^{n-1} \hat{\eta}_j, \; n \ge 0 \right)$$

given that

$$Q_0 = \max_{n \ge 0} \left(- \sum_{j=0}^{n-1} \hat{\eta}_j \right) = 0.$$

By (4) this is the same as the law of $2\tilde{M} - \tilde{X}$ given that $Q_0 = 0$ or, equivalently, that $\eta_0 = 1$; but this is the same as the law of $2M - X$, so we are done. □

In the queueing interpretation, $\hat{\eta} = -1$ whenever there is a departure from the queue and $\hat{\eta} = 1$ otherwise. Thus, Corollary 3 states that the process of departures from the queue has the same law as the process of arrivals to the queue; it can therefore be regarded as a non-Markovian analogue of the celebrated theorem in queueing theory, due to Burke [3], which states that the output of a stable $M/M/1$ queue in equilibrium has the same law as the input (both are Poisson processes). Note, however, that in this non-Markovian queueing process, the arrivals and services are not independent (being mutually exclusive). Our proof of Theorem 2 is inspired by the kind of reversibility arguments used often in queueing theory, although usually in a Markovian setting. For general discussions on the role of reversibility in queueing theory, see [2, 10, 17]; the idea of using reversibility to prove Burke's theorem is originally due to Reich [16].

Finally, we remark that the following analogue of Theorem 1 holds in continuous time: let $(\xi_t, \; t \ge 0)$ be a continuous-time Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$, and set $X_t = \int_0^t \xi_s ds$, $M_t = \max_{0 \le s \le t} X_s$. We assume that the transition rates of the chain are such that event that X remains non-negative forever has positive probability. Then $2M - X$ has the same law as that of X conditioned to stay non-negative. The proof is identical to that of Theorem 1; in particular, the following analogues of Theorem 2 and Corollary 3 also hold: if we let $(\eta_t, \; t \in \mathbb{R})$ be a stationary version of ξ and, for $t \in \mathbb{R}$, set

$$Q_t = \max_{s \le t} \left(- \int_s^t \eta_s ds \right),$$

the $\bar{Q}$ (defined as $\bar{Q}_t = Q_{-t}$) has the same law as Q, and $\hat{\eta}$, defined by

$$\hat{\eta}_t = \begin{cases} -1 & \text{if } \eta_t = 1 \text{ and } Q_t > 0 \\ 1 & \text{otherwise,} \end{cases} \tag{6}$$

has the same law as η. The process X in this setting is sometimes called the *telegrapher's random process*, because it is connected with the telegrapher equation. It was introduced by Kac [9], where it is also shown to be related to the Dirac equation. There is a considerable literature on this process and its connections with relativistic quantum mechanics (see, for example, [4, 5] and references therein).

For other variants and multidimensional extensions of Pitman's theorem see [1, 7, 8, 11, 6, 12, 13, 14, 15] and references therein.

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