Large Deviations at Equilibrium For A Large Star-Shaped Loss Network

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large networks, stationary distribution, partition function, large deviations

We consider a symmetric network composed of N links, each with capacity C. Calls arrive according to a Poisson process and each call requires L distinct links (chosen at random). If each of these links has free capacity, the call is held for an exponential time; otherwise it is lost. The semi-explicit stationary distribution for this process is similar to a Gibbs measure: it involves a normalizing factor, the partition function, which is very difficult to evaluate. We consider the limit $N \to \infty$ with the offered arrival rate to a link fixed. We use asymptotic combinatorics and recent techniques involving the law of large numbers to obtain the logarithmic equivalent for the partition function, and deduce the large deviation principle for the empirical measure of the occupancies of the links. We give an explicit formula for the rate function and examine its properties.

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KEYWORDS: large networks, stationary distribution, partition function, large deviations.

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1 Introduction

We consider a large star-shaped network composed of links numbered 1 to N, each with capacity C. Calls arrive as a Poisson flow. Each call chooses a route, uniformly at random, in

$$\mathcal{R}^N = \{\text{subsets of } L \text{ distinct links among } 1, 2, \dots, N\} \tag{1.1}$$

and is lost if any link on the route is at capacity; otherwise it holds one channel on each of these links for an exponential time with parameter λ. The call arrival gives rise to

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L-body mean-field interaction, while the simultaneous release of L channels introduces strong interaction.

This may model many situations of simultaneous service, such as telecommunication or computer networks, locking of items in data-bases, parallel computing or job processing in factories; see Whitt [13], Ziedins and Kelly [14], Kelly [9], and Hunt [8].

A (huge) Markovian description of the network is given by $\mathbf{Y}^N = (Y_r^N)_{r\in\mathcal{R}^N}$, where $Y_r^N \in I\!D(\mathrm{I\!R}_+, \{0,1,\dots,C\})$ counts the number of ongoing calls on route r. The process

$$X_i^N = \sum_{r\in\mathcal{R}^N : i\in r} Y_r^N \tag{1.2}$$

counts the number of occupied channels on link i, and the simultaneous releases prevent $(X_i^N)_{1\le i\le N}$ from being Markovian. A relevant tractable quantity is the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N} \in \mathcal{P}(I\!D(\mathrm{I\!R}_+, \{0,1,\dots,C\}))$ and its flow of time-marginals

$$\bar{X}^N = (\bar{X}_t^N)_{t\ge 0} \in I\!D(\mathrm{I\!R}_+, \mathcal{P}), \quad \bar{X}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)} \tag{1.3}$$

where $\mathcal{P} = \mathcal{P}(\{0,1,\dots,C\})$ can be naturally identified with the C-dimensional simplex. This records the occupancies of the links, averaged over the network.

In this paper we obtain a large deviation principle for the stationary distribution of $\bar{X}^N$, as N goes to infinity. We keep the arrival rate seen by each link equal to ν, thus the arrival rate on any route r in $\mathcal{R}^N$ is

$$\nu_N = \nu \Big/ \binom{N-1}{L-1}. \tag{1.4}$$

The starting point is a description of the stationary distribution given in the paper of Ziedins and Kelly [14]. A key ingredient is to make use of the law of large numbers, which was obtained in this context by Whitt [13]. This gives us a convenient asymptotic representation for the partition function, which otherwise would have been difficult to evaluate. It also leads to an explicit formula for the rate function. We show that the rate function is strictly positive everywhere except at its minimum, and uniformly convex at this point. The rate function is not convex in general; we give a complete description of its curvature.

The idea of using laws of large numbers to obtain large deviation principles seems to be quite useful, and has been exploited to an even greater extent than here, for a variety of applications, in [10, 11, 12].

We remark that large deviations results have been obtained for a class of networks, which includes this model, under a different asymptotic regime, by Chang and Wang [1]. In this regime, the topology of the network remains fixed while the capacity and arrival rate go to infinity.

The outline of the paper is as follows. In Section 2, we present some preliminary material. In Section 3, we simplify the asymptotic evaluation of the partition function using Whitt's law of large numbers. In Section 4, we compute the necessary combinatorial asymptotics. We present the large deviation principle in Section 5, and a detailed analysis of the rate function in Section 6. In Section 7 we present some pictures.

2 Preliminaries

Some pathwise results

Whitt [13] gives a functional law of large numbers (LLN) on $\bar{X}^N$, given that $\bar{X}_0^N$ satisfies a LLN; he deduces a LLN for the stationary distribution. Graham and Méléard [3] prove propagation of chaos in total variation norm on path space for an initially empty network: there is a law Q on $\mathbb{D}([0,T],\{0,1,\dots,C\})$, defined by a tree construction and unique solution to a non-linear martingale problem, such that $|\mathcal{L}(X_1^N,\dots,X_k^N) - Q^{\otimes k}| \leq k^2 C(T)/N$. This implies the convergence in probability of μ^N to Q and of $\bar{X}^N = (\bar{X}_t^N)_{t\geq 0}$ to $(Q_t)_{t\geq 0}$ and can be extended to more general initial conditions satisfying a LLN. Graham and Méléard give a Gaussian fluctuation result for $\bar{X}^N$ in [4], and large deviation results (complete only for $C = 1$) in [5] and [6], for initial conditions satisfying proper asymptotics.

The limit flow $(Q_t)_{t\geq 0}$ satisfies the nonlinear ordinary differential equation (ODE)

(obtained by taking the expectation of the nonlinear martingale problem)

$$\begin{cases} \dot{Q}_t\{0\} &= -\nu(1-Q_t\{C\})^{L-1}Q_t\{0\} + \lambda Q_t\{1\} \\ \quad\dots & \\ \dot{Q}_t\{k\} &= \nu(1-Q_t\{C\})^{L-1}(Q_t\{k-1\} - Q_t\{k\}) + \lambda((k+1)Q_t(k+1) - kQ_t\{k\}) \\ \quad\dots & \\ \dot{Q}_t\{C\} &= \nu(1-Q_t\{C\})^{L-1}Q_t(C-1) - \lambda C Q_t\{C\} \end{cases} \quad (2.1)$$

on $\mathcal{P}$. We set $\rho = \nu/\lambda$ and $\rho_N = \nu_N/\lambda$. Any fixed point q_ρ of this ODE satisfies

$$kq_\rho\{k\} = \rho(1-q_\rho\{C\})^{L-1}q_\rho\{k-1\}\,, \quad k = 1,2,\dots,C \quad (2.2)$$

which is solved in terms of $q_\rho\{0\}$ as

$$q_\rho\{k\} = q_\rho\{0\}\frac{\left(\rho(1-q_\rho\{C\})^{L-1}\right)^k}{k!}\,, \; k = 0,1,\dots,C \quad (2.3)$$

and such a q_ρ is in $\mathcal{P}$ if and only if we have $q_\rho\{k\} \geq 0$ and

$$\sum_{k=0}^{C} q_\rho\{k\} = 1 \Leftrightarrow q_\rho\{0\} = \left(\sum_{k=0}^{C} \frac{\left(\rho(1-q_\rho\{C\})^{L-1}\right)^{k-C}}{k!}\right)^{-1}. \quad (2.4)$$

Thus q_ρ is a fixed point in $\mathcal{P}$ if and only if the blocking probability $q_\rho\{C\}$ satisfies

$$q_\rho\{C\} = \left(\sum_{k=0}^{C} \frac{\left(\rho(1-q_\rho\{C\})^{L-1}\right)^k}{k!}\right)^{-1} \frac{\left(\rho(1-q_\rho\{C\})^{L-1}\right)^C}{C!} \quad (2.5)$$

in $[0,1]$. This corresponds to the Erlang fixed point approximation, see Kelly [9] and Ziedins and Kelly [14]. The r.h.s. of (2.5) is a continuous decreasing function of $q_\rho\{C\}$ which is strictly positive for $q_\rho\{C\} = 0$ and tends to 0 as $q_\rho\{C\}$ tends to 1, hence there is a unique solution $q_\rho\{C\}$ which is in $]0,1[$. Thus there is a unique fixed point q_ρ to (2.1), determined by (2.3) and (2.5), and q_ρ belongs to $\mathcal{P}^o$. We obtain using (2.2)

$$\langle q_\rho \rangle = \sum_{k=0}^{C} kq_\rho\{k\} = \rho(1-q_\rho\{C\})^L \quad (2.6)$$

where we denote the first moment or mean occupancy of α in $\mathcal{P}$ by $\langle \alpha \rangle = \sum_{k=0}^{C} k\alpha\{k\}$.

The stationary distribution

The unique stationary distribution of $(Y_r^N)_{r\in\mathcal{R}^N}$ on

$$\mathcal{A}^N = \left\{\mathbf{m} = (m_r)_{r\in\mathcal{R}^N} : m_r \in \mathbb{N}\,, \sum_{r\in\mathcal{R}^N : i\in r} m_r \le C\,,\ \forall i \in \{1,\dots,N\}\right\}$$

is given, using the notation $|\mathbf{m}| = \sum_{r\in\mathcal{R}^N} m_r$, by

$$\tilde{\pi}^N(\mathbf{m}) = P_{st}(\mathbf{Y}_t^N = \mathbf{m}) = \frac{1}{Z_\rho^N}\prod_{r\in\mathcal{R}^N}\frac{\rho_N^{m_r}}{m_r!} = \frac{1}{Z_\rho^N}\frac{\rho^{|\mathbf{m}|}}{\binom{N-1}{L-1}^{|\mathbf{m}|}}\prod_{r\in\mathcal{R}^N}\frac{1}{m_r!}\,,\qquad \mathbf{m}\in\mathcal{A}^N \tag{2.7}$$

where the normalizing factor or partition function Z_ρ^N is given by

$$Z_\rho^N = \sum_{\mathbf{m}\in\mathcal{A}^N}\frac{\rho^{|\mathbf{m}|}}{\binom{N-1}{L-1}^{|\mathbf{m}|}}\prod_{r\in\mathcal{R}^N}\frac{1}{m_r!}\,. \tag{2.8}$$

Computing this factor is a NP-complete problem, and good approximations are needed; for further discussions and references, see Ziedins and Kelly [14].

This gives the distribution π^N on $\mathcal{P}$ of $(\bar{X}_t^N)_{t\ge 0}$ in equilibrium. There is a well-defined function $f^N : \mathcal{A}^N \mapsto \mathcal{P}$ such that $\bar{X}_t^N = f^N(\mathbf{Y}_t^N)$, see (1.2) and (1.3). We set

$$\mathcal{A}^N(\alpha) = \left\{\mathbf{m}\in\mathcal{A}^N \,:\, f^N(\mathbf{m}) = \alpha\right\},\qquad \alpha\in\mathcal{P}$$

and obtain

$$\pi^N(\alpha) = P_{st}(\bar{X}_t^N = \alpha) = \frac{1}{Z_\rho^N}\frac{\rho^{\frac{N\langle\alpha\rangle}{L}}}{\binom{N-1}{L-1}^{\frac{N\langle\alpha\rangle}{L}}}\sum_{\mathbf{m}\in\mathcal{A}^N(\alpha)}\prod_{r\in\mathcal{R}^N}\frac{1}{m_r!}\,,\qquad \alpha\in\mathcal{P} \tag{2.9}$$

where we use that (considering the total number of occupied links in the network)

$$\forall\mathbf{m}\in\mathcal{A}^N(\alpha)\,,\quad N\langle\alpha\rangle = L|\mathbf{m}| \le NC\,. \tag{2.10}$$

3 The law of large numbers and the partition function

Pathwise results do not extend directly to the stationary distributions: it is difficult to exchange the $N\to\infty$ and $t\to\infty$ limits. Whitt [13] achieves this by a compactness-uniqueness method, see Theorem 3 and Section III in [13] or Section 4 in Kelly [9] (with notation closer to ours). This gives the following law of large numbers, which can also be proved from direct computations on the partition function (see the extensions in Section 4 of Theorem 2.4 and Corollary 2.6 in Ziedins and Kelly [14].) For completeness we present a short proof.

Theorem 3.1 *We have equivalently, with q_ρ the fixed point (2.3),*

$$\lim_{N\to\infty} \pi^N = \delta_{q_\rho} \quad \textit{weakly}\,, \qquad \lim_{N\to\infty} \bar{X}_t^N = q_\rho \quad \textit{in probability at equilibrium.}$$

This convergence is uniform in t on bounded intervals.

Proof. Let a sequence of networks start at their stationary distributions. Classical criteria show that the sequence μ^N is tight in $\mathcal{P}(I\!D(\mathbb{R}_+, \{0,1,\ldots,C\}))$. Any limit point is in equilibrium (by consideration of the finite-dimensional marginals) and satisfies a limiting non-linear martingale problem with unique solution (by a simple limiting procedure). Taking the expectation yields that the limit points of $(\bar{X}_t^N)_{t\geq 0}$ satisfy the ODE (2.1), with unique fixed point (2.3). Hence the martingale problem starts at q_ρ, and by uniqueness we have convergence. Classical Ascoli estimates give relative compactness for the flow of marginals, hence the uniform convergence on compact sets (which is in fact implied by the convergence of μ^N). □

We now investigate the asymptotics of $\log Z_\rho^N$ for large N. We set

$$\sigma(N,\alpha) = \sum_{\mathbf{m}\in\mathcal{A}^N(\alpha)} \prod_{r\in\mathcal{R}^N} \frac{1}{m_r!} \tag{3.1}$$

and for B in $\mathcal{B}(\mathcal{P})$ we have following (2.9)

$$\log \pi^N(B) = \log \sum_{\alpha\in B} \frac{\rho^{\frac{N(\alpha)}{L}}}{\binom{N-1}{L-1}^{\frac{N(\alpha)}{L}}} \sigma(N,\alpha) - \log Z_\rho^N\,. \tag{3.2}$$

The sum has a support of cardinality less than $(N+1)^C$, since it is included in the set of $\alpha \in \mathcal{P}$ such that $N\alpha(0)$, $N\alpha(1)$, ..., $N\alpha(C)$ are integers. We bound this sum between its maximal term and its maximal term multiplied by $(N+1)^C$ and set

$$K(N,\alpha) = \langle\alpha\rangle\frac{L-1}{L}\log N - \frac{\langle\alpha\rangle}{L}\log(\rho(L-1)!) - \frac{1}{N}\log\sigma(N,\alpha) \tag{3.3}$$

and obtain

$$\log\pi^N(B) = -N\inf_{\alpha\in B} K(N,\alpha) - \log Z_\rho^N + O(\log N) \tag{3.4}$$

in which $\alpha \in B$ not in the support satisfy $\sigma(N,\alpha) = 0$ and hence $K(N,\alpha) = +\infty$.

Theorem 3.2 *For any neighborhood B of q_ρ given by (2.3)*

$$\log Z_\rho^N = -N\inf_{\alpha\in B} K(N,\alpha) + O(\log N) + o_B(1)\,.$$

Hence an asymptotic evaluation of $K(N,\alpha)$ continuous and non-zero at q_ρ yields an equivalent for $\log Z_\rho^N$, by considering open balls B shrinking to q_ρ.

Proof. Theorem 3.1 implies that for any neighborhood B of q_ρ, the left-hand side of (3.4) goes to 0 when N goes to infinity. Hence the result. □

4 Some asymptotic combinatorics

We restrict our attention at first to α in $\mathcal{P}^N$, where

$$\mathcal{P}^N = \{\alpha\in\mathcal{P} : N\alpha(0), N\alpha(1), \ldots, N\alpha(C), N\langle\alpha\rangle/L \in \mathbb{N}\}\,.$$

Lemma 4.1 *Let $d(N,\alpha)$ be the number of ways of setting up $N\langle\alpha\rangle/L$ distinguishable calls so that the resulting network occupancy $\mathbf{m}$ is in $\mathcal{A}^N(\alpha)$ (with $|\mathbf{m}| = N\langle\alpha\rangle/L$). Then*

$$d(N,\alpha) = \sum_{\mathbf{m}\in\mathcal{A}^N(\alpha)} \frac{|\mathbf{m}|!}{\prod_{r\in\mathcal{R}^N} m_r!} = |\mathbf{m}|!\,\sigma(N,\alpha)\,.$$

Proof. For each $\mathbf{m}\in\mathcal{A}^N(\alpha)$, the multinomial coefficient $|\mathbf{m}|!\,(\prod_{r\in\mathcal{R}^N} m_r!)^{-1}$ counts the number of ways of partitioning $|\mathbf{m}|$ distinguishable balls in successive subsets of size m_r

for r in $\mathcal{R}^N$ (which we order for this purpose). Each possible set-up of $|\mathbf{m}|$ calls giving occupancy $\mathbf{m}$ is clearly in one-to-one correspondence with such a partitioning. □

Proposition 4.2 *Let $w(N,\alpha)$ be the number of ways of dropping $|\mathbf{m}| = N\langle\alpha\rangle/L$ distinguishable groups of L distinguishable balls in N distinguishable boxes, so that the balls in each group fall in distinct boxes and that there are $N\alpha\{k\}$ boxes with k balls, $k = 0, 1, \ldots, C$. Then $w(N,\alpha) = L!^{|\mathbf{m}|}d(N,\alpha)$ and hence*

$$\sigma(N,\alpha) = w(N,\alpha)\left(\frac{N\langle\alpha\rangle}{L}!\,L!^{\frac{N\langle\alpha\rangle}{L}}\right)^{-1}.$$

Proof. Each box corresponds to a link, and each group of L balls to a call. There are $L!$ different ways to settle these L balls in a given subset of L boxes (corresponding to a given route), hence $w(N,\alpha) = L!^{|\mathbf{m}|}d(N,\alpha)$. We express $d(N,\alpha)$ using Lemma 4.1 to obtain the formula for $\sigma(N,\alpha)$. □

The computation of $w(N,\alpha)$ recalls the occupancy problem, see Feller [7] II-5.

Proposition 4.3 *Let $w_+(N,\alpha)$ be the number of ways of dropping $N\langle\alpha\rangle/L$ distinguishable groups of L distinguishable balls in N distinguishable boxes, so that there are $N\alpha\{k\}$ boxes with k balls, $k = 0, 1, \ldots, C$. This is simply the number of ways of dropping $N\langle\alpha\rangle$ balls in N boxes with the given occupancy of the boxes. For any such configuration, let $a(N,\alpha)$ be the number of permutations of the balls for which balls in a group do not fall in the same box. Then*

$$w_+(N,\alpha) = \frac{N!}{\prod_{k=0}^{C}(N\alpha\{k\})!}\frac{(N\langle\alpha\rangle)!}{\prod_{k=0}^{C}(k!)^{N\alpha\{k\}}} \geq w(N,\alpha) = \frac{a(N,\alpha)}{(N\langle\alpha\rangle)!}w_+(N,\alpha).$$

Proof. This formula expresses $w_+(N,\alpha)$ as the product of two multinomial coefficients, the first counting all possible partitions of the N boxes in subsets of $N\alpha\{k\}$ boxes which are to contain k balls, the second all possible possible partitions of the $N\langle\alpha\rangle$ balls in $N\alpha\{k\}$ subsets of k balls, $k = 0, 1, \ldots, C$, see Feller [7] II-5. More precisely, the second multinomial coefficient is the quotient of the number $(N\langle\alpha\rangle)!$ of permutations of the balls in a given configuration counted in $w_+(N,\alpha)$ by the number of such permutations which keep any ball in its original box, and hence give rise to the same global configuration. We can express $w(N,\alpha)$ similarly as $w_+(N,\alpha)$, only replacing $(N\langle\alpha\rangle)!$ by the number $a(N,\alpha)$ of permutations for which balls in a group do not fall in the same box. □

We wish to show that $w(N,\alpha)$ and $w_+(N,\alpha)$ are asymptotically close, with some uniformity on α. We first give a loose lower bound for $w(N,\alpha)$, then an appropriately tight and uniform lower bound on $w(N,\alpha)/w_+(N,\alpha)$.

Remark. For arbitrarily large N we may find α such that $w(N,\alpha) = 0$ and $w_+(N,\alpha) \geq 1$: for $L = 2$ and $C = 2$ we take $\alpha(2) = 1/N$ and $\alpha(0) = 1 - 1/N$.

Lemma 4.4 *Let α in $\mathcal{P}^N$ be such that $N\langle\alpha\rangle \geq CL$. Then*

$$w(N,\alpha) \geq \frac{N!}{\prod_{k=0}^{C}(N\alpha\{k\})!} L!^{\frac{N\langle\alpha\rangle}{L}} \geq 1\,.$$

Proof. The multinomial number counts the possible choices on the boxes, and $L!^{\frac{N\langle\alpha\rangle}{L}}$ counts the permutations of balls within each group (balls in a group are in distinct boxes, hence these permutations give distinct configurations). Thus, it is sufficient to prove that for $N\langle\alpha\rangle \geq CL$ there is at least one way to place the balls as in the definition of $w(N,\alpha)$, once we have fixed the $N\alpha\{k\}$ boxes which should hold k balls, for $k = 0, 1, \ldots, C$.

We fix these boxes, and call any box which should hold k balls "a box of type k". We now prove by induction on C that for $N\langle\alpha\rangle \geq CL$ there is at least one way to place k balls in every box of type k, so that balls in a group do not fall in the same box. This is obvious for $C = 1$. Let us assume it true for $C - 1 \geq 1$.

For $N\alpha(C) \geq L$, we place the balls by layers. We place balls successively in each box of type 1, then in each box of type 2, and so on until we place balls successively in each box of type C, thus completing the first layer. The boxes of type 1 now hold 1 ball each. We go back and place balls successively in each box of type 2, then in each box of type 3, and so on until we place balls successively in each box of type C, thus completing the second layer. The type 2 boxes now hold 2 ball each. We continue in a similar manner until there are no balls left. Since there are at least L boxes of type C, we never place balls from the same group in the same box, and we eventually fill up all the boxes appropriately.

For $N\alpha(C) \leq L-1$, we take a group of L balls and place one ball in each box of type C. There remains $L - N\alpha(C)$ balls in the group to place properly. There is a total of $N\langle\alpha\rangle - N\alpha(C) \geq CL - L + 1$ balls left to place; since the boxes are either of type k with $k \leq C-1$ or of type C and contain already one ball, there must be at least $(CL - L + 1)/(C-1) > L$ boxes which can each accept in the future at least one ball. So we can place the remaining $L - N\alpha(C)$ balls of the group in separate boxes distinct from the $N\alpha(C)$ ones already used. Then we are left with $N\langle\alpha\rangle - L \geq (C-1)L$ balls to place according to a configuration in which the maximal number of balls in a box is $C-1$, and by induction we know there is at least a way of doing so. □

Proposition 4.5 *We have for any α in $\mathcal{P}^N$*

$$0 \geq \log \frac{w(N,\alpha)}{w_+(N,\alpha)} \geq O(\log N)$$

with a $O(\log N)$ term uniform for $N \geq 2$ and α in $\mathcal{P}^N$ such that $N\langle\alpha\rangle \geq CL$.

Proof. The upper bound is obvious, see Proposition 4.3. We have $w(N,\delta_0) = w_+(N,\delta_0) = 1$. For $\alpha \neq \delta_0$ we consider N large enough so that $N\langle\alpha\rangle \geq CL$, and bound below $a(N,\alpha)$. The boxes are fixed, and the order of placement of the balls in the boxes is taken into account: we call "spot" the conjunction of a box and an order of placement in the box. After the $(j-1)$–th group of L balls has been placed, $(j-1)L$ spots have been occupied. The first ball in the j–th group can thus be placed in at least $(N\langle\alpha\rangle - (j-1)L)$ spots, the second in $(N\langle\alpha\rangle - (j-1)L - C)$ spots since the placement of the first ball in a box prevents the placement of the second ball in the at most C spots in the box, and so on, until the last ball in the group can be placed in at least $(N\langle\alpha\rangle - jL - (L-1)C) \geq 1$ spots since only the spots in $L-1$ boxes are forbidden. After thus placing groups of L balls for $j = 1, 2, \ldots, \frac{N\langle\alpha\rangle}{L} - C$, there are C groups left, and Lemma 4.4 applied to this restricted placement problem with fixed boxes states that there is at least $(L!)^C$ ways to do so. Thus

$$a(N,\alpha) \geq (L!)^C \prod_{j=1}^{\frac{N\langle\alpha\rangle}{L}-C} (N\langle\alpha\rangle-(j-1)L)(N\langle\alpha\rangle-(j-1)L-C)\cdots(N\langle\alpha\rangle-(j-1)L-(L-1)C)$$

and considering Proposition 4.3, we express $(N\langle\alpha\rangle)!$ as we did $a(N,\alpha)$ and use a simple bound to obtain

$$\frac{w(N,\alpha)}{w_+(N,\alpha)} = \frac{a(N,\alpha)}{(N\langle\alpha\rangle)!} \geq \frac{L!^C}{(LC)!} \prod_{j=1}^{\frac{N\langle\alpha\rangle}{L}-C} \left(1 - \frac{(L-1)(C-1)}{N\langle\alpha\rangle - jL + 1}\right)^{L-1}.$$

We take the logarithm. Classically

$$\sum_{j=1}^{\frac{N\langle\alpha\rangle}{L}-C} \log\left(1 - \frac{(L-1)(C-1)}{N\langle\alpha\rangle - jL+1}\right) \geq \int_{C+\frac{1}{L}-1}^{\frac{N\langle\alpha\rangle}{L}+\frac{1}{L}-1} \log\left(1 - \frac{(L-1)(C-1)}{xL}\right) dx$$

which we integrate to obtain

$$\begin{aligned}\log\frac{w(N,\alpha)}{w_+(N,\alpha)} \geq \frac{L-1}{L}\Big(& (N\langle\alpha\rangle + 1 - L)\log\left(1 - \frac{(L-1)(C-1)}{N\langle\alpha\rangle + 1 - L}\right) \\ & + (CL+1-L)\log(CL+1-L) \\ & - (L-1)(C-1)\log(N\langle\alpha\rangle + C - CL) - C\log C\Big) \\ & + C\log L! - \log(LC)! = O(\log N)\end{aligned}$$

with the uniformity we have stated. □

For α and β in $\mathcal{P}$ we define the entropy and relative entropy (or Kullback information)

$$H(\alpha) = -\sum_{k=0}^{C} \alpha\{k\}\log\alpha\{k\}, \quad H(\alpha\,|\,\beta) = \sum_{k=0}^{C} \alpha\{k\}\log\frac{\alpha\{k\}}{\beta\{k\}} \tag{4.1}$$

(with the conventions $0\log 0 = 0$, etc.) and the continuous function

$$K(\alpha) = -H(\alpha) + \sum_{k=0}^{C} \alpha\{k\}\log k! - \langle\alpha\rangle\frac{L-1}{L}(\log\langle\alpha\rangle - 1) - \frac{\langle\alpha\rangle}{L}\log\rho. \tag{4.2}$$

Theorem 4.6 *We have for any α in $\mathcal{P}^N$*

$$K(N,\alpha) = K(\alpha) + O\Big(\frac{\log N}{N}\Big)$$

see (3.3) and (4.2), with a $O(\frac{\log N}{N})$ remainder term uniformly bounded below, and bounded above uniformly for $N \geq 2$ and α in $\mathcal{P}^N$ such that $N\langle\alpha\rangle \geq CL$.

Proof. The statement for $\alpha = \delta_0$ is obvious. Else Propositions 4.2 and 4.3 give

$$\begin{aligned}\log\sigma(N,\alpha) &= \log N! - \sum_{k=0}^{C}\log\,(N\alpha\{k\})! + \log\,(N\langle\alpha\rangle)! - N\sum_{k=0}^{C}\alpha\{k\}\log k! \\ &\quad - \log\frac{N\langle\alpha\rangle}{L}! - \frac{N\langle\alpha\rangle}{L}\log L! + \log\frac{w(N,\alpha)}{w_+(N,\alpha)}\,.\end{aligned}$$

We recall the Stirling formula $\log n! = n(\log n - 1) + O(\log n)$, see Feller [7]. We have $\log 0! = 0(\log 0 - 0)$ and $\log 1! = 1(\log 1 - 1) + 1$. Since the function log is increasing and $\log 2 > 0$, we have $\log NC \geq \log N\langle\alpha\rangle > 0$ for $N\langle\alpha\rangle \geq 2$ and $\log N \geq \log N\alpha\{k\} > 0$ for $N\alpha\{k\} \geq 2$. We obtain using these results and Proposition 4.5 that

$$\begin{aligned}\log\sigma(N,\alpha) &= N(\log N - 1) - N\sum_{k=0}^{C}\alpha\{k\}(\log N\alpha\{k\} - 1) \\ &\quad + N\langle\alpha\rangle(\log N\langle\alpha\rangle - 1) - N\sum_{k=0}^{C}\alpha\{k\}\log k! \\ &\quad - \frac{N\langle\alpha\rangle}{L}\Big(\log\frac{N\langle\alpha\rangle}{L} - 1\Big) - \frac{N\langle\alpha\rangle}{L}\log L! + O(\log N)\end{aligned}$$

with an $O(\log N)$ remainder term uniformly bounded above, and bounded below uniformly for $N \geq 2$ and α in $\mathcal{P}^N$ such that $N\langle\alpha\rangle \geq CL$. Since $\sum_{k=0}^{C}\alpha\{k\} = 1$ we have

$$\begin{aligned}\log\sigma(N,\alpha) &= NH(\alpha) - N\sum_{k=0}^{C}\alpha\{k\}\log k! + N\langle\alpha\rangle\frac{L-1}{L}(\log N\langle\alpha\rangle - 1) \\ &\quad - \frac{N\langle\alpha\rangle}{L}\log\,(L-1)! + O(\log N)\end{aligned}$$

and we conclude considering (3.3) and (4.2). $\square$

5 The large deviation principle

Theorem 5.1 *We have (see (2.3), (2.6) and (4.2))*

$$\lim_{N\to\infty}\frac{1}{N}\log Z_\rho^N = -K(q_\rho) = -\log q_\rho\{0\} - \frac{L-1}{L}\langle q_\rho\rangle\,.$$

Proof. We use Theorem 3.2 for a sequence of open balls B shrinking to q_ρ and not containing δ_0. We may then use Theorem 4.6 with a uniform remainder. We obtain the result using the continuity of K at q_ρ and expliciting $K(q_\rho)$. □

Using the notations (4.1) and (4.2) we define the rate function

$$J(\alpha) = K(\alpha) - \log q_\rho\{0\} - \frac{L-1}{L}\langle q_\rho\rangle = H(\alpha\,|\,q_\rho) - \frac{L-1}{L}\left(\langle q_\rho\rangle - \langle\alpha\rangle + \langle\alpha\rangle\log\frac{\langle\alpha\rangle}{\langle q_\rho\rangle}\right) \quad (5.1)$$

which is continuous in $\mathcal{P}$ and C^∞ in its interior $\mathcal{P}^o$. Note that $J(q_\rho) = 0$.

Theorem 5.2 *A large deviation principle with continuous rate function J holds for $(\pi^N)_{N\geq 1}$: for any Borel set B included in $\mathcal{P}$*

$$-\inf_{\alpha\in B^o} J(\alpha) \leq \liminf_{N\to\infty}\frac{1}{N}\log\pi^N(B) \leq \limsup_{N\to\infty}\frac{1}{N}\log\pi^N(B) \leq -\inf_{\alpha\in B} J(\alpha)\,.$$

We have $J(\alpha) = H(\alpha\,|\,q_\rho)$ if and only if $\langle\alpha\rangle = \langle q_\rho\rangle$, else $J(\alpha) < H(\alpha\,|\,q_\rho)$.

Proof. Let a Borel set B be given. We divide (3.4) by N and obtain

$$\frac{1}{N}\log\pi^N(B) = -\inf_{\alpha\in B} K(N,\alpha) - \frac{1}{N}\log Z_\rho^N + O\left(\frac{\log N}{N}\right)$$

and use Theorems 4.6 and 5.1; we recall that $K(N,\alpha) = +\infty$ for $\alpha\notin\mathcal{P}^N$.

The LDP lower bound follows easily from the classical

$$\liminf_{N\to\infty} -\inf_{\alpha\in B} K(N,\alpha) = -\limsup_{N\to\infty}\inf_{\alpha\in B} K(N,\alpha) \geq -\inf_{\alpha\in B}\limsup_{N\to\infty} K(N,\alpha)$$

and hence using the continuity of K

$$\liminf_{N\to\infty} -\inf_{\alpha\in B} K(N,\alpha) \geq -\inf_{\alpha\in B^o} \limsup_{N\to\infty} K(N,\alpha) = -\inf_{\alpha\in B^o} K(\alpha)\,.$$

For the upper bound, we use uniform convergence. Theorem 4.6 yields

$$-\inf_{\alpha\in B} K(N,\alpha) \leq -\inf_{\alpha\in B} K(\alpha) + O\Big(\frac{\log N}{N}\Big)$$

with an uniform remainder, hence

$$\limsup_{N\to\infty} -\inf_{\alpha\in B} K(N,\alpha) \leq -\inf_{\alpha\in B} K(\alpha)$$

form which we deduce the LDP upper bound.

The last statement follows from the study of the function $y \mapsto 1 - y - y\log y$. □

6 The shape of the rate function

So that the LDP may be of any practical use, we give an appropriate description of the rate function J. We already know that it is non-negative continuous on $\mathcal{P}$ and C^∞ in $\mathcal{P}^o$, that $J(q_\rho) = 0$, and that $J(\alpha) = H(\alpha\,|\,q_\rho)$ if and only if $\langle\alpha\rangle = \langle q_\rho\rangle$, else $J(\alpha) < H(\alpha\,|\,q_\rho)$.

We differentiate twice in $\mathcal{P}^o$ considered as a subset of $\mathbb{R}^{C+1}$. Only the action of these differentials on the tangent space $\mathcal{T} = \{h : h_0 + h_1 + \cdots + h_C = 0\}$ is intrinsic. We use the last formulation in (5.1) and obtain for $i, j = 0, 1, \ldots, C$

$$\partial_i J(\alpha) = 1 + \log\frac{\alpha\{i\}}{q_\rho\{i\}} - i\frac{L-1}{L}\log\frac{\langle\alpha\rangle}{\langle q_\rho\rangle}\,, \quad \partial_{ij} J(\alpha) = \frac{\delta_{ij}}{\alpha\{i\}} - \frac{L-1}{L}\frac{ij}{\langle\alpha\rangle}\,. \tag{6.1}$$

As one expects $DJ(q_\rho).h = (1,1,\ldots,1)^* h = 0$ for h in $\mathcal{T}$. More surprisingly, $D^2 J$ does not depend on ρ.

Theorem 6.1 *The law q_ρ is the only α in $\mathcal{P}^o$ such that $DJ(\alpha).h = 0$ for all h in $\mathcal{T}$ and hence the only locus in $\mathcal{P}^o$ of a local extremum of J. Moreover $J(\alpha) > 0$ for any α in $\mathcal{P} - \{q_\rho\}$, while $J(q_\rho) = 0$. Hence there is exponential decay of $\pi^N(A)$ for any closed set A not containing q_ρ and $(\pi^N)_{N\geq 1}$ converges a.s. to q_ρ.*

Proof. If $DJ(\alpha)$ is tangent to $\mathcal{P}$ then $\partial_i J(\alpha)$ does not depend on i, hence

$$\log \frac{\alpha\{i\}}{q_\rho\{i\}} = \log \frac{\alpha\{i-1\}}{q_\rho\{i-1\}} + \frac{L-1}{L} \log \frac{\langle\alpha\rangle}{\langle q_\rho\rangle}$$

from which follows using (2.2) and (2.6)

$$\frac{\alpha\{i\}}{\alpha\{i-1\}} = \frac{q_\rho\{i\}}{q_\rho\{i-1\}} \left(\frac{\langle\alpha\rangle}{\langle q_\rho\rangle}\right)^{\frac{L-1}{L}} = i^{-1} \langle\alpha\rangle^{\frac{L-1}{L}} \rho^{\frac{1}{L}}$$

hence

$$i\alpha\{i\} = \langle\alpha\rangle^{\frac{L-1}{L}} \rho^{\frac{1}{L}} \alpha\{i-1\}. \tag{6.2}$$

By summation over i we obtain

$$\langle\alpha\rangle = \langle\alpha\rangle^{\frac{L-1}{L}} \rho^{\frac{1}{L}} (1-\alpha\{C\}) \Rightarrow \langle\alpha\rangle = \rho(1-\alpha\{C\})^L$$

and going back to (6.2), we see that α solves the recurrence relation (2.2) of which we know that the unique solution in $\mathcal{P}$ is q_ρ.

We have $J \geq 0$. If there is a zero of J in $\mathcal{P}^o$ then there is a local minimum at that point, which must then be q_ρ. For any boundary point α of $\mathcal{P}$ except δ_0, $\partial_i J(\alpha) = -\infty$ whenever $\alpha\{i\} = 0$, and $J(\alpha) > 0$. Moreover $\partial_1 J(\delta_0) = -\infty$ and again $J(\delta_0) > 0$.

Exponential decay follows by the compactness of $\mathcal{P}$, the continuity of J, and the LDP. The a.s. convergence follows a standard Borel-Cantelli argument. □

For v the column vector $(0,1,\ldots,C)^*$ and $\text{diag}(\alpha)$ and $\text{diag}(\alpha)^{-1} = \text{diag}(\alpha^{-1})$ the diagonal matrices $\text{diag}(\alpha\{0\},\alpha\{1\},\ldots,\alpha\{C\})$ and $\text{diag}(\alpha\{0\}^{-1},\alpha\{1\}^{-1},\ldots,\alpha\{C\}^{-1})$,

$$D^2 J(\alpha) = \text{diag}(\alpha)^{-1} - \frac{L-1}{L} \frac{1}{\langle\alpha\rangle} vv^* \tag{6.3}$$

is a rank 1 perturbation of a positive definite matrix of which we know the inverse. This particular structure enables us to study its invertibility and signature.

Let $Q(\alpha)$ be the restriction of $D^2 J(\alpha)$ to $\mathcal{T} = \{h : h_0 + h_1 + \cdots h_C = 0\}$, and $\mathcal{N} = v^\perp \cap \mathcal{T} = \{h \in \mathcal{T} : \langle h\rangle = 0\}$. We are actually interested only in $\mathcal{T}_\alpha = \{\beta - \alpha : \beta \in \mathcal{P}\}$

and $\mathcal{N}_\alpha = v^\perp \cap \mathcal{T}_\alpha = \{\beta - \alpha : \beta \in \mathcal{P},\ \langle\beta\rangle = \langle\alpha\rangle\}$ which has a natural interpretation in terms of mean occupancy.

Denote by $\langle\alpha\rangle_2$ and $\mathrm{Var}(\alpha)$ the second moment and variance of α, respectively. For $\alpha \in \mathcal{P}^o$ we denote by $\bar\alpha$ the probability measure

$$\bar\alpha\{i\} = \frac{i\alpha\{i\}}{\langle\alpha\rangle}, \quad i = 0, 1, \ldots, C. \tag{6.4}$$

Note that $\bar\alpha\{0\} = 0$ and $\bar\alpha\{i\} > 0$ for $i = 1, \ldots, C$, and that $\langle\bar\alpha\rangle = \langle\alpha\rangle^{-1}\langle\alpha\rangle_2$,

$$\langle\bar\alpha - \alpha\rangle = \frac{\mathrm{Var}(\alpha)}{\langle\alpha\rangle} > 0, \quad (\bar\alpha - \alpha)^*\mathrm{diag}(\alpha)^{-1}(\bar\alpha - \alpha) = \frac{\mathrm{Var}(\alpha)}{\langle\alpha\rangle^2} > 0. \tag{6.5}$$

For $L \geq 2$ we define on $\mathcal{P}$ and IR^{C+1} the second degree polynomial

$$F(\alpha) = \frac{L}{L-1}\langle\alpha\rangle - \mathrm{Var}(\alpha) = \langle\alpha\rangle^2 + \left(1 + \frac{1}{L-1}\right)\langle\alpha\rangle - \langle\alpha\rangle_2. \tag{6.6}$$

The equation $F(\alpha) = 0$ defines a cylinder with parabolic base, delimiting a convex open set $\{F(\alpha) < 0\}$. Clearly $F(\delta_0) = 0$ and $F(\delta_i) = iL/(L-1) > 0$ for $i = 1, \ldots, C$.

Theorem 6.2 *For any $\alpha \in \mathcal{P}^o$ the decomposition*

$$\mathcal{T} = \mathcal{N} + \mathrm{span}(\bar\alpha - \alpha)$$

is orthogonal for $Q(\alpha)$. The restriction of $Q(\alpha)$ to $\mathcal{N}$ coincides with the restriction of $\mathrm{diag}(\alpha)^{-1}$ *and hence is is positive definite (of rank $C-1$), and*

$$(\bar\alpha - \alpha)^*Q(\alpha)(\bar\alpha - \alpha) = \left(1 - \frac{L-1}{L}\frac{\mathrm{Var}(\alpha)}{\langle\alpha\rangle}\right)\frac{\mathrm{Var}(\alpha)}{\langle\alpha\rangle^2} = F(\alpha)\frac{L-1}{L}\frac{1}{\langle\alpha\rangle}\frac{\mathrm{Var}(\alpha)}{\langle\alpha\rangle^2}.$$

Proof. We represent $\mathcal{T}$ in a (non-orthogonal) basis by associating to $h \in \mathcal{T}$ the vector $(h_1, \ldots, h_C)$, and naturally $h_0 = -(h_1 + \cdots + h_C)$. Since $\mathrm{diag}(\alpha)^{-1}$ is positive definite, the restriction of the corresponding quadratic form on $\mathcal{T}$ is positive definite. We denote by B^{-1} its matrix. Both B^{-1} and B can be readily explicited: for $h \in \mathcal{T}$,

$$h^*\mathrm{diag}(\alpha)^{-1}h = \sum_{i=0}^{C}\frac{1}{\alpha\{i\}}h_i^2 = \frac{1}{\alpha\{0\}}(h_1 + \cdots + h_C)^2 + \sum_{i=1}^{C}\frac{1}{\alpha\{i\}}h_i^2$$

and setting $u = (1, \dots, 1)^*$ and $A = \mathrm{diag}(\alpha\{1\}, \dots, \alpha\{C\})$, we have

$$B^{-1} = A^{-1} + \frac{1}{\alpha\{0\}} uu^*, \quad B = A - Au(Au)^* = A - Auu^*A, \tag{6.7}$$

where $BB^{-1} = I_C$ is easily checked using $u^*Au = \alpha\{1\} + \cdots + \alpha\{C\} = 1 - \alpha\{0\}$.

We set $w = (1, \dots, C)^*$ in this basis of $\mathcal{T}$. Then (6.3) implies

$$Q(\alpha) = B^{-1} - \frac{L-1}{L} \frac{1}{\langle\alpha\rangle} ww^* \tag{6.8}$$

and the restrictions of Q and B^{-1} to $w^\perp$ are equal. Since $w^*Bw > 0$ then $Bw \notin w^\perp$ for the canonical scalar product in this basis. Since $x \in w^\perp$ implies $x^*Qx = x^*B^{-1}x$ and $x^*QBw = 0$, the decomposition of $\mathcal{T}$ into $w^\perp + \mathrm{span}(Bw)$ is orthogonal for Q. Then

$$Bw = Aw - (u^*Aw)Au = Aw - \langle\alpha\rangle Au = \langle\alpha\rangle(\bar{\alpha} - \alpha),$$

$$(Bw)^*QBw = \left(1 - \frac{L-1}{L}\frac{1}{\langle\alpha\rangle} w^*Bw\right) w^*Bw,$$

$$w^*Bw = w^*Aw - (u^*Aw)^2 = \langle\alpha\rangle_2 - \langle\alpha\rangle^2 = \mathrm{Var}(\alpha),$$

and we conclude by writing everything intrinsically, in particular $w^\perp = \mathcal{N}$. □

This allows us to study the shape of J. In the independent case $L = 1$ the Hessian Q is everywhere positive definite, and J is uniformly convex at each α in $\mathcal{P}^o$ as it should be.

Theorem 6.3 *Let $L \geq 2$. If $C = 1$ or if $C = 2$ and $L = 2$, then Q is positive definite and J is uniformly convex at each α in $\mathcal{P}^o$. Else, Q has signature $(C-1, 1)$ in the non-empty convex open set $\{F(\alpha) < 0\} \cap \mathcal{P}^o$, has signature $(C-1, 0)$ on the parabolic cylinder $\{F(\alpha) = 0\} \cap \mathcal{P}^o$, and has signature $(C, 0)$ in the non-empty open set $\{F(\alpha) > 0\} \cap \mathcal{P}^o$, on which J is uniformly convex at each point.*

Proof. If $C = 1$ then $\langle\alpha\rangle_2 = \langle\alpha\rangle$ and $F(\alpha) > 0$ in $\mathcal{P}^o$. For $C = 2$ we have

$$F(\alpha) = \alpha\{1\}^2 + 4\alpha\{1\}\alpha\{2\} + 4\alpha\{2\}^2 + \frac{1}{L-1}\alpha\{1\} - \left(2 - \frac{2}{L-1}\right)\alpha\{2\}.$$

Thus if $L = 2$ then $F(\alpha) > 0$ in $\mathcal{P}^o$, and if $L \geq 3$ then

$$F(\alpha) \leq \alpha\{1\}(\alpha\{1\} + 4\alpha\{2\} + 1/2) - \alpha\{2\}(1 - 4\alpha\{2\})$$

and $F(\alpha) < 0$ for $0 < \alpha\{2\} < 1/4$ and $\alpha\{1\}$ sufficiently small. For $C = 3$ and $L \geq 2$,

$$\begin{aligned} F(\alpha) &\leq \langle\alpha\rangle^2 + 2\langle\alpha\rangle - \langle\alpha\rangle_2 = (\alpha\{1\} + 2\alpha\{2\} + 3\alpha\{3\})^2 + \alpha\{1\} - 3\alpha\{3\} \\ &\leq (\alpha\{1\} + 2\alpha\{2\})^2 + 6(\alpha\{1\} + 2\alpha\{2\})\alpha\{3\} + \alpha\{1\} - 3\alpha\{3\}(1 - 3\alpha\{3\}) \end{aligned}$$

and $F(\alpha) < 0$ for $0 < \alpha\{3\} < 1/3$ and $\alpha\{1\}$ and $\alpha\{2\}$ sufficiently small. We conclude for all $C \geq 3$ and $L \geq 2$ by a continuity argument. □

We now obtain a good understanding of the behavior of J near q_ρ.

Theorem 6.4 *The Hessian matrix $Q(q_\rho)$ is definite positive (of rank C), and the rate function J is uniformly convex at q_ρ.*

Proof. We obtain using (2.2)

$$\begin{aligned} \langle q_\rho\rangle_2 &= \sum_{k=1}^{C} k^2 q_\rho\{k\} = \rho(1 - q_\rho\{C\})^{L-1} \sum_{k=1}^{C} k q_\rho\{k-1\} \\ &= \rho(1 - q_\rho\{C\})^{L-1}\Big(\langle q_\rho\rangle - C q_\rho\{C\} + 1 - q_\rho\{C\}\Big) \end{aligned}$$

and using (2.6) and (6.6)

$$\begin{aligned} F(q_\rho) &= \rho(1 - q_\rho\{C\})^{L-1}\Big((1 - q_\rho\{C\})\langle q_\rho\rangle + \frac{1}{L-1}(1 - q_\rho\{C\}) - \langle q_\rho\rangle + C q_\rho\{C\}\Big) \\ &= \rho(1 - q_\rho\{C\})^{L-1}\Big((C - \langle q_\rho\rangle) q_\rho\{C\} + \frac{1}{L-1}(1 - q_\rho\{C\})\Big) > 0 \end{aligned} \tag{6.9}$$

where we conclude using the capacity constraint $\langle q_\rho\rangle \leq C$. □

This enables us to give a local estimate at q_ρ.

Theorem 6.5 *Let $\mathcal{T}$ be furnished with a norm $\|\cdot\|$ and $\mathcal{P}$ with the corresponding distance. Let $B(q_\rho, \varepsilon)$ denote the ball of radius $\varepsilon > 0$ centered at q_ρ. Then*

$$\lim_{N\to\infty} \frac{1}{N} \log \pi^N (B(q_\rho, \varepsilon)^c) = -\theta\varepsilon^2 + O_{\varepsilon\to 0^+}(\varepsilon^3)\,, \quad \theta = \inf_{h\in\mathcal{T}\,:\,\|h\|=1} h^* Q(q_\rho) h > 0\,.$$

If the norm is Euclidean, then θ is the least eigenvalue of $Q(q_\rho)$ in an orthonormal basis.

Proof. Since J is non-negative continuous on $\mathcal{P}$, vanishes only at q_ρ, and is convex in a neighborhood of q_ρ, then for $\varepsilon > 0$ small enough the infimum of $J(\alpha)$ for $\alpha \notin B(q_\rho, \varepsilon)$ will be attained at the boundary. A Taylor expansion gives for any vector h of norm 1 $J(q_\rho + \varepsilon h) = \varepsilon^2 h^* Q(q_\rho) h + O_{\varepsilon\to 0^+}(\varepsilon^3)$. □

7 Some pictures

We present plots of the rate function J for $C = 2$ and $\rho = 10$, as a function of $\alpha\{1\}$ and $\alpha\{2\}$. In Figure 1, $L = 5$ and in Figure 2, $L = 50$. The non-convexity is quite apparent.

In Figure 1 the origin is at the lower left, the $\alpha\{1\}$ axis points right, and the $\alpha\{2\}$ axis points to the rear. In Figure 2 we have rotated the perspective, and the origin is the bottommost point (at the middle), the $\alpha\{1\}$ points to the right, and the $\alpha\{2\}$ axis points to the left.

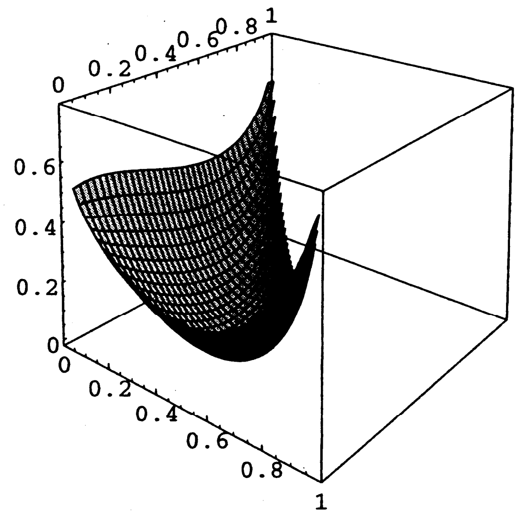

Figure 1. $C = 2$, $\rho = 10$, $L = 5$.

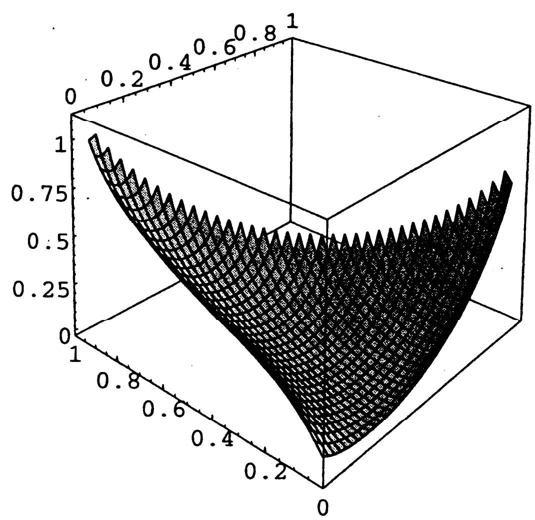

Figure 2. $C = 2$, $\rho = 10$, $L = 50$.

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