Quantum Boundary Conditions for Torus Maps

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The quantum states of a dynamical system whose phase space is the two-torus are periodic up to phase factors under translations by the fundamental periods of the torus in the position and momentum representations. These phases, θ_1 and θ_2, are conserved quantities of the quantum evolution. We show that for a large and important class of quantum maps, θ_1 and θ_2, are restricted to being the co-ordinates of the fixed points of the automorphism induced on the fundamental group of the torus by the underlying classical dynamics. As a consequence, if the classical map commutes with lattice translations in $\mathbf{R}^2$ it can be quantized for any choice of the phases, but otherwise it can be quantized for only a finite set. This result is a special case of a more general condition on the phases, which is also derived. The cat maps, perturbed cat maps, and the kicked Harper map are discussed as specific examples.

1 Introduction

The quantum kinematics of maps whose phase space is the two-torus was developed by Hannay and Berry [13], who also investigated the quantum dynamics of a particular class of examples: the Arnold cat maps. Since then the quantization [12, 7, 10, 11, 18] and semiclassical properties [16, 17, 21] of the cat maps have been studied in a wide variety of contexts, principally to investigate the influence of classical chaos on quantum eigenstates. Other torus maps that have been quantized and studied in the same way include nonlinear perturbations of the cat maps [3, 6, 5, 9], the baker's transformation [2, 23, 14], and the kicked Harper map [19].

Our purpose here is to describe an unusual connection between the quantum boundary conditions satisfied by the wavefunctions of torus maps and a topological property of the underlying classical dynamics. Let $T_Q(x) = e^{-\frac{ix\hat{P}}{\hbar}}$ and $T_P(y) = e^{\frac{iy\hat{Q}}{\hbar}}$ be the translation operators in position and momentum respectively. A quantum state must be physically invariant under translations in both position and momentum by a fundamental period of the torus. Therefore, in units where both periods are equal to one,

$$T_Q(1)\,|\psi\rangle = e^{-2\pi i\theta_2}\,|\psi\rangle\,, \tag{1.1a}$$

and

$$T_P(1)\,|\psi\rangle = e^{2\pi i\theta_1}\,|\psi\rangle\,. \tag{1.1b}$$

The phases θ_1 and θ_2 can be thought of as representing boundary conditions on the edge of the fundamental square whose sides are identified to form the torus. We shall be concerned here with the values these phases can take.

It is straightforward to show that, on the one hand, the quantum propagator for the cat maps is consistently defined only for a finite number of phase vectors $\boldsymbol{\theta} = (\theta_1, \theta_2)$ [18, 10], while on the other, some systems, such as the kicked Harper map, can be quantized for all values of $\boldsymbol{\theta} \mod 1$. Our main result, which we now outline, is a general condition on $\boldsymbol{\theta}$, for the quantization of any canonical map on the torus (modulo nonintegral translations), that explains the difference between these two examples and determines the allowed values of the phases when not all are possible.

First, associated with every canonical map $\boldsymbol{\phi}$ of the two-torus there is an integer unimodular matrix A, which generates the transformation of the winding numbers of any closed curve under the action of $\boldsymbol{\phi}$. This connection is reviewed in Section 2. Our result is that the quantum propagator related to $\boldsymbol{\phi}$ is defined only for phase vectors $\boldsymbol{\theta}$ satisfying

$$A \cdot \boldsymbol{\theta} - \frac{N}{2}\mathbf{v} = \boldsymbol{\theta} \mod 1, \tag{1.2}$$

where N is the dimension of the Hilbert space in which the propagator acts, and $\mathbf{v}$ is an integer vector determined by A. The quantum kinematics needed to prove this is described in Section 3.1, and the proof itself is given in Section 3.2. In many cases (in particular, when N is even, or when $AA^T = I \mod 2$), the term $N/2\,\mathbf{v} \mod 1$ vanishes, and so (1.2) simplifies to

$$A \cdot \boldsymbol{\theta} = \boldsymbol{\theta} \mod 1. \tag{1.3}$$

The quantum propagator is then defined only if θ_1 and θ_2 are the coordinates of a fixed point of the cat map $\boldsymbol{\theta} \mapsto A \cdot \boldsymbol{\theta}$ associated with the topology of the torus map $\boldsymbol{\phi}$. Naturally the result just stated is rendered meaningful only when the term quantization is defined, and so we give an explicit construction of the quantum propagator associated with a very general class of maps $\boldsymbol{\phi}$ in Section 3.3. Finally, some specific examples, including the cat maps, perturbed cat maps, and the kicked Harper map are discussed in Section 4.

The phases θ_1 and θ_2 play the role of Aharonov-Bohm-flux-like parameters and so are connected with fundamental geometrical and topological features of the quantum mechanics of torus maps, such as geometric (Berry) phases and Chern numbers. There is thus a close mathematical analogy between our analysis of quantum maps, the physics of the integer and fractional quantum Hall effects [25], and the influence of guage fields in mesoscopic devices [24, 4], to give but two examples. Some of the implications of our result, and questions it gives rise to, will be discussed in this context in Section 3.2.

2 Lifts of torus maps

We regard the -torus as a classical phase space, with canonical coordinates $\mathbf{z} = (q, p)$ defined modulo one, and consider a canonical map $\boldsymbol{\phi}(\mathbf{z})$ on the torus. Here canonical means that $\boldsymbol{\phi}$ is smooth, area-preserving and orientation-preserving, so that its Jacobian matrix, $D\boldsymbol{\phi} = [\partial\phi_i/\partial z_j]$, has determinant one. By a *lift* of the torus map, we mean a smooth canonical map $\boldsymbol{\Phi}(\mathbf{Z})$ of the plane, with canonical coordinates $\mathbf{Z} = (Q, P)$, whose action modulo integer translations is given by $\boldsymbol{\phi}$. That is,

$$\boldsymbol{\Phi}(\mathbf{Z}) \mod 1 = \boldsymbol{\phi}(\mathbf{Z} \mod 1). \tag{2.1}$$

It is straightforward to show that a lift exists, as follows. (2.1) determines $\boldsymbol{\Phi}(\mathbf{Z})$ up to an integer vector. This indeterminacy can be removed by fixing the value of $\mathbf{\Phi}$ at one point, say the origin, and determining its value for all other points $\mathbf{Z}$ by requiring $\mathbf{\Phi}$ to be continuous along paths from the origin to $\mathbf{Z}$. (This determination is unambiguous, because any two such paths can be continuously deformed into each other.) The property of being canonical, which is local, is then inherited from $\boldsymbol{\phi}$. It is clear that two lifts of the same torus map must differ by an integer vector.

If the argument of $\mathbf{\Phi}$ is shifted by an integer vector, (2.1) implies that its value must also be shifted by an integer vector; thus $\Phi(\mathbf{Z}+\mathbf{m}) = \Phi(\mathbf{Z})+\mathbf{m}'$, where $\mathbf{m}'$ depends on $\mathbf{m}$ but not (by continuity) on $\mathbf{Z}$. Since $(\mathbf{m}+\mathbf{n})' = \mathbf{m}'+\mathbf{n}'$ (this is easily seen by writing $\mathbf{\Phi}(\mathbf{Z}+(\mathbf{m}+\mathbf{n})) = \mathbf{\Phi}((\mathbf{Z}+\mathbf{m})+\mathbf{n})$), $\mathbf{m}'$ must be linearly related to $\mathbf{m}$. Therefore

$$\mathbf{\Phi}(\mathbf{Z}+\mathbf{m}) = \mathbf{\Phi}(\mathbf{Z}) + A\cdot\mathbf{m}, \tag{2.2}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.3}$$

is an integer matrix.

We now show that

$$\det A = 1. \tag{2.4}$$

First, we note that $\mathbf{\Phi}$ is invertible; its inverse $\mathbf{\Phi}^{-1}$ can be taken to be a lift of the torus map $\boldsymbol{\phi}^{-1}$. As a lift, $\mathbf{\Phi}^{-1}$ satisfies a relation $\mathbf{\Phi}^{-1}(\mathbf{Z}+\mathbf{m}) = \mathbf{\Phi}^{-1}(\mathbf{Z}) + B\cdot\mathbf{m}$ analogous to (2.2), where B is an integer matrix. It is easily checked that $AB = BA = I$, so that A is invertible. As A is also integral, it follows that $\det A$ is equal to ± 1.

To establish that $\det A$ is equal to $+1$, we consider the map $\mathbf{F}(\mathbf{Z}) = A^{-1}\cdot\mathbf{\Phi}(\mathbf{Z}) - \mathbf{Z}$, which, from (2.2), is periodic in $\mathbf{Z}$ with period one. Its Jacobian matrix is, of course, also periodic, and satisfies the relation $D\mathbf{F}+I = A^{-1}\cdot D\mathbf{\Phi}$. Taking determinants (and noting that $\det D\mathbf{\Phi} = 1$), we get that

$$\operatorname{tr} D\mathbf{F} + \det D\mathbf{F} = \det A^{-1} - 1. \tag{2.5}$$

On integration over the unit square, the right-hand side of (2.5) is unchanged ($\det A^{-1}$ is constant), while the left-hand side,

$$\int_0^1\int_0^1 (\operatorname{tr} D\mathbf{F} + \det \mathbf{F})\, d^2\mathbf{Z} = \tag{2.6}$$

$$\int_0^1\int_0^1 (\partial_Q F_1 + \partial_P F_2 + \partial_Q F_1 \partial_P F_2 - \partial_Q F_2 \partial_P F_1)\, dQdP, \tag{2.7}$$

vanishes by periodicity (for the determinant terms this follows after integrating by parts). (2.4) then follows.

As a simple example, let us consider the cat map $\boldsymbol{\phi}(\mathbf{z}) = T \cdot \mathbf{z} \mod 1$, where T is an integer matrix with unit determinant. A lift is just the linear map $\boldsymbol{\Phi}(\mathbf{Z}) = T \cdot \mathbf{Z}$, and the matrix A of (2.2) is just T itself.

More generally, associated to every canonical map $\boldsymbol{\phi}$ of the two-torus is an integer unimodular matrix A. For those maps which can be continuously connected to the identity by a one-parameter family of torus maps, $A = I$, by continuity. This is the case for time-one flows of (possibly time-dependent) torus Hamiltonians, as well as for translations on the torus, $\mathbf{z} \mapsto \mathbf{z} + \mathbf{z}_0 \mod 1$. More generally, for those maps which can be continuously connected to a cat map $\boldsymbol{\phi}(\mathbf{z}) = T \cdot \mathbf{z} \mod 1$, including, for example, Anosov perturbations of hyperbolic cat maps, the associated matrix A is equal to T.

The associated matrix A has the following topological interpretation. Let $\mathbf{z}_\sigma$, $0 \le \sigma \le 1$, denote a closed curve on the torus with winding numbers r and s about the q and p directions. Its image under the torus map, $\mathbf{z}'_\sigma = \boldsymbol{\phi}(\mathbf{z}_\sigma)$, is also a closed curve, whose winding numbers we denote by r' and s'. Then A describes the linear transformation of the winding numbers, ie

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = A \begin{pmatrix} r \\ s \end{pmatrix}. \tag{2.8}$$

(2.8) may be obtained by considering a *lifted curve* $\mathbf{Z}_\sigma$, a continuous curve in the plane equal to $\mathbf{z}_\sigma$ modulo 1, whose endpoints $\mathbf{Z}_0$ and $\mathbf{Z}_1$ therefore differ precisely by the vector of winding numbers (r, s). From (2.2),

$$\boldsymbol{\Phi}(\mathbf{Z}_1) = \boldsymbol{\Phi}(\mathbf{Z}_0) + A \cdot \begin{pmatrix} r \\ s \end{pmatrix}. \tag{2.9}$$

The curve $\mathbf{Z}'_\sigma = \boldsymbol{\Phi}(\mathbf{Z}_\sigma)$ is a lift of $\mathbf{z}'_\sigma$, and (2.9) implies that the difference between its endpoints, $\boldsymbol{\Phi}(\mathbf{Z}_1) - \boldsymbol{\Phi}(\mathbf{Z}_0)$, which is just the vector of transformed winding numbers (r', s'), is given by (2.8).

The preceding discussion shows that canonical torus maps are equivalent to canonical maps of the plane satisfying (2.2). As discussed in the next section, this equivalence is the basis of our quantization prescription. In anticipation of this discussion, it will be useful to introduce *classical lattice translation operators* $\boldsymbol{T}^{\mathrm{cl}}_{\mathbf{m}}$, defined by

$$\boldsymbol{T}^{\mathrm{cl}}_{\mathbf{m}}(\mathbf{Z}) = \mathbf{Z} + \mathbf{m} \tag{2.10}$$

and to re-express (2.2) in terms of them, as follows:

$$\boldsymbol{\Phi} \circ \boldsymbol{T}^{\mathrm{cl}}_{\mathbf{m}} = \boldsymbol{T}^{\mathrm{cl}}_{A \cdot \mathbf{m}} \circ \boldsymbol{\Phi}. \tag{2.11}$$

Finally, we note that the lifted map $\mathbf{\Phi}$ is an example of a more general construction, familiar in the theory of covering spaces (see, eg, [15]). Let M be a simply connected manifold, G a discrete group which acts properly discontinuously on M, and p the canonical projection from M to M/G. Then M is a universal covering space of M/G, whose fundamental group $\pi_1(M/G)$ is isomorphic to G. If ϕ is a continuous map on M/G, then there exists a continuous map Φ on M such that

$$p \circ \Phi = \phi \circ p. \tag{2.12}$$

ϕ induces a homomorphism ϕ^* on $\pi_1(M/G)$, and the lifted map Φ satisfies

$$\Phi(g \cdot m) = \phi^*(g) \cdot \Phi(m) \tag{2.13}$$

When M is the plane $\mathbb{R}^2$ and $G = \mathbb{Z}^2$ acts on M by lattice translations, we recover the example above – M/G is the two-torus, the induced homomorphism ϕ^* is described by the relation (2.8) between winding numbers, and (2.12) and (2.13) coincide with (2.1) and (2.2).

3 Quantization of torus maps

A torus map describes the classical dynamics of a system whose physical properties are invariant under unit translations in position and momentum. Quantum mechanically, such a system is represented by a state vector quasiperiodic under unit translations in position and momentum, ie invariant up to phase factors $\exp(-2\pi i\theta_2)$ and $\exp(2\pi i\theta_1)$. (For later convenience, the phases $\boldsymbol{\theta} = (\theta_1, \theta_2)$ are defined modulo one rather than modulo 2π.) In order for superpositions of states to be quasiperiodic, the phases $\boldsymbol{\theta} = (\theta_1, \theta_2)$ must be the same for all states representing the system. Let $\mathcal{H}(\boldsymbol{\theta})$ denote the Hilbert space of vectors quasiperiodic under unit translations with phases $\boldsymbol{\theta}$.

In Section 3.1 we construct the Hilbert spaces $\mathcal{H}(\boldsymbol{\theta})$. The quantized torus map is then obtained in Section 3.2 as a unitary operator on $\mathcal{H}(\boldsymbol{\theta})$. There we also derive our main result, namely that the allowed values of the phases $\boldsymbol{\theta}$ are constrained to be fixed points of either the cat map $\boldsymbol{\theta} \mapsto A \cdot \boldsymbol{\theta} \mod 1$ induced by the associated matrix A, or else, in certain cases, a map related to the induced cat map in a simple way. The quantization procedure itself is discussed in greater detail in Section 3.3.

3.1 Hilbert space of quasiperiodic states

The quantized cat map of Hannay and Berry [13] is defined on states whose position and momentum wavefunctions are both periodic. The generalization to the space $\mathcal{H}(\boldsymbol{\theta})$ of quasiperiodic states is straightforward. Our discussion is similar to that of Knabe [18] and Lebeouf et al [19]. When appropriate, we use a carat ^ to distinguish quantum objects from classical; for example, $\hat{\mathbf{Z}} = (\hat{Q}, \hat{P})$ denotes the position and momentum operators.

Corresponding to the $\mathbf{Z}$-phase-plane is the Hilbert space $L^2(\mathbb{R})$ of square-integrable complex-valued wavefunctions $\psi(Q)$. Unitary operators representing unit translations in position and momentum are given by $T_Q = \exp(-i\hat{P}/\hbar)$ and $T_P = \exp(i\hat{Q}/\hbar)$, respectively. These generate the quantum lattice translation operators $T(\mathbf{m})$, which we define to be

$$T(\mathbf{m}) = T_Q^{m_1} T_P^{m_2}. \tag{3.1}$$

The quantum lattice translations satisfy

$$T(\mathbf{m})\,\hat{\mathbf{Z}}\,T^{\dagger}(\mathbf{m}) = \hat{\mathbf{Z}} + \mathbf{m}, \tag{3.2}$$

in analogy with (2.10), but unlike the classical lattice translations, they do not, in general, commute; instead,

$$T(\mathbf{m})T(\mathbf{n}) = \exp(-i\mathbf{m}\wedge\mathbf{n}/\hbar)T(\mathbf{n})T(\mathbf{m}), \tag{3.3}$$

where $\mathbf{m}\wedge\mathbf{n} = m_1 n_2 - m_2 n_1$. To make the quantum lattice translations commute, we impose hereafter the quantization condition

$$h = 1/N, \tag{3.4}$$

where N is an integer, on Planck's constant.

With the condition (3.4), an arbitrary state $|\psi\rangle \in L^2(\mathbb{R})$ can be expressed as a continuous superposition of quasiperiodic states $|\psi(\boldsymbol{\theta})\rangle$ satisfying

$$T(\mathbf{m})\,|\psi(\boldsymbol{\theta})\rangle = \exp(-2\pi i\mathbf{m}\wedge\boldsymbol{\theta})\,|\psi(\boldsymbol{\theta})\rangle\,. \tag{3.5}$$

These quasiperiodic states may be characterised as follows. The quasiperiodicity of the momentum wavefunction $\tilde{\psi}(P;\boldsymbol{\theta}) = \langle P\,|\psi(\boldsymbol{\theta})\rangle$, namely that

$$\tilde{\psi}(P-1;\boldsymbol{\theta}) = \exp(2\pi i\theta_1)\tilde{\psi}(P;\boldsymbol{\theta}), \tag{3.6}$$

implies that the position wavefunction $\psi(Q;\boldsymbol{\theta}) = \langle Q\,|\psi(\boldsymbol{\theta})\rangle$ is supported on the lattice

$$Q_j = \frac{j+\theta_1}{N}. \tag{3.7}$$

The quasiperiodicity of the position wavefunction, namely that

$$\psi(Q_j - 1;\boldsymbol{\theta}) = \psi(Q_{j-N};\boldsymbol{\theta}) = \exp(-2\pi i\theta_2)\psi(Q_j;\boldsymbol{\theta}), \tag{3.8}$$

implies that $\psi(Q;\boldsymbol{\theta})$ is completely determined by its values at the N lattice points $Q_1, Q_2, \ldots, Q_N$. Thus, the space $\mathcal{H}(\boldsymbol{\theta})$ of quasiperiodic states is N-dimensional. A basis is given by the states $|j(\boldsymbol{\theta})\rangle$ whose position wavefunctions are nonzero at just one of the points $Q_j, 1 \le j < N$; explicitly,

$$\langle Q\, |j:\boldsymbol{\theta}\rangle = \sum_{m=-\infty}^{\infty} \exp(2\pi i m\theta_2)\delta(Q - Q_{j+mN}), \tag{3.9}$$

The states $|j(\boldsymbol{\theta})\rangle$ are, of course, unnormalizable with respect to the L^2-norm, and satisfy the generalized orthonormality conditions

$$\langle j(\boldsymbol{\theta})\, |k(\boldsymbol{\theta}')\rangle = N\delta_{jk}\delta_{T^2}(\boldsymbol{\theta} - \boldsymbol{\theta}') \tag{3.10}$$

(δ_{T^2} denotes the delta-function on the two-torus). We introduce a renormalized inner product on $\mathcal{H}(\boldsymbol{\theta})$ by taking the basis states $|j(\boldsymbol{\theta})\rangle$ to be orthonormal, and thereby make $\mathcal{H}(\boldsymbol{\theta})$ a Hilbert space.

In some of the discussion to follow, it will be necessary to know how the spaces $\mathcal{H}(\boldsymbol{\theta})$ transform under nonintegral translation operators $T(\mathbf{Z})$, defined in analogy with (3.1) by

$$T(\mathbf{Z}) = \exp(-2\pi i N Q\hat{P})\exp(2\pi i N P\hat{Q}). \tag{3.11}$$

The commutation relations

$$T(\mathbf{Z})T(\mathbf{Z}') = \exp(-2\pi i N\mathbf{Z}\wedge\mathbf{Z}')T(\mathbf{Z}')T(\mathbf{Z}) \tag{3.12}$$

imply that $T(\mathbf{m})T(\mathbf{Z})\,|\psi(\boldsymbol{\theta})\rangle = \exp(-2\pi i\mathbf{m}\wedge(\boldsymbol{\theta}+N\mathbf{Z}))T(\mathbf{Z})\,|\psi(\boldsymbol{\theta})\rangle$, so that the state $T(\mathbf{Z})\,|\psi(\boldsymbol{\theta})\rangle$ belongs to $\mathcal{H}(\boldsymbol{\theta}+N\mathbf{Z})$. Therefore

$$T(\mathbf{Z})\mathcal{H}(\boldsymbol{\theta}) = \mathcal{H}(\boldsymbol{\theta}+N\mathbf{Z}). \tag{3.13}$$

Thus, typical translations map the spaces $\mathcal{H}(\boldsymbol{\theta})$ into each other, while rational translations of the form $T(\mathbf{m}/N)$ leave $\mathcal{H}(\boldsymbol{\theta})$ invariant. We remark that, for $1 \le m_1, m_2 \le N$, the rational translations $T(\mathbf{m}/N)$ constitute a basis for the N^2-dimensional space of operators on $\mathcal{H}(\boldsymbol{\theta})$ (with respect to a basis, this is just the space of $N \times N$ matrices).

3.2 Allowed phases for the quantized torus map

By a quantization of the torus map $\boldsymbol{\phi}$, we mean a unitary operator $U(\boldsymbol{\theta})$ defined on the Hilbert space $\mathcal{H}(\boldsymbol{\theta})$ whose action in the classical limit is given by $\boldsymbol{\phi}$. Of course, this requirement does not determine $U(\boldsymbol{\theta})$ uniquely; the quantization procedure must be prescribed.

We shall consider the following explicit scheme, which can be applied to all torus maps, modulo translations. Given a torus map $\phi(\mathbf{z})$, we first construct its lift $\mathbf{\Phi}(\mathbf{Z})$. $\mathbf{\Phi}$ is then quantized to give a unitary operator U on $L^2(\mathbb{R})$. Finally $U(\boldsymbol{\theta})$ is defined, where possible, by restricting U to $\mathcal{H}(\boldsymbol{\theta})$.

The first step in the procedure, the construction of the lifted map, was described in Section 2. The next step, the quantization of the lift, is described in the following Section 3.3. There it is shown that the quantized lift U satisfies the quantum analogue of the kinematic property (2.2), an analogue most transparently expressed in terms of the *Heisenberg translation operators* $T_H(\mathbf{Z})$, defined by

$$T_H(\mathbf{Z}) = \exp(-2\pi i N \mathbf{Z} \wedge \hat{\mathbf{Z}}). \tag{3.14}$$

(These differ from the translations $T(\mathbf{Z})$ of (3.11) by a phase factor.) The quantum analogue of (2.2) is just

$$T_H(\mathbf{m})U = U T_H(A^{-1} \cdot \mathbf{m}). \tag{3.15}$$

The Heisenberg operator $T_H(\mathbf{m})$ differs from the quantum lattice translation $T(\mathbf{m})$ by the sign factor $(-1)^{N m_1 m_2}$, so (3.15) can also be written as

$$T(\mathbf{m})U = (-1)^{\mathbf{m}\wedge\mathbf{v}} U T(A^{-1} \cdot \mathbf{m}), \tag{3.16}$$

where the integer vector $\mathbf{v}$ is given by

$$\mathbf{v} = \begin{pmatrix} ab \\ cd \end{pmatrix}. \tag{3.17}$$

(One needs to check, making use of $\det A = ad - bc = 1$, that $m_1 m_2 - m_1' m_2' = \mathbf{m} \wedge \mathbf{v} \mod 2$, where $\mathbf{m}' = A^{-1} \cdot \mathbf{m}$.)

Here we examine the last step in the quantization procedure, the construction of $U(\boldsymbol{\theta})$. As will be apparent, our considerations depend only on U satisfying the kinematic condition (3.15), and not on any other features of its construction.

Provided $\mathcal{H}(\boldsymbol{\theta})$ is invariant under U, $U(\boldsymbol{\theta})$ is taken to be simply the restriction of U to $\mathcal{H}(\boldsymbol{\theta})$. Therefore, with respect to the basis $|j(\boldsymbol{\theta})\rangle$, $U(\boldsymbol{\theta})$ is represented by a matrix $U_{jk}(\boldsymbol{\theta})$, defined by

$$U\,|k(\boldsymbol{\theta})\rangle = \sum_{j=1}^{N} U_{jk}(\boldsymbol{\theta})\,|j(\boldsymbol{\theta})\rangle\,, \tag{3.18}$$

whose components may be determined explicitly using (3.10). The fact that U is unitary implies that $U_{jk}(\boldsymbol{\theta})$ is a unitary matrix. For $\boldsymbol{\theta} = \mathbf{0}$, this gives, for cat maps, just the Hannay-Berry quantization.

To determine whether $\mathcal{H}(\boldsymbol{\theta})$ is invariant under U, we consider how the once-mapped state $|\psi'\rangle = U\,|\psi(\boldsymbol{\theta})\rangle$ transforms under integer translations. From (3.16) and (3.5),

$$\begin{aligned} T(\mathbf{m})\,|\psi'\rangle &= T(\mathbf{m})U\,|\psi(\boldsymbol{\theta})\rangle \\ &= (-1)^{N\mathbf{m}\wedge\mathbf{v}}UT(A^{-1}\cdot\mathbf{m})\,|\psi(\boldsymbol{\theta})\rangle \\ &= (-1)^{N\mathbf{m}\wedge\mathbf{v}}\exp(-2\pi i(A^{-1}\cdot\mathbf{m})\wedge\boldsymbol{\theta})U\,|\psi(\boldsymbol{\theta})\rangle \\ &= (-1)^{N\mathbf{m}\wedge\mathbf{v}}\exp(-2\pi i\mathbf{m}\wedge A\cdot\boldsymbol{\theta})\,|\psi'\rangle\,, \end{aligned} \tag{3.19}$$

where in the second equality we have used the identity $(A^{-1}{\cdot}\mathbf{a}){\wedge}\mathbf{b} = \mathbf{a}{\wedge}(A{\cdot}\mathbf{b})$. Thus $|\psi'\rangle$ belongs to $\mathcal{H}(A\cdot\boldsymbol{\theta} - N/2\mathbf{v})$, and

$$U\cdot\mathcal{H}(\boldsymbol{\theta}) = \mathcal{H}(A\cdot\boldsymbol{\theta} - N/2\mathbf{v}). \tag{3.20}$$

From (3.20), it follows that $\mathcal{H}(\boldsymbol{\theta})$ is invariant under U, and therefore, that $U(\boldsymbol{\theta})$ is defined, only for

$$A\cdot\boldsymbol{\theta} - \frac{N}{2}\mathbf{v} = \boldsymbol{\theta} \mod 1 \tag{3.21}$$

If N is even, or if A has one of the checkerboard forms

$$\begin{pmatrix}\text{even} & \text{odd}\\ \text{odd} & \text{even}\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}\text{odd} & \text{even}\\ \text{even} & \text{odd}\end{pmatrix}. \tag{3.22}$$

described by Hannay and Berry [13] (a condition equivalent to $AA^T = I \mod 2$), then $N/2\,\mathbf{v} = \mathbf{0} \mod 1$, and (3.21) simplifies to

$$A\cdot\boldsymbol{\theta} = \boldsymbol{\theta} \mod 1. \tag{3.23}$$

In this case, $U(\boldsymbol{\theta})$ is defined only for phases $\boldsymbol{\theta}$ which are fixed points of the cat map $\boldsymbol{\theta} \mapsto A\cdot\boldsymbol{\theta}$ associated with the topology of the torus map $\boldsymbol{\phi}$. The condition (3.21) and the special case (3.23) constitute our main results.

If A is the identity, as is the case for time-one flows of time-dependent torus Hamiltonians, then all values of $\boldsymbol{\theta}$ are allowed. In this case, the eigenstates $|\psi_j(\boldsymbol{\theta})\rangle$ of $U(\boldsymbol{\theta})$ are defined for all $\boldsymbol{\theta}$, and in the absence of degeneracies depend continuously on $\boldsymbol{\theta}$, up to an overall phase factor. The family of eigenstates $|\psi_j(\boldsymbol{\theta})\rangle$ is characterized by its *Chern number*, an integer given by $1/2\pi$ times the $\boldsymbol{\theta}$-integral of the Berry curvature

$$V_j(\boldsymbol{\theta}) = \frac{\partial^2}{\partial\theta_1'\partial\theta_2''}\mathrm{Im}\,\left\langle T^\dagger(\boldsymbol{\theta}'/N)\cdot\psi_j(\boldsymbol{\theta})\,\middle|\,T^\dagger(\boldsymbol{\theta}''/N)\cdot\psi_j(\boldsymbol{\theta}'')\right\rangle\Bigg|_{\boldsymbol{\theta}'=\boldsymbol{\theta}''=\boldsymbol{\theta}}. \tag{3.24}$$

In this expression, the states $T^\dagger(\boldsymbol{\theta}'/N)\,|\psi_j(\boldsymbol{\theta}')\rangle$ and $T^\dagger(\boldsymbol{\theta}''/N)\,|\psi_j(\boldsymbol{\theta}'')\rangle$ both belong to $\mathcal{H}(\mathbf{0})$, and $\langle\cdot\,|\cdot\rangle$ denotes the (renormalized) inner product on $\mathcal{H}(\mathbf{0})$. As shown by Leboeuf et al. [19], nonzero Chern numbers are characteristic of eigenstates delocalized in both position and momentum, and are signatures of underlying classical chaos.

For A not equal to the identity, the quantum map is in general defined only for a finite set of values of $\boldsymbol{\theta}$ (including, if A has the checkerboard form (3.22), $\boldsymbol{\theta} = \mathbf{0}$). This is the case, for example, for hyperbolic cat maps (Section 4.1) and their perturbations (Section 4.2). It would be interesting to know if, even in this case, the dependence of the eigenstates on the discrete phases $\boldsymbol{\theta}$ carries any information concerning the localization of the wavefunction and the chaoticity of the underlying classical map.

One might ask whether by relaxing the kinematic condition (3.15), so that $UT_H(\mathbf{m}) = T_H(A\cdot\mathbf{m})U$ holds semiclassically rather than exactly, one could construct a quantized torus map for every $\boldsymbol{\theta}$, regardless of A. Let us describe two possible constructions, both of which have unsatisfactory features. For simplicity let us assume that A is of the form (3.22).

First, we consider

$$U^{(i)}(\boldsymbol{\theta}) = T^\dagger((A\cdot\boldsymbol{\theta}-\boldsymbol{\theta})/N)U\big|_{\mathcal{H}(\boldsymbol{\theta})}. \tag{3.25}$$

For $A = I$ this is just $U(\boldsymbol{\theta})$, but for $A \neq I$, (3.13) and (3.20) imply that $U^{(i)}(\boldsymbol{\theta})$ leaves $\mathcal{H}(\boldsymbol{\theta})$ invariant – the translation $T^\dagger((A\cdot\boldsymbol{\theta}-\boldsymbol{\theta})/N)$ compensates for the change in $\boldsymbol{\theta}$ induced by U. Since the translation is semiclassically small (ie, of order $1/N$), it does not alter the dynamics in the classical limit. However, the scheme (3.25) prescribes physically distinct maps for $\boldsymbol{\theta}+\mathbf{m}$ and $\boldsymbol{\theta}$; indeed, from (3.12), $U^{(i)}(\boldsymbol{\theta}+\mathbf{m}) = \text{phase factor} \times T^\dagger(\mathbf{m}/N)U^{(i)}(\boldsymbol{\theta})$, so that $U^{(i)}(\boldsymbol{\theta}+\mathbf{m})$ is not, in general, unitarily related to $U^{(i)}(\boldsymbol{\theta})$.

As a second example, consider

$$U^{(ii)}(\boldsymbol{\theta}) = T(\boldsymbol{\theta}/N)UT^\dagger(\boldsymbol{\theta}/N)\big|_{\mathcal{H}(\boldsymbol{\theta})}. \tag{3.26}$$

This too leaves $\mathcal{H}(\boldsymbol{\theta})$ invariant for all $\boldsymbol{\theta}$ and has the correct dynamics in the classical limit. Moreover, $U^{(ii)}(\boldsymbol{\theta}+\mathbf{m})$ and $U^{(ii)}(\boldsymbol{\theta})$ are unitarily related. However, in this case, the $\boldsymbol{\theta}$-dependence is trivial; the eigenphases of $U^{(ii)}(\boldsymbol{\theta})$ are independent of $\boldsymbol{\theta}$ while the Berry curvatures (3.24) vanish.

These two examples suggest the possibility that, regardless of the quantization prescription, continuous families of quantum maps $U(\boldsymbol{\theta})$ with nontrivially periodic $\boldsymbol{\theta}$-dependence might exist only for $A = I$.

3.3 Quantization of the lifted map

By a quantization of the lifted map, we mean a unitary operator U defined on the Hilbert space $L^2(\mathbb{R})$ of square-integrable wavefunctions $\psi(Q)$ whose action in the classical limit is given by $\mathbf{\Phi}(\mathbf{Z})$. We now give an explicit prescription, in which U satisfies the condition (3.15), and discuss the class of maps to which the prescription can be applied.

Let

$$\mathbf{\Phi}_A(\mathbf{Z}) = A \cdot \mathbf{Z} \tag{3.27}$$

be the linear canonical map generated by the matrix A associated to $\mathbf{\Phi}$. Then (2.2) implies that the map $\mathbf{\Phi}_1$, defined by

$$\mathbf{\Phi}_1 = \mathbf{\Phi}_A^{-1} \circ \mathbf{\Phi}, \tag{3.28}$$

satisfies

$$\mathbf{\Phi}_1(\mathbf{Z} + \mathbf{m}) = \mathbf{\Phi}_1(\mathbf{Z}) + \mathbf{m}. \tag{3.29}$$

As $\mathbf{\Phi} = \mathbf{\Phi}_A \circ \mathbf{\Phi}_1$, we quantize $\mathbf{\Phi}$ by taking

$$U = U_A U_1, \tag{3.30}$$

where U_A and U_1 are quantizations of the factors $\mathbf{\Phi}_A$ and $\mathbf{\Phi}_1$, respectively. Thus, the quantization problem is reduced to the separate consideration of linear canonical maps and nonlinear canonical maps which commute with lattice translations.

It is well known how to quantize the linear canonical map ϕ_A. In the coordinate representation, U_A is given by

$$\langle q_2 | \, U_A \, | q_1 \rangle = \left(\frac{N}{ib} \right)^{1/2} \exp \left(\frac{i\pi N}{b} (a q_1^2 - 2 q_1 q_2 + d q_2^2) \right) \tag{3.31}$$

(the limit $b \to 0$ gives the correct delta-function kernal). Given this expression, one can verify explicitly that

$$U_A T_H(\mathbf{m}) = T_H(A \cdot \mathbf{m}) U_A. \tag{3.32}$$

The quantization (3.31) is exact, in the sense that U_{AB} is equal to $U_A U_B$, up to a sign factor. Indeed, (3.31) defines the metaplectic operators, the group of unitary operators generated by Hamiltonians quadratic in $\hat{\mathbf{Z}}$. These constitute a projective representation of the group $SL(2, \mathbb{R})$ of linear canonical maps. This point of view is discussed further in [20], in which the kinematic condition (3.32) is seen to follow from the multiplication law for the inhomogeneous metaplectic operators (these are generated by Hamiltonians containing linear as well as quadratic terms).

Next, we consider $\mathbf{\Phi}_1$. Its quantization is straightforward if $\mathbf{\Phi}_1$ is the time-one flow of a Hamiltonian $H(\mathbf{Z}, t)$ periodic under lattice translations. Explicitly, we mean that $\mathbf{\Phi}_1(\mathbf{Z}) = \mathbf{\Psi}(\mathbf{Z}, 1)$, where $\mathbf{\Psi}(\mathbf{Z}, t)$ satisfies Hamilton's equations,

$$\partial_t \mathbf{\Psi}(\mathbf{Z}, t) = J \cdot \nabla_{\mathbf{Z}} H(\mathbf{\Psi}(\mathbf{Z}, t), t), \tag{3.33}$$

with initial condition $\mathbf{\Psi}(\mathbf{Z}, 0) = \mathbf{Z}$. In (3.33), $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$H(\mathbf{Z} + \mathbf{m}, t) = H(\mathbf{Z}, t). \tag{3.34}$$

The unitary operator U_1 is then taken to be the time-one evolution generated by the Weyl quantization $\hat{H}(t)$ of the classical Hamiltonian $H(\mathbf{Z}, t)$. That is, $U_1 = W(1)$, where $W(t)$ satisfies the Schrödinger equation

$$2\pi i N \dot{W}(t) = \hat{H}(t) W(t) \tag{3.35}$$

with initial condition $W(0) = I$. The Weyl-quantized Hamiltonian is given explicitly by

$$\hat{H}(t) = \int\int H(\mathbf{Z}, t) \delta(\mathbf{Z} - \hat{\mathbf{Z}})\, d^2\mathbf{Z}, \tag{3.36}$$

where the delta-function $\delta(\mathbf{Z} - \hat{\mathbf{Z}})$ is defined by

$$\delta(\mathbf{Z} - \hat{\mathbf{Z}}) = N^2 \int\int \exp\left(2\pi i N \mathbf{Z}' \wedge \mathbf{Z}\right) T_H(\mathbf{Z}')\, d^2\mathbf{Z}'. \tag{3.37}$$

From (3.2) we get that $T_H(\mathbf{m})\delta(\mathbf{Z} - \hat{\mathbf{Z}})T_H^\dagger(\mathbf{m}) = \delta(\mathbf{Z} - \mathbf{m} - \hat{\mathbf{Z}})$. Thus, if $H(\mathbf{Z}, t)$ is periodic under lattice translations, its Weyl quantization is similarly periodic, ie $T_H(\mathbf{m})\hat{H}(t)T_H^\dagger(\mathbf{m}) = \hat{H}(t)$. This implies in turn that $W(t)$ is periodic under lattice translations, so that U_1 satisfies

$$U_1 T_H(\mathbf{m}) = T_H(\mathbf{m}) U_1. \tag{3.38}$$

Together with the corresponding result (3.32) for the quantized linear map, this implies that the quantized lift $U = U_A U_1$ satisfies the kinematic condition (3.15), as claimed.

It remains to consider when $\mathbf{\Phi}_1$ can be expressed as the time-one flow of a periodic Hamiltonian. First, we note that this is not always possible; a necessary condition is that $\mathbf{\Phi}_1$ be *translation-free*, by which we mean that $\mathbf{\Phi}_1$ must leave the centre-of-mass of the unit square invariant. Explicitly, letting

$$\mathbf{\Delta} = \int_0^1 \int_0^1 (\mathbf{\Phi}_1(\mathbf{Z}_0) - \mathbf{Z}_0)\, d^2\mathbf{Z}_0, \tag{3.39}$$

we require that

$$\mathbf{\Delta} = \mathbf{0}. \tag{3.40}$$

(3.40) follows from differentiating the quantity

$$\mathbf{\Delta}(t) = \int_0^1 \int_0^1 (\mathbf{\Psi}(\mathbf{Z}_0, t) - \mathbf{Z}_0)\, d^2\mathbf{Z}_0 \tag{3.41}$$

to obtain

$$\begin{aligned} \dot{\mathbf{\Delta}}(t) &= J \cdot \textstyle\int_0^1 \int_0^1 \nabla_{\mathbf{Z}} H(\mathbf{\Psi}(\mathbf{Z}_0, t), t) d^2\mathbf{Z}_0 \\ &= J \cdot \textstyle\int \int_{\Sigma_t} \nabla_{\mathbf{Z}} H(\mathbf{Z}, t) d^2\mathbf{Z}, \end{aligned} \tag{3.42}$$

where Σ_t is the image under $\mathbf{\Psi}(\mathbf{Z}, t)$ of the unit square (we are using the fact that the flow $\mathbf{\Psi}(\mathbf{Z}, t)$ is area-preserving). Because $H(\mathbf{Z}, t)$ is periodic under lattice translations, the integral of its gradient over the unit square vanishes. The integral of $\nabla_{\mathbf{Z}} H$ also vanishes over any domain, such as Σ_t, which is mapped diffeomorphically onto the two-torus by the projection $\mathbf{Z} \mapsto (\mathbf{Z} \bmod 1)$. Thus $\dot{\mathbf{\Delta}}(t) = 0$, implying that $\mathbf{\Delta}(t) = 0$ for all t ($\mathbf{\Delta}(0)$ vanishes trivially), and in particular for $t = 1$.

The nonintegral translations $\mathbf{Z} \mapsto \mathbf{Z} + \mathbf{Z}_0$ (ie, the lifts of nontrivial translations on the torus) clearly do not satisfy (3.40). This is consistent with the fact that the Hamiltonians $H(\mathbf{Z}) = Q_0 P - P_0 Q$, whose time-one flows generate them, are clearly not periodic under lattice translations.

It turns out that the condition (3.40) is sufficient as well; Conley and Zehnder [8] have shown that any translation-free canonical map of the plane which commutes with lattice translations can be realized as the time-one flow of a periodic Hamiltonian. Apparently, it is not known whether an analogous result holds in higher dimensions.

Of course, if $\mathbf{\Phi}_1$ is not translation-free, then the map $\mathbf{\Phi}_1 \circ \boldsymbol{T}^{\mathrm{cl}}(-\mathbf{\Delta})$ is. Thus, every lifted torus map has a unique decomposition of the form

$$\mathbf{\Phi} = \mathbf{\Phi}_A \circ \mathbf{\Psi}(1) \circ \boldsymbol{T}^{\mathrm{cl}}(\mathbf{\Delta}), \tag{3.43}$$

where $\mathbf{\Psi}(1)$ is the time-one flow of a periodic Hamiltonian, and $\mathbf{\Delta}$ is given by (3.39). The quantization prescription embodied in (3.30), which preserves the kinematic condition (3.15), may be applied provided $\mathbf{\Delta} = \mathbf{0}$, and therefore applies to all canonical torus maps, modulo translations.

4 Examples

We now discuss three representative examples which illustrate different aspects of the general analysis of Section 3, and the results derived there. These are the cat maps, their (nonlinear) perturbations, and the kicked Harper map.

4.1 Cat maps

We have already noted that for the cat map $\phi(\mathbf{z}) = T \cdot \mathbf{z} \mod 1$, where T is an integer matrix with unit determinant, the matrix A of (2.2) is just T itself (Section 2), and that the kinematic condition (3.15) is a direct consequence of the fact that the corresponding quantum propagator is a metaplectic operator (Section 3.3). Our main results, (3.21) and (3.23), then follow immediately, and coincide with those found previously in this case [18, 10]. There is, however, a more elementary derivation of (3.21) and (3.23) for the cat maps which illustrates clearly the kinematic and dynamical origins of these conditions. This we now describe.

The Wigner function for a pure quantum state is a real function on phase space, defined by

$$W(Q,P) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \psi(Q+Q')\psi^*(Q-Q')^* e^{\frac{-i2PQ'}{\hbar}} dQ'. \tag{4.1}$$

It has two important properties that we shall use. The first, which is kinematic, is that integrating the Wigner function with respect to P gives the probability density in the position representation $|\psi(Q)|^2$, and vice-versa when the integration is with respect to Q. The second, which is dynamical, is that when a quantum wavefunction is propagated by a unitary transformation corresponding to the quantization of a linear canonical map, the associated Wigner function itself maps classically [13].

The Wigner function for an arbitrary state $|\psi(\boldsymbol{\theta})\rangle \in \mathcal{H}(\boldsymbol{\theta})$ can be obtained by expanding its wavefunction $\psi(Q;\boldsymbol{\theta})$ in (4.1) as a linear combination of basis vectors (3.9):

$$W(Q,P) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\delta\left(Q - \frac{\theta_1}{N} - \frac{j}{2N}\right)\delta\left(P - \frac{\theta_2}{N} - \frac{k}{2N}\right) \tag{4.2}$$

where

$$c_{j+2N,k} = c_{j,k+2N} = c_{j,k}.$$

This represents a $2N \times 2N$ periodic 'δ-brush' in the classical phase space, and is exactly periodic. It has delta-functions on each of the $(2N)^2$ points whose coordinates have the form

$$\left(\frac{\text{integer}}{2N}+\frac{\theta_1}{N},\frac{\text{integer}}{2N}+\frac{\theta_2}{N}\right). \tag{4.3}$$

However, the weightings of these delta-functions are not all independent: the four situated at the corners of any square of side N spacings have weightings which differ at most by a sign [13]:

$$\begin{aligned} W(Q,P) &= (-1)^{2N(P-\frac{\theta_2}{N})}W\left(Q+\frac{1}{2},P\right) = (-1)^{2N(Q-\frac{\theta_1}{N})}W\left(Q,P+\frac{1}{2}\right) \\ &= (-1)^{[2(Q-\frac{\theta_1}{N})+2(P-\frac{\theta_2}{N})+1]N}W\left(Q+\frac{1}{2},P+\frac{1}{2}\right). \end{aligned}$$

Any two on a given side that extends to intersect either the Q- or P-axis at a 'halfway' coordinate

$$\frac{\text{odd}}{2N}+\theta_i \quad i=1,2 \tag{4.4}$$

must differ in sign. This guarantees that the projection integrals of the Wigner function are zero unless either Q or P are of the form

$$\frac{\text{integer}}{N}+\theta_i \quad i=1,2,$$

as must be the case for the first (kinematic) of the two general properties listed above to hold.

The second (dynamical) general property of the Wigner function implies that the Wigner lattice (4.3) be invariant under the action of the cat map T. Let $\mathbf{m}$ and $\mathbf{n}$ denote two integer vectors, and let $T\cdot\mathbf{m}=\mathbf{m}'$. This condition then becomes

$$T\cdot\frac{1}{2N}\mathbf{m}+T\cdot\frac{1}{N}\boldsymbol{\theta}=\frac{1}{2N}\mathbf{m}'+\frac{1}{N}\boldsymbol{\theta}+\frac{1}{2N}\mathbf{n}.$$

If the integers in $\mathbf{m}$ are both odd, then those of $\mathbf{m}'+\mathbf{n}$ must also be odd - this is necessary and sufficient to ensure that the projection of the Wigner function onto the Q- and P-axes at 'halfway' coordinates (4.4) remains zero after the mapping - and so (4.1) reduces to the form (3.21) with $A=T$, as claimed.

Another direct proof of (3.21) for the cat maps follows from the Hannay-Berry construction of the quantum propagator [13]. Their method involves averaging (3.31) over all positions equivalent to (3.7) under lattice translations. A lengthy but straightforward calculation confirms that this average is zero unless (3.21) holds.

4.2 Perturbed cat maps

Perturbed cat maps are maps whose lifts are of the form

$$C(\mathbf{Z}) = T \cdot \mathbf{Z} + \mathbf{F}(\mathbf{Z}), \tag{4.5}$$

where $\mathbf{F}(\mathbf{Z})$ is a periodic function on the torus. Their quantization has been the subject of a number of studies [3, 6, 5, 9], because they appear to be free of the number-theoretical peculiarities of the quantum cat maps themselves.

It follows from the periodicity of $\mathbf{F}(\mathbf{Z})$ that

$$C(\mathbf{Z} + \mathbf{m}) = C(\mathbf{Z}) + T \cdot \mathbf{m}.$$

Comparing with (2.2) then implies immediately that $A = T$ in (3.21). Thus the condition on the quantum phase vectors is unaffected by the perturbation (i.e. is independent of $\mathbf{F}(\mathbf{Z})$).

It is interesting to compare this with Anosov's theorem [1], which implies that if

$$||\mathbf{F}|| = \max\left(\frac{|\partial \mathbf{F}/\partial \mathbf{Z} \cdot \mathbf{Z}|}{|\mathbf{Z}|}\right) < 1 - \lambda,$$

where λ is the smaller eigenvalue of T, then C is topologically equivalent to T and can be written

$$C = H^{-1} \circ T \circ H = T \circ T^{-1} \circ H^{-1} \circ T \circ H = T \circ P = T \circ (I + \epsilon)\,,$$

where H is a homeomorphism. Thus, as regards the conditions on the phase-vector $\boldsymbol{\theta}$, the structural stability of the cat maps may be viewed as persisting under quantization, even to the extent that it is independent of the size of the perturbation.

4.3 The kicked Harper map

To conclude, we discuss, briefly, a much-studied example, the kicked Harper map, that commutes with lattice translations (i.e. $A = I$ in (2.2)), and which therefore can be quantized for all $\boldsymbol{\theta}$. This map is associated with the kicked Harper Hamiltonian

$$H(Q, P, t) = -V_2 \cos(2\pi P) - V_1 \cos(2\pi Q) K(t), \tag{4.6}$$
$$K(t) = \tau \sum_n \delta(t - n\tau),$$

(for a review see Lebouf et al. [19]). Defining $\gamma_i = 2\pi V_i \tau$, the classical map, obtained by integrating the equations of motion between successive kicks, is given by

$$Q_{n+1} = Q_n + \gamma_2 \sin(2\pi P_{n+1}) \tag{4.7a}$$
$$P_{n+1} = P_n - \gamma_1 \sin(2\pi Q_n). \tag{4.7b}$$

It follows from the general arguments of Section 3.3 that A in (2.2) must be the 2×2 identity matrix, because the map is derived from the time-one flow of a Hamiltonian that is periodic under lattice translations in Q and P. This may also be verified directly using the explicit expressions (4.7). The quantum propagator thus exists for all boundary conditions $\boldsymbol{\theta}$, and takes the general form

$$U = \exp[iN\pi\gamma_1 \cos(2\pi\hat{Q})] \exp[i\pi N\gamma_2 \cos(2\pi\hat{P})]. \tag{4.8}$$

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