The Cycle Time Vector of D-A-D Functions

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It is of interest to find conditions under which the cycle time vector, $\mathcal{X}(F) = \lim_{k\to\infty} (F^k(x)/k) \in \mathbf{R}^n$, of a topical function F, that is a function from $\mathbf{R}^n$ into itself which is homogenous and nonexpansive in the l_∞ norm, exists. For a class of topical functions associated with matrix scaling problems, we show that the cycle time vector exists and that it can be computed from the spectral radius of the function.

Introduction

Given a nonnegative $m \times n$ matrix A and vectors $r \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ with positive entries, M.V. Menon and Hans Schneider defined in [7] a nonlinear operator $T = T(A, r, c)$ on the positive cone of $\mathbb{R}^n$. They determined the spectrum of T as well as the zero pattern of its eigenvectors. The operator T was constructed so that an associated matrix scaling problem has a solution if and only if T has a strictly positive eigenvector. The underlying scaling problem is the matrix D-A-D problem for (A, r, c). This problem asks if there are diagonal matrices D_1 and D_2, with positive entries in the diagonal such that D_1AD_2 has row sums r_i, $i = 1, \ldots, m$ and column sums c_j, $j = 1, \ldots, n$. An alternative method for solving a more general form of this scaling problem can be found in [10].

Our interest in the operator T (whose definition is given in the next section) was triggered by the fact that it maps the interior of the positive cone into itself, and it is homogeneous and order preserving (see [7] for more details). By applying on T the functional $\mathcal{E}$, where $\mathcal{E}(T) = \log(T(\exp))$, we can transport it from the positive cone into the whole of $\mathbb{R}^n$. We shall call this new map $\mathcal{E}(T)$, a D-A-D function.

In general a function $F : \mathbb{R}^n \to \mathbb{R}^n$ which is homogeneous and monotonic, is called topical ([4]). Topical maps belong to the class of nonexpansive functions (in the l_∞ norm) and appear in different areas of mathematics. Among others in the theory of nonnegative matrices, Bellman operators of games and of Markov decision process, mathematical biology and discrete event systems. The reader is referred to [4], [6], [5], [1], [8], [9].

The existence of fixed points is a question of general interest. In the case of topical functions the approach to answering this question is dynamical rather than metric. One reason for taking this approach is that the existence of a generalized fixed point of F implies that the cycle time vector $\chi(F) = \lim_{k\to\infty} \frac{F^k(x)}{k}$ (for some $x \in \mathbb{R}^n$) exists and has the same value in each component. In particular for min-max functions, a class of topical maps, it turns out that the inverse implication also holds ([1]).

In this paper using the characterization of the spectrum of T, we compute the cycle time vector of the topical map $\mathcal{E}(T)$, (Theorem 8). We see that as in the case of $\mathcal{E}(A)$, where A is a square nonnegative matrix ([5]), χ can be thought of as a vector generalization of the spectral radius, (Proposition 3).

It is hoped that this present work will bring us one step closer to answering the question of characterizing the cycle time vector of topical functions and the conditions under which it exists.

Preliminaries

We denote the positive cone of $\mathbb{R}^n$ by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$ and its interior by $(\mathbb{R}^n_+)^\circ = \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}$. If x, y are vectors in $\mathbb{R}^n$ we say that $x \leq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$. For $h \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $x+h$ is the vector whose ith coordinate is $x_i + h$. A vector x is said to be constant if all its coordinates are equal. In this case we shall write $x = c$, $c \in \mathbb{R}$.

A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called topical ([4]) if it satisfies the following properties:

$$F(x+h) = F(x) + h, \qquad x \in \mathbb{R}^n, \qquad h \in \mathbb{R} \quad (\textit{homogeneity}) \tag{1}$$

and

$$x \leq y \Longrightarrow F(x) \leq F(y), \qquad x, y \in \mathbb{R}^n \quad (\textit{monotonicity}) \tag{2}$$

It follows by Proposition 2 in [2] that topical finctions are nonexpansive in the l_∞ norm. The cycle time vector $\chi(F) \in \mathbb{R}^n$ of a topical function F, ([4]), is defined as the $\lim_{k\to\infty} \frac{F^k(x)}{k}$, if this limit exist for some $x \in \mathbb{R}^n$, and is undefined otherwise.

Applying the nonexpansiveness of F, it is easy to see that if $\lim_{k\to\infty} \frac{F^k(x)}{k}$ exists for some x in $\mathbb{R}^n$ then it exists everywhere and has thc same value.

Recall that a topical function F has a generalized fixed point if there are $x \in \mathbb{R}^n$ and $h \in \mathbb{R}$ such that $F(x) = x + h$. Since $F - h$ is also a topical function we may talk about fixed points instead of generalized ones. It is clear that if $F(x) = x + h$ then the cycle time vector of F exists and it is constant, $\chi(F) = h$.

Define $\exp : \mathbb{R}^n \longrightarrow (\mathbb{R}^n_+)^\circ$ and $\log : (\mathbb{R}^n_+)^\circ \longrightarrow \mathbb{R}^n$ componentwise, i.e. if x is a vector, $\exp(x)_i = \exp x_i$ and $\log(x)_i = \log x_i$. Let Φ be a map of $(\mathbb{R}^n_+)^\circ$ into itself. We define the functional $\mathcal{E}(\Phi) : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathcal{E}(\Phi) = \log(\Phi(\exp))$. Since $\mathcal{E}(\Phi\Psi) = \mathcal{E}(\Phi)\mathcal{E}(\Psi)$, the dynamic behaviours of Φ and $\mathcal{E}(\Phi)$ are equivalent.

We shall say that a vector $x = (x_1, \ldots, x_n)$ is positive, written $x > 0$ (respectively strictly positive, written $x \gg 0$) if $x_i \geq 0$ for all $i = 1, \ldots, n$ but $x \neq 0$ (resp. $x_i > 0$ for all $i = 1, \ldots, n$). Also following the notation in [7], A will always be an $m \times n$ nonnegative matrix, with no zero row or column and $r \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ will denote strictly positive vectors. Moreover the triple (A, r, c) is said to be a matrix-rowsum-columnsum triple, or for short an mrc.

For a strictly positive vector p in $\mathbb{R}^k$, define $L_p : (\mathbb{R}^k_+)^\circ \longrightarrow (\mathbb{R}^k_+)^\circ$ to be the function

$$L_p(w_1, \ldots, w_k) = (\frac{p_1}{w_1}, \ldots, \frac{p_k}{w_k}).$$

Let (A, r, c) be an mrc. The operator $T = T(A, r, c) : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$ is defined by

$$T = T(A, r, c) = L_c A^T L_r A$$

where A^T is the transpose of A. Note here that we use the conventions $0^{-1} = \infty$, $\infty^{-1} = 0$, $\infty + \infty = \infty$, $0 \cdot \infty = 0$ and $a \cdot \infty = \infty$, for $a > 0$.

Observe that T has the following properties ([7])

1. T is homogeneous on $\mathbb{R}^n_+$

$$T(\lambda x) = \lambda Tx, \quad \lambda \geq 0$$

2. T is monotonic on $\mathbb{R}^n_+$

$$x \leq y \Longrightarrow Tx \leq Ty$$

3. T maps $(\mathbb{R}^n_+)^\circ$ into itself and is continuous on $\mathbb{R}^n_+$.

Definition 1. Let (A, r, c) be an mrc and $T = T(A, r, c)$. The function $\mathcal{E}(T) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be a D-A-D function. It is clear that $\mathcal{E}(T)$ is a topical function.

Following Menon and Schneider ([7]), an mrc (A, r, c) is said to be consistent if for all nonempty proper subsets I, J of M, N respectively, with $A[I'|J] = 0$ the following are satisfied

$$\omega(I, J) < \omega(M, N) \quad \text{if} \quad A[I|J'] \neq 0 \tag{3}$$

and

$$\omega(I, J) = \omega(M, N) \quad \text{if} \quad A[I|J'] = 0. \tag{4}$$

Where $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$, I', J' are the complements of I, J in M, N respectively and $\omega(I, J) = \frac{\Sigma_{j \in J} c_j}{\Sigma_{i \in I} r_i}$. Also $A[I|J]$ denotes the submatrix of A consisting of all elements a_{ij} where $i \in I$ and $j \in J$.

By Lemma 2.3 in [7], given an mrc, (A, r, c), the operator $T = T(A, r, c)$ has a largest positive eigenvalue

$$\rho(T) = \sup\{\lambda : \exists x > 0, Tx \geq \lambda x\}. \tag{5}$$

Furthermore if (A, r, c) is consistent then $\rho(T) = \omega(M, N)$. Before we state the next theorem recall that a matrix A is said to be indecomposable if $A[I'|J] = 0$ implies that $A[I|J'] \neq 0$. Also $A[I|J]$ is a maximal zero submatrix of A if $A[I|J] = 0$ and $A[I_1|J_1] \neq 0$ when $I_1 \times J_1 \supset I \times J$. For $x \in \mathbb{R}^n$ and $J \subset N$ we denote by x_J the corresponding subvector of x ($\subset$ denotes proper containment).

Theorem 2. [7]. Let (A, r, c) be an mrc. The spectrum of $T = T(A, r, c)$ consists of all λ for which there are nonempty subsets I, J of M, N respectively such that

either (i) $\quad I \times J = M \times N, \quad$ or

(ii) $\quad \emptyset \subset I \subset M, \quad \emptyset \subset J \subset N \quad$ and $\quad A[I'|J] \quad$ is a maximal zero submatrix of A,

$$(A[I|J], r_I, c_J) \quad \text{is consistent}$$

and

$$\lambda = \omega(I, J).$$

If both these conditions are satisfied then there is an associated eigenvector x with $x_J \gg 0$ and (for $J \subset N$) $x_{J'} = 0$.

The following notation will also be needed. Let $\{I_\alpha\}_{\alpha=1}^{\sigma}$ and $\{J_\alpha\}_{\alpha=1}^{\sigma}$ be partitions of M and N respectively. We denote $A_\alpha = A[I_\alpha|J_\alpha]$, $r_{I_\alpha} = r_\alpha$, $c_{J_\alpha} = c_\alpha$ and $T_\alpha = T(A[I_\alpha|J_\alpha], r_{I_\alpha}, c_{J_\alpha})$ for $\alpha = 1, \ldots, \sigma$.

Main Results

The computation of the cycle time vector of $T = T(A, r, c)$ in the case where (A, r, c) is a consistent mrc, is straightforward and it is given in Proposition 3. Moreover if A is decomposable, that is $A = A_1 \oplus A_2 \oplus \ldots \oplus A_\sigma$, where each A_α is indecomposable, then it is easy to check that $\chi(\mathcal{E}(T)) = \chi(\mathcal{E}(T_1)) \oplus \chi(\mathcal{E}(T_2)) \oplus \ldots \oplus \chi(\mathcal{E}(T_\sigma))$. Therefore our interest lies in the case where (A, r, c) is non-consistent and A is indecomposable. The main result is that in the latter case, we can "partition" A (by permuting rows and columns) as follows

$$A = \begin{bmatrix} A_1 & B_{12} & \ldots & B_{1\sigma} \\ & A_2 & \ldots & B_{2\sigma} \\ & 0 & \ddots & \vdots \\ & & & A_\sigma \end{bmatrix},$$

where $(A_\alpha, r_\alpha, c_\alpha)$ is consistent and A_α is indecomposable for each $\alpha = 1, \ldots, \sigma$. In Lemmas 4 and 6 we show that this "partition" is unique up to a permutation and is so that $\omega(I_1, J_1) \geq \ldots \geq \omega(I_\sigma, J_\sigma)$. Finally in Theorem 8 we prove that $\chi(\mathcal{E}(T))_{J_\alpha} = \log \omega(I_\alpha, J_\alpha)$, for $\alpha = 1, \ldots, \sigma$.

Proposition 3. Let (A, r, c) be an mrc. Then $\mathcal{E}(T)$, where $T = T(A, r, c)$, has a fixed point if and only if (A, r, c) is consistent. In this case the cycle time vector of $\mathcal{E}(T)$ is constant and $\chi(\mathcal{E}(T)) = \log \omega(M, N)$.

Proof: By Theorem 3.5 in [7], (A, r, c) is consistent if and only if $T = T(A, r, c)$ has a strictly positive eigenvector with corresponding eigenvalue $\omega(M, N)$. It follows immediately from the definitions that $Tx = \omega(M, N)x$, $x \gg 0$, is equivalent to the existence of a fixed point of $\mathcal{E}(T)$, namely $\mathcal{E}(T)(\log x) = \log \omega(M, N) + \log x$. Moreover by applying the definition of the cycle time vector we get that $\chi(\mathcal{E}(T)) = \log \omega(M, N)$. □

Since in the case where (A, r, c) is consistent, $\omega(M, N)$ is the largest eigenvalue of $T = T(A, r, c)$ we see that χ can be thought of as a vector generalization of the spectral radius. Note here that if (A, r, c) is a consistent mrc and $\emptyset \subset I \subset M$ and $\emptyset \subset J \subset N$ are such that

$A[I'|J] = 0$ then $\omega(I,J) = \omega(M,N)$ implies that $A[I|J'] = 0$ and that $(A[I|J], r_I, c_J)$ is also consistent.

It will be useful to observe that in order to check consistency it is enough to consider those $\emptyset \subset I \subset M$ and $\emptyset \subset J \subset N$ for which $A[I'|J]$ is a maximal zero submatrix of A.

Lemma 4. Let (A, r, c) be a non-consistent mrc and A be indecomposable. Suppose $\emptyset \subset I_1 \subset M$ and $\emptyset \subset J_1 \subset N$, are such that $A[I_1'|J_1]$ is a maximal zero submatrix of A, $(A[I_1|J_1]; r_{I_1}, c_{J_1})$ is consistent and $\omega(I_1, J_1)$ is the maximum eigenvalue of $T = T(A, r, c)$. If $\omega(I_2, J_2)$ is the maximum eigenvalue of $T' = T(A[I_1'|J_1'], r_{I_1'}, c_{J_1'})$, then

$$\omega(I_1, J_1) \geq \omega(I_2, J_2) \tag{6}$$

Proof: By the indecomposability of A it is true that $A[I_1|J_1'] \neq 0$. Denote $I_3 = I_1' \backslash I_2$, $J_3 = J_1' \backslash J_2$ and $(A[I_1'|J_1'], r_{I_1'}, c_{J_1'}) = (A_1', r_1', c_1')$.

If (A_1', r_1', c_1') is consistent then $\omega(I_2, J_2) = \omega(I_1', J_1')$. By the assumption we have that $\omega(I_1, J_1) \geq \omega(M, N)$ which implies the inequality, $\omega(I_1, J_1) \geq \omega(I_1', J_1')$.

Assume now that (A_1', r_1', c_1') is non-consistent. Then by Theorem 2, I_2, J_2 are nonempty proper subsets of I_1', J_1' respectively. Furthermore $A[I_3|J_2]$, is a maximal zero submatrix of A_1', (A_2, r_2, c_2) is consistent and $\omega(I_2, J_2) \geq \omega(I_1', J_1')$. Let $L = I_1 \cup I_2$ and $K = J_1 \cup J_2$. Consider the following two cases:

Case 1. $(A[L|K], r_L, c_K)$ is consistent. In this case $\omega(L, K)$ is an eigenvalue of $T(A, r, c)$ and thus $\omega(I_1, J_1) \geq \omega(L, K)$. On the other hand the consistency of $(A[L|K], r_L, c_K)$ implies that $\omega(I_1, J_1) \leq \omega(L, K)$. Therefore we must have that $\omega(I_1, J_1) = \omega(L, K)$ and consequently $\omega(I_1, J_1) = \omega(I_2, J_2)$ and $A[I_1|J_2] = 0$.

Case 2. $(A[L|K], r_L, c_K)$ is non-consistent. If $A[I_1|J_2] = 0$ then $A[I_1 \cup I_3|J_2]$ is a maximal zero submatrix of A and since (A_2, r_2, c_2) is consistent, $\omega(I_2, J_2)$ is an eigenvalue of $T(A, r, c)$. Therefore $\omega(I_1, J_1) \geq \omega(I_2, J_2)$.

Now assume that $A[I_1|J_2] \neq 0$. Then since $\omega(I_1, J_1)$ is also the maximum eigenvalue of $T(A[L|K], r_L, c_K)$, we must have that $\omega(I_1, J_1) \geq \omega(L, K)$. From this we deduce that (6) holds. □

Remark 5. From the proof of Lemma 4, it is clear that in case where (A, r, c) is non-consistent the following inequality is satisfied

$$\omega(I_1, J_1) \geq \omega(I_2, J_2) \geq \omega(I_3, J_3).$$

Lemma 6. Let (A, r, c) be a non-consistent mrc and A be indecomposable. If $\emptyset \subset I, L \subset M$ and $\emptyset \subset J, K \subset N$ are such that

1. $A[I'|J]$ and $A[L'|K]$ are maximal zero submatrices of A,

2. $A[I|J]$ and $A[L|K]$ are indecomposable,

3. $(A[I|J], r_I, c_J)$ and $(A[L|K], r_L, c_K)$ are consistent,

4. $\omega(I,J) = \omega(L,K)$ is the maximum eigenvalue of $T = T(A,r,c)$, and

5. $I \times J \neq L \times K$,

then $I \cap L = \emptyset$ and $J \cap K = \emptyset$.

Proof: We can either have $(I \times J) \cap (L \times K) = \emptyset$ or $(I \times J) \cap (L \times K) \neq \emptyset$. So one of the following holds true:
(a) $I \cap L = \emptyset$ and $J \cap K \neq \emptyset$, (b) $I \cap L \neq \emptyset$ and $J \cap K = \emptyset$,
(c) $I \cap L \neq \emptyset$ and $J \cap K \neq \emptyset$, (d) $I \cap L = \emptyset$ and $J \cap K = \emptyset$.
Let $I_1 = I \cap L'$, $I_2 = I \cap L$, $I_3 = L \cap I'$ and $I_4 = I' \cap L'$. Also $J_1 = J \cap K'$, $J_2 = J \cap K$, $J_3 = K \cap J'$ and $J_4 = J' \cap K'$. Then by (1) we can see that A has the following form

$$\begin{array}{c} \\ I_1 = I \cap L' \\ I_2 = I \cap L \\ I_3 = L \cap I' \\ I_4 = I' \cap L' \end{array} \begin{array}{cccc} J_1 = J \cap K' & J_2 = J \cap K & J_3 = K \cap J' & J_4 = J' \cap K' \\ \left[\begin{array}{c} A_1 \\ C_1 \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} 0 \\ A_2 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ C_2 \\ A_3 \\ 0 \end{array} & \left. \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \right] \end{array}.$$

It is easy to see that (a) is not possible, for otherwise by (1) we would have that $A[M|J_2] = 0$, the latter contradicting the fact that A has no zero column.
We make the following claim

$$\omega(I_1 \cup I_2, J_1 \cup J_2) \geq \omega(I_3, J_3) \tag{7}$$

and

$$\omega(I_2 \cup I_3, J_2 \cup J_3) \geq \omega(I_1, J_1). \tag{8}$$

Proof of claim: Observe that by assumption 4 and by symmetry, inequalities (7) and (8) are equivalent.
If (d) holds then (7) is obviously true. Thus assume that either (b) or (c) hold. Consider the mrc $(A', r', c') = (A[\cup_{k=1}^3 I_k | \cup_{k=1}^3 J_k], r_{\cup_{k=1}^3 I_k}, c_{\cup_{k=1}^3 J_k})$.
If (A', r', c') is consistent, then $\omega(I_1 \cup I_2 \cup I_3, J_1 \cup J_2 \cup J_3)$ is an eigenvalue of $T(A,r,c)$ and thus (7) follows by the maximality of $\omega(I_1 \cup I_2, J_1 \cup J_2)$.
Suppose that (A', r', c') is non-consistent. By considering carefully assumptions 1, 2 and 3, we see that in order to ensure the non-consistency of (A', r', c') we must either have (7) (equivalently (8)), or $\omega(I_2, J_2) \geq \omega(I_1 \cup I_3, J_1 \cup J_3)$. Since $A[I_2|J_1]$ and $A[I_2|J_3]$ are non-zero and $(A[I_1 \cup I_2|J_1 \cup J_2], r_{I_1 \cup I_2}, c_{J_1 \cup J_2})$ and $(A[I_2 \cup I_3|J_2 \cup J_3], r_{I_2 \cup I_3}, c_{J_2 \cup J_3})$ are consistent,

we have that $\omega(I_2, J_2) < \omega(I_1 \cup I_2, J_1 \cup J_2)$ and $\omega(I_2, J_2) < \omega(I_2 \cup I_3, J_2 \cup J_3)$. From these we conclude that $\omega(I_2, J_2) < \omega(I_1 \cup I_3, J_1 \cup J_3)$. Therefore (7) must be true. This completes the proof of the claim.

Now assume that (b) holds. In this case we have $J_2 = \emptyset$ and thus

$$\begin{aligned} \omega(I, J) &= \omega(I_1 \cup I_2, J_1) \\ &= \omega(I_2 \cup I_3, J_3) = \omega(L, K). \end{aligned}$$

Also by the claim $\omega(I_1 \cup I_2, J_1) \geq \omega(I_3, J_3)$. Therefore

$$\omega(I_2 \cup I_3, J_3) \geq \omega(I_3, J_3). \tag{9}$$

On the other hand

$$\omega(I_3, J_3) > \omega(I_2 \cup I_3, J_3). \tag{10}$$

This is a contradiction, so (b) cannot hold. Note that (9) and (10) can both be true if $I_2 = I \cap L = \emptyset$.

Finally suppose that (c) is true. By (7) and since $\omega(I, J) = \omega(L, K)$ we have that $\omega(I_2 \cup I_3, J_2 \cup J_3) \geq \omega(I_3, J_3)$ which implies

$$\omega(I_2, J_2) \geq \omega(I_3, J_3). \tag{11}$$

By consistency of $(A[I_2 \cup I_3 | J_2 \cup J_3], r_{I_2 \cup I_3}, c_{J_2 \cup J_3})$ and since of $A[I_2|J_3] \neq 0$ we have that $\omega(I_2 \cup I_3, J_2 \cup J_3) > \omega(I_2, J_2)$ and so

$$\omega(I_3, J_3) > \omega(I_2, J_2). \tag{12}$$

Inequalities (11) and (12) lead to a contradiction. Hence (c) does not hold either. We conclude that we can only have (d).□

Corollary 7. Let (A, r, c) be an mrc and A be indecomposable. If $\{I_\alpha : \alpha = 1, \ldots, \sigma\}$ and $\{J_\alpha : \alpha = 1, \ldots, \sigma\}$ are partitions of M and N respectively such that for $\alpha = 1, \ldots \sigma$

1. $(A[I_\alpha | J_\alpha], r_{I_\alpha}, c_{J_\alpha})$ is consistent,

2. $A[I_\alpha | J_\alpha]$ is indecomposable,

3. $A[\cup_{k=\alpha+1}^{\sigma} I_k | J_\alpha]$ is a maximal zero submatrix of $A[\cup_{k=\alpha}^{\sigma} I_k | \cup_{k=\alpha}^{\sigma} J_k]$ and,

4. $\omega(I_\alpha, J_\alpha)$ is the maximum eigenvalue of $T(A[\cup_{k=\alpha}^{\sigma} I_k | \cup_{k=\alpha}^{\sigma} J_k], r_{\cup_{k=\alpha}^{\sigma} I_k}, c_{\cup_{k=\alpha}^{\sigma} J_k})$.

Then

(a) $\omega(I_1, J_1)$ is the maximum eigenvalue of $T(A, r, c)$.

(b) $\omega(I_1, J_1) \geq \ldots \geq \omega(I_\sigma, J_\sigma)$.

(c) This partition, which we shall call the maximum eigenvalue partition of $T = T(A, r, c)$ and denote by $\mathcal{P}(A, r, c) = \{I_\alpha \times J_\alpha : \alpha = 1, \ldots, \sigma\}$, is unique up to a permutation.

Proof: If (A, r, c) is consistent then $\sigma = 1$ and there is nothing to show. In the case when (A, r, c) is non-consistent the proof follows immediately from Lemmas 4 and 6.□

Theorem 8. Let (A, r, c) be an mrc and A be indecomposable. If the maximum eigenvalue partition of $T = T(A, r, c)$ is $\mathcal{P}(A, r, c) = \{I_\alpha \times J_\alpha : \alpha = 1, \ldots, \sigma\}$ then the cycle time vector $\chi(\mathcal{E}(T))$ of the operator $\mathcal{E}(T)$, where $T = T(A, r, c)$, is

$$\chi(\mathcal{E}(T))_{J_\alpha} = \log \omega(I_\alpha, J_\alpha) = \log \rho(T_\alpha), \qquad \alpha = 1, \ldots, \sigma \tag{13}$$

where $T_\alpha = T(A[I_\alpha | J_\alpha], r_{I_\alpha}, c_{J_\alpha})$.

Proof: If (A, r, c) is consistent then $\sigma = 1$ and (13) was already proven in Proposition 3. Suppose that (A, r, c) is non-consistent. For $x \in (\mathbb{R}^n_+)^\circ$, define $\tilde{\chi}(T) = \lim_{k \to \infty} (T^k x)^{\frac{1}{k}}$. It is easy to see that $\chi(\mathcal{E}(T)) = \log \tilde{\chi}(T)$. Thus it is enough to show that

$$\tilde{\chi}(T)_{J_\alpha} = \omega(I_\alpha, J_\alpha) = \rho(T_\alpha), \qquad \alpha = 1, \ldots, \sigma. \tag{14}$$

We shall prove (14) for $\sigma = 2$. By assumption the matrix A has the form

$$A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where A_1 has no zero row or column. So for $x_1 \gg 0$ we have that $T_1 x_1 \gg 0$. For any strictly positive vector $x = (x_1, x_2)^T$ we have

$$Tx = \begin{bmatrix} L_{c_1} A_1^T L_{r_1}(A_1 x_1 + B x_2) \\ L_{c_2}[B^T L_{r_1}(A_1 x_1 + B x_2) + A_2^T L_{r_2} A_2 x_2] \end{bmatrix}. \tag{15}$$

Observe that $T_\alpha x_\alpha = L_{c_\alpha} A_\alpha^T L_{r_\alpha} A_\alpha x_\alpha, \quad \alpha = 1, 2$. It is straightforward to see that for any $x = (x_1, x_2)^T \gg 0$

$$T_1 x_1 \leq (Tx)_1 \tag{16}$$

$$(Tx)_2 \leq T_2 x_2. \tag{17}$$

Since $(A_\alpha, r_\alpha, c_\alpha)$ is consistent there is $x_\alpha \gg 0$ such that $T_\alpha x_\alpha = \rho(T_\alpha) x_\alpha$, $\alpha = 1, 2$. Given $\epsilon > 0$ we can find a real number $\lambda^\epsilon > 0$ such that $B \lambda^\epsilon x_2 \leq A_1 \epsilon x_1$. Thus for $x^\epsilon = (x_1, x_2^\epsilon)^T \gg 0$, where $x_2^\epsilon = \lambda^\epsilon x_2$, we have

$$(Tx^\epsilon)_1 \leq (1 + \epsilon) T_1 x_1 = (1 + \epsilon) \rho(T_1) x_1. \tag{18}$$

From (17) we get

$$(Tx^\epsilon)_2 \leq T_2 x_2^\epsilon < (1+\epsilon)\rho(T_2)x_2^\epsilon. \tag{19}$$

Therefore

$$Tx^\epsilon < (1+\epsilon)(\rho(T_1)x_1, \rho(T_2)x_2^\epsilon)^T. \tag{20}$$

Using the fact that $\rho(T_2) \leq \rho(T_1)$ we can see that inequalities (18) and (19) hold also for the vector $(\rho(T_1)x_1, \rho(T_2)x_2^\epsilon)^T$. The latter together with the monotonicity of T and (20) give

$$T^2 x^\epsilon < (1+\epsilon)^2(\rho^2(T_1)x_1, \rho^2(T_2)x_2^\epsilon)^T.$$

So by induction we have that for $k \geq 1$

$$T^k x^\epsilon < (1+\epsilon)^k(\rho^k(T_1)x_1, \rho^k(T_2)x_2^\epsilon)^T. \tag{21}$$

Thus

$$\lim_{k\to\infty} (T^k x^\epsilon)^{\frac{1}{k}} \leq (1+\epsilon)(\rho(T_1), \rho(T_2))^T$$

which implies that

$$\tilde{\chi}(T) \leq (1+\epsilon)(\rho(T_1), \rho(T_2))^T.$$

Since $\epsilon > 0$ was arbitrary

$$\tilde{\chi}(T) \leq (\rho(T_1), \rho(T_2))^T. \tag{22}$$

Now suppose that $0 < \epsilon < 1$ is given and that $x_1 \gg 0$ and $x_2 \gg 0$ are as above. We can find a real number $\mu^\epsilon > 0$ satisfying $B^T L_{r_1}(\mu^\epsilon A_1 x_1 + B x_2) \leq \frac{\epsilon}{1-\epsilon} A_2^T L_{r_2} A_2 x_2$. Let $x_1^\epsilon = \mu^\epsilon x_1$, then for $x^\epsilon = (x_1^\epsilon, x_2)^T \gg 0$ we obtain,

$$(Tx^\epsilon)_2 \geq (1-\epsilon)T_2 x_2 = (1-\epsilon)\rho(T_2)x_2. \tag{23}$$

By (16) we see that

$$(Tx^\epsilon)_1 \geq T_1 x_1^\epsilon > (1-\epsilon)\rho(T_1)x_1^\epsilon. \tag{24}$$

So (23) and (24) imply $Tx^\epsilon > (1-\epsilon)(\rho(T_1)x_1^\epsilon, \rho(T_2)x_2)^T$. In a similar way as above we can show that

$$(1-\epsilon)^k(\rho^k(T_1)x_1^\epsilon, \rho^k(T_2)x_2) < T^k x^\epsilon. \tag{25}$$

By taking the kth root in (25) and letting $k \to \infty$ and then using the fact that $0 < \epsilon < 1$ was arbitrary, we obtain a lower bound for $\tilde{\chi}(T)$, i.e.

$$(\rho(T_1), \rho(T_2))^T \leq \tilde{\chi}(T). \tag{26}$$

Finally inequalities (22) and (26) give that

$$\tilde{\chi}(T) = (\rho(T_1), \rho(T_2))^T.$$

For $\sigma > 2$ we show (14) using induction. More specifically, we have that for any $x = (x_1, \ldots, x_\sigma)^T \gg 0$

$$\begin{aligned} T_1 x_1 &\leq (Tx)_1 \\ & (Tx)_\alpha \leq [T_{2,\ldots,\sigma}(x_2, \ldots, x_\sigma)^T]_\alpha, \quad \text{for } \alpha = 2, \ldots, \sigma - 1 \\ \text{and} \quad & (Tx)_\sigma \leq T_\sigma x_\sigma, \end{aligned}$$

where $T_{2,\ldots,\sigma} = T(A[\cup_{\alpha=2}^{\sigma} I_\alpha | \cup_{\alpha=2}^{\sigma} J_\alpha], r_{\cup_{\alpha=2}^{\sigma} I_\alpha}, c_{\cup_{\alpha=2}^{\sigma} J_\alpha})$. Given $x_\alpha \gg 0$ with $T_\alpha x_\alpha = \rho(T_\alpha) x_\alpha$ and $\epsilon > 0$ we can find inductively positive real numbers $\lambda_2^\epsilon, \ldots, \lambda_\sigma^\epsilon$ such that, if $x_\alpha^\epsilon = \lambda_\alpha^\epsilon x_\alpha$ for $\alpha = 2, \ldots, \sigma$ then, for $x^\epsilon = (x_1, x_2^\epsilon, \ldots, x_\sigma^\epsilon)^T \gg 0$ we have

$$T^k x^\epsilon \leq (1+\epsilon)^k (\rho^k(T_1) x_1, \ldots, \rho^k(T_\sigma) x_\sigma^\epsilon)^T.$$

Therefore

$$\tilde{\chi}(T) \leq (\rho(T_1), \ldots, \rho(T_\sigma))^T.$$

Similarly given $0 < \epsilon < 1$ we can construct $x^\epsilon = (x_1^\epsilon, \ldots, x_{\sigma-1}^\epsilon, x_\sigma)^T \gg 0$ such that

$$(1-\epsilon)^k (\rho^k(T_1) x_1^\epsilon, \ldots, \rho^k(T_\sigma) x_\sigma)^T \leq T^k x^\epsilon.$$

The latter implies that

$$(\rho(T_1), \ldots, \rho(T_\sigma))^T \leq \tilde{\chi}(T).$$

Hence

$$\tilde{\chi}(T)_{J_\alpha} = \rho(T_\alpha) = \omega(I_\alpha, J_\alpha), \quad \alpha = 1, \ldots, \sigma.$$

This completes the proof of the theorem.□

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