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Indistinguishability for Quantum Particles: Further Considerations

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Our earlier arguments (Berry, M.V. & Robbins, J.M. 1997. *Proc Roy. Soc. Lond.* A453 1771-1790) leading to the spin-statistics relation are summarized and then revisited. Correcting a minor logical error in the double application of exchange reinforces the parallel transport requirement for the basis incorporating spin exchange. There are certain 'perverse' constructions, satisfying all our previous requirements but leading to the wrong exchange sign (e.g. replacing commutators by anticommutators in the Schwinger representation of the spins of the particles); we suggest why these are unacceptable. In our previous arguments, no use was made of relativity; we comment on this, and on the fact that spin is equally a consequence of galilean as of einsteinian relativity. Finally, we obtain an extended spin-statistics relation for identical particles that possess properties additional to position and spin.

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1. Introduction

Following our earlier nonrelativistic derivation (Berry and Robbins, 1997, hereinafter called I) of the connection between spin and statistics for identical quantum particles, and comments from several people, we find it necessary to elaborate and extend our arguments in several directions.

In section 2, we repeat, for later convenience, the outline of the way we obtained the spin-statistics connection for two particles with spin S (integer or half-integer). In section 3 we correct a minor logical error in the derivation of the exchange sign. Then, in section 4, we draw attention to the existence of alternative ('perverse') constructions, satisfying the conditions in I, that lead to the wrong spin-statistics connection in certain cases, and speculate on how these can be excluded. Next, in section 5, we comment on the fact that our argument made no use of einsteinian relativity. Finally, in section 6, we extend our derivation to identical particles with additional properties (e.g. isospin) in addition to position and spin.

2. Reprise

In I, the state of the particles is represented by

$$|\Psi(\mathbf{r})\rangle = \sum_M \psi_M(\mathbf{r})|M(\mathbf{r})\rangle \tag{1}$$

In this equation, $\mathbf{r}\equiv\mathbf{r}_1\text{-}\mathbf{r}_2$ is the relative position vector for the particles (the dependence on the centre of mass is implicit); therefore exchange of positions corresponds to $\mathbf{r}\rightarrow -\mathbf{r}$. $M\equiv\{m_1,m_2\}$ labels the spin state of the particles, with m denoting the z component of spin; exchange of spins corresponds to $M \rightarrow \overline{M} \equiv \{m_2, m_1\}$. The aim is to incorporate the indistinguishability of the particles by identifing **r** and -**r**. In order to do this, it is necessary to exchange the spins along with the positions. To this end, we construct a transported spin basis $|M(\mathbf{r})\rangle$, with the exchange property

$$|M(-\mathbf{r})\rangle = \exp\{i\gamma(\mathbf{r})\}|\overline{M}(\mathbf{r})\rangle \tag{2}$$

(this extension of I, to include a position-dependent phase, will be discussed in section 3). The transported basis must be singlevalued and smooth.

An additional requirement in I was that the basis be parallel-transported, that is

$$\mathbf{A}_{M,M'} \equiv i\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = 0 \tag{3}$$

for all M, M', **r**. This condition was automatically satisfied by the $|M(\mathbf{r})\rangle$ constructed in section 4 of I. However, it was imposed *a priori* in order to exclude a certain counter-construction (appendix D of I) whose results were physically incoherent. At the end of section 3 we will discuss parallel transport further.

With the basis $|M(\mathbf{r})\rangle$, $\mathbf{r}\rightarrow -\mathbf{r}$ corresponds to complete exchange of the particles, motivating the central step, which is to impose singlevaluedness on $|\Psi(\mathbf{r})\rangle$, regarded as a function on the product of the projective plane (sphere with identified antipodes) with the radial coordinate $r=|\mathbf{r}|$, taking values in the Hilbert space of the two spins. Thus

$$|\Psi(\mathbf{r})\rangle = |\Psi(-\mathbf{r})\rangle \tag{4}$$

With (1) and (2), singlevaluedness implies that the exchange phase acquired by the basis $|M(\mathbf{r})\rangle$ is inherited by the coefficients $\psi_M(\mathbf{r})$, that is

$$\psi_{\overline{M}}(-\mathbf{r}) = \exp\{\text{-i}\gamma(\mathbf{r})\}\psi_M(\mathbf{r}) \tag{5}$$

We emphasize that this is a natural extension to two identical particles of the requirement of singlevaluedness of wavefunctions familiar in one-body quantum mechanics. Because of the way we defined the configuration space, **r** and –**r** are the same point. (Some other nonrelativistic accounts of the spin-statistics relation (Broyles, 1976, Bacry, 1995, see also Hilborn, 1995) also invoke notions of singlevaluedness; these and other treatments are discussed in comprehensive reviews of the spin-statistics relation by Duck and Sudarshan (1997, 1998).)

A calculation of the transported basis (section 4 of I) gave

$$\gamma(\mathbf{r}) = 2\pi S \tag{6}$$

that is

$$|\overline{M}(-\mathbf{r})\rangle = (-1)^{2S}|M(\mathbf{r})\rangle \tag{7}$$

so that (5) gives

$$\psi_{\overline{M}}(-\mathbf{r}) = (-1)^{2S}\psi_M(\mathbf{r}) \tag{8}$$

This is precisely the familiar spin-statistics relation, since the coefficients $\psi_M(\mathbf{r})$ were shown to be the same as those in the conventional formulation, in which the state is represented in terms of the fixed spin basis states $|M\rangle$, that is

$$|\Psi_{\text{fixed}}(\mathbf{r})\rangle = \sum_M \psi_M(\mathbf{r})|M\rangle \tag{9}$$

Central to I was the calculation of the sign (7) acquired by the transported basis under exchange. The basis was constructed using the Schwinger representation (Schwinger, 1965, Sakurai, 1994, and section 4 of I), in which each of the spins 1 and 2 is regarded as made from two harmonic oscillators, that is a_1, b_1, a_2, b_2, with annihilation and creation operators a_1, $a_1^\dagger$, b_1, $b_1^\dagger$, a_2, $a_2^\dagger$, b_2, $b_2^\dagger$. The resulting transported basis $|M(\mathbf{r})\rangle$ required an augmented space of spins, whose dimension is $d\equiv(4S+1)(4S+2)(4S+3)/6$ (the number of ways that 4S quanta can be distributed among four oscillators), which is larger than the dimension $(2S+1)^2$ of the fixed spin space. We argued (at the end of section 7 of I) that this enlargement (also required by the parallel transport (3) of the basis states) is physically natural, even though the expansion of $|M(\mathbf{r})\rangle$ involves fixed-basis states where the two particles would have different spins (the formalism guarantees that these unphysical states are never realized).

It will be convenient to exhibit this explicitly by writing the transported basis for two spin-1/2 particles in terms of a basis for the ten-dimensional augmented spin space. The first four states are the original fixed-basis states of the two particles, which in an obvious notation are

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle \tag{10}$$

In the remaining six, one particle has spin zero and the other has spin one, that is, using the notation $|S_1, S_2; m_1, m_2\rangle$,

$$\begin{array}{l} |0,1;0,1\rangle, |0,1;0,0\rangle, |0,1;0,-1\rangle, \\ |1,0;1,0\rangle, |1,0;0,0\rangle, |1,0;-1,0\rangle \end{array} \tag{11}$$

Then the transported states, as functions of the polar angles θ, ϕ of $\mathbf{r}$, are

$$
\begin{aligned}
|++(\mathbf{r})\rangle &= \frac{1}{\sqrt{2}}\sin\theta\left(\exp\{-\mathrm{i}\phi\}|0,1;0,1\rangle - \exp\{\mathrm{i}\phi\}|1,0;1,0\rangle\right) \\
&\quad + \cos\theta|++\rangle \\
|+-(\mathbf{r})\rangle &= \tfrac{1}{2}\left[\sin\theta\left(-\exp\{-\mathrm{i}\phi\}|1,0;0,0\rangle + \exp\{\mathrm{i}\phi\}|0,1;0,0\rangle\right)\right] \\
&\quad + \tfrac{1}{2}\left[(\cos\theta+1)|+-\rangle + (\cos\theta-1)|-+\rangle\right] \\
|-+(\mathbf{r})\rangle &= \tfrac{1}{2}\left[\sin\theta\left(-\exp\{-\mathrm{i}\phi\}|1,0;0,0\rangle + \exp\{\mathrm{i}\phi\}|0,1;0,0\rangle\right)\right] \\
&\quad + \tfrac{1}{2}\left[(\cos\theta-1)|+-\rangle + (\cos\theta+1)|-+\rangle\right] \\
|--(\mathbf{r})\rangle &= \frac{1}{\sqrt{2}}\sin\theta\left(\exp\{-\mathrm{i}\phi\}|0,1;0,-1\rangle - \exp\{\mathrm{i}\phi\}|1,0;-1,0\rangle\right) \\
&\quad + \cos\theta|--\rangle
\end{aligned}
\tag{12}
$$

It is easy to confirm that these states are singlevalued and smooth functions of $\mathbf{r}$, and also orthonormal and parallel-transported according to (3) (that is, the derivatives of each state with respect to θ and ϕ are orthogonal to all four states).

3. Derivation of the exchange sign

In section 2 of I we argued that singlevaluedness of the transported basis, embodied in a double application of $\mathbf{r}\rightarrow -\mathbf{r}$, requires the exchange phase factor to be a sign, that is $\exp\{\mathrm{i}\gamma(\mathbf{r})\}=(-1)^K$. This was wrong. In fact, singlevaluedness under double exchange requires

$$
\begin{aligned}
|M(\mathbf{r})\rangle &= |M(-(-\mathbf{r}))\rangle = \exp\{\mathrm{i}\gamma(-\mathbf{r})\}|\overline{M}(-\mathbf{r})\rangle \\
&= \exp\{\mathrm{i}[\gamma(-\mathbf{r})+\gamma(\mathbf{r})]\}|M(\mathbf{r})\rangle
\end{aligned}
\tag{13}
$$

implying

$$\gamma(\mathbf{r}) = \pi K + \mu(\mathbf{r}) \tag{14}$$

where K is an integer and $\mu(-\mathbf{r})=-\mu(\mathbf{r})$.

The exchange rule generated by (14) is

$$|M(-\mathbf{r})\rangle = \exp\{\mathrm{i}(\pi K + \mu(\mathbf{r}))\}|\overline{M}(\mathbf{r})\rangle \tag{15}$$

However, $\mu(\mathbf{r})$ is forced to vanish by the parallel transport requirement (3). This is because

$$\mathbf{A}_{M,M'}(-\mathbf{r}) = \mathbf{A}_{\overline{M},\overline{M'}}(\mathbf{r}) - \nabla\mu(\mathbf{r})\delta_{M,M'} \tag{16}$$

so that satisfying (3) for all $\mathbf{r}$, M, M' requires that μ be a constant, which must vanish since the function $\mu(\mathbf{r})$ is odd. Thus the exchange phase factor is a sign, and the resulting physics (spin-statistics relation) is isotropic, that is independent of the direction of the line joining the particles, as it must be.

This argument reinforces the requirement of parallel transport in the form used here, namely the vanishing of the connection one-form $\mathbf{A}_{MM'}(\mathbf{r})$. We wish to elaborate on this. Equation (3) describes at once two separate requirements of our construction. The first is that the curvature vanishes: this is an intrinsic property of the geometry. The second is a property specific to the transported basis, namely that the connection one-form expressed in terms of it should vanish. It is the latter that forces the vanishing of the phase μ(r).

To clarify the distinction, the conditions in I can be reformulated in the following more abstract way. The wavefunction $|\Psi(\mathbf{r})\rangle$ can be regarded as a section of a vector bundle over the space of relative coordinates R^3 with the origin removed (this is the distinguished configuration space, not the identified space as in a related discussion in

I). The vector bundle is supposed to be endowed with a connection (perhaps, as in the case of the Schwinger construction, inherited from its embedding in a trivial bundle $\{R^3\text{-}0\}\mathrm{x}C^d$). We impose the following requirements: (i) The relative momentum operator, acting on wavefunctions $|\Psi(\mathbf{r})\rangle$, should be the covariant derivative, and (ii) the connection should be flat. The first condition is motivated by simplicity. The second condition, taken in conjunction with the first, has physical content: it means there are no gauge fields (e.g. magnetic) intrinsic to the kinematics - of course, gauge fields can be introduced later in the hamiltonian as the dynamics dictates.

Vanishing curvature implies not (3) but the vanishing of the nonabelian curl:

$$\nabla \times \mathbf{A}_{M,M'} - \mathrm{i} \sum_{M''} \mathbf{A}_{M,M''} \times \mathbf{A}_{M'',M'} = 0 \tag{3a}$$

However, if the curvature vanishes, it is always possible to construct a basis that is parallel-transported (provided the base space is simply connected, as the distinguished configuration space is). In terms of a parallel-transported basis, the connection one-form $\mathbf{A}_{M,M'}(\mathbf{r})$ itself vanishes, and the relative momentum, by virtue of (ii) above, acts as $-\mathrm{i}\hbar\nabla$ with respect to the coefficients $\psi_M(\mathbf{r})$. We took this route in I, by introducing from the outset the transported basis $|M(\mathbf{r})\rangle$.

We repeat that parallel transport is a consequence of the Schwinger construction of I; it does not have to be imposed separately.

4. Perverse constructions

Contrary to what we conjectured at the end of I, it is possible to construct 'perverse' transported bases $|M(\mathbf{r})\rangle$ that are singlevalued, smooth, orthonormal and parallel-transported, but which generate the wrong exchange sign. One such perverse basis results from a modification of the Schwinger representation, where in the argument of section 4 all commutators involving the operators a_1, $a_1^\dagger$, b_1, $b_1^\dagger$, a_2, $a_2^\dagger$, b_2, $b_2^\dagger$ are replaced by anticommutators. Because anticommutation implies $a_1^2=0$, etc, this particular modification works only for spin-1/2 particles. After some calculation, it leads to the following 'anti-Schwinger' transported basis, which should be compared with the Schwinger-generated basis (12):

$$
\begin{aligned}
&|++(\mathbf{r})\rangle = |++\rangle, \quad |--(\mathbf{r})\rangle = |--\rangle \\
&|+-(\mathbf{r})\rangle = \tfrac{1}{2}\sin\theta\left(-\exp\{i\phi\}|1,0;0,0\rangle + \exp\{-i\phi\}|0,1;0,0\rangle\right) \\
&\qquad\qquad + \cos^2\tfrac{1}{2}\theta|+-\rangle + \sin^2\tfrac{1}{2}\theta|-+\rangle \\
&|-+(\mathbf{r})\rangle = \tfrac{1}{2}\sin\theta\left(\exp\{i\phi\}|1,0;0,0\rangle - \exp\{-i\phi\}|0,1;0,0\rangle\right) \\
&\qquad\qquad + \sin^2\tfrac{1}{2}\theta|+-\rangle + \cos^2\tfrac{1}{2}\theta|-+\rangle
\end{aligned}
\tag{17}
$$

These states are singlevalued, smooth, and parallel-transported, and they satisfy the fundamental requirement that position exchange ($\mathbf{r}\to-\mathbf{r}$) is equivalent to spin exchange ($|M\rangle \to |\overline{M}\rangle$). But the sign is wrong: a plus, that is the bosonic exchange rule $|M(-\mathbf{r})\rangle = +|\overline{M}(\mathbf{r})\rangle$, instead of the fermionic minus.

Another perverse basis can be constructed for spin-0 particles, where the single transported state can be represented as the unit vector

$$|M(\mathbf{r})\rangle = \mathbf{r}/|\mathbf{r}| \tag{18}$$

This is singlevalued, smooth, and parallel-transported, but changes sign fermionically under $\mathbf{r}\rightarrow-\mathbf{r}$, rather than being bosonically invariant.

If our arguments leading to the spin-statistics connection are to be regarded as a derivation, these perverse constructions must be excluded. The following 'exclusion principles' are worth further consideration.

a. Constructions should be irreducible: if $|M(\mathbf{r})\rangle$ can be expressed as a tensor product $|M'(\mathbf{r})\rangle \otimes |\Sigma_M(\mathbf{r})\rangle$, in which $|M'(\mathbf{r})\rangle$ satisfies the properties required of the transported basis (exchange under $\mathbf{r}\rightarrow-\mathbf{r}$, smoothness, parallel transport), then $|\Sigma_M(\mathbf{r})\rangle$ must be a constant vector, independent of M and $\mathbf{r}$. This excludes (18) and related more general constructions.

b. Constructions should be general, that is, they should work for all S. The construction (17) is restricted to S=1/2 and so fails this test.

c. Operators associated with different particles should commute. This is the case with the Schwinger but not the anti-Schwinger construction, because operators with labels 1 and 2 anticommute. (It is interesting to note that a related requirement, that "all physical quantities commute for spacelike separations" was imposed by Pauli (1940) in his field-theoretic proof of the spin-statistics relation, although later authors have relaxed this requirement (Duck and Sudarshan, 1997).)

d. Quantum mechanics is a fundamental theory, but its application (like that of newtonian physics) is not restricted to fundamental particles. In particular, it can be applied to identical composites (e.g. atoms and α-particles), whose statistics must be those calculated from their constituents. In this opinion we differ from some authors (including Schwinger) of proofs of spin-statistics using field theory, who argue (Duck and Sudarshan, 1998) that the only fundamental fields are those with S=0 and S=1/2 and so it is necessary to prove the connection only for these fields. Anti-Schwinger fails this 'compositeness' test, because the restriction to spin-1/2 makes it impossible to build up such composites.

e. The physical hypothesis can be made that quantum spins are built from Schwinger's oscillators. The spin-statistics connection follows from this hypothesis, by a slight rephrasing of the arguments in I.

5. Spin and relativity

First we should point out that our derivation was nonrelativistic in the sense that it made no use of relativity, and not in the sense of being a low-velocity approximation. Since time never entered our considerations, the exchanges we considered can be regarded as taking place at fixed time. But fixed time is not relativistically invariant. Regarded relativistically, our exchanges were spacelike. This makes our arguments appear complementary to the some of the quantum field theoretic ones (Balachandran et al., 1993, Feynman, 1987), which involve the creation of pairs of antiparticles, and therefore are based on timelike exchanges.

Second, although we considered only the relation between spin and statistics, and not the origin of spin itself, the widespread belief that spin is unavoidably relativistic has led to doubts about our arguments involving exchange. Here we point out that the existence of spin is equally a consequence of galilean relativity as of einsteinian relativity. This point has been well made by Levy-Leblond (1967, 1974) and it is not necessary to repeat the general arguments. However, we think it worth outlining the galilean spin-1/2 case in the simplest and least technical way, in an argument attributed to Feynman (Mackintosh, 1983).

For a free particle without spin, with hamiltonian $H=p^2/2m$, the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2\psi(\mathbf{r},t) \tag{19}$$

is Galilean-invariant in the following sense. Under the transformation to

$$\begin{gathered} t \to t_1 \equiv t - T, \quad \mathbf{r} \to \mathbf{r}_1 \equiv \mathbf{R}\mathbf{r} - \mathbf{v}t - \mathbf{a}, \\ \psi(\mathbf{r},t) \to \psi_1(\mathbf{r}_1,t_1) \equiv \psi(\mathbf{r},t)\exp\left\{-\mathrm{i}\frac{m}{\hbar}\left(\mathbf{v}\cdot\mathbf{r}_1 + \tfrac{1}{2}v^2 t_1\right)\right\} \end{gathered} \tag{20}$$

where T is a constant scalar, $\mathbf{a}$ and $\mathbf{v}$ are constant vectors, and $\mathbf{R}$ is a constant rotation matrix, the equation preserves its form:

$$\mathrm{i}\hbar\frac{\partial}{\partial t_1}\psi_1(\mathbf{r}_1,t_1) = -\frac{\hbar^2}{2m}\nabla_{\mathbf{r}_1}^2\psi_1(\mathbf{r}_1,t_1) \tag{21}$$

For a particle with spin 1/2, this invariance is obviously shared by the two-spinor Schrödinger equation generated by the free 2x2 matrix Hamiltonian

$$\mathbf{H} = \frac{1}{2m}(\mathbf{S}\cdot\mathbf{p})^2 = \frac{1}{2m}p^2\mathbf{1} \tag{22}$$

where $\mathbf{S}$ is the vector of Pauli matrices.

In both cases, external fields with potentials $\mathbf{A}(\mathbf{r},t)$, $V(\mathbf{r},t)$ can then be introduced by the minimal coupling to the particle's charge q through

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A}(\mathbf{r},t) \tag{23}$$

and addition of $qV(\mathbf{r},t)$ to H. In the spin 1/2 case, coupling to the first equation in (22) leads to

$$\begin{aligned}\mathbf{H} &= \frac{1}{2m}(\mathbf{S}\cdot(\mathbf{p}-q\mathbf{A}))^2 + qV(\mathbf{r},t) \\ &= \frac{1}{2m}[\mathbf{p}-q\mathbf{A}(\mathbf{r},t)]^2\mathbf{1} - \frac{q\hbar}{2m}\mathbf{S}\cdot\mathbf{B}(\mathbf{r},t) + qV(\mathbf{r},t)\end{aligned} \tag{24}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. This is the Pauli equation, with the spin operator $\hbar\mathbf{S}/2$ coupled to the magnetic field with a magnetic moment **m**, that is

$$\frac{q\hbar}{2m}\mathbf{S}\cdot\mathbf{B} = \mathbf{m}\cdot\mathbf{B}, \quad \text{where} \quad \mathbf{m} = \frac{q\hbar}{2m}\mathbf{S} \tag{25}$$

This **m**, originating in a free equation that is invariant under the galilean transformations, is the same – that is, it has the same gyromagnetic ratio - as that in the corresponding Dirac equation, which is Lorentz-invariant.

Anandan (1998) has presented a relativistic generalization of our construction, in what may be a first step in establishing a bridge to the field-theoretic arguments.

6. Extended spin-statistics relations for particles with additional properties

Now we consider particles characterised not only by position and spin but by one or more further quantum properties, that we denote by P. Two examples of P are the isospin of nucleons and the colour of quarks. We denote the values of P by p (assumed discrete), and the pair of values for two particles, and the associated exchanged pair, by

$$P \equiv \{p_1, p_2\}, \quad \overline{P} \equiv \{p_2, p_1\} \tag{26}$$

If we regard P as describing different states of identical particles, the argument we employed in I to derive the spin-statistics relation can be extended by requiring the state to be singlevalued under full exchange, including $P \to \overline{P}$ as well as $\mathbf{r} \to -\mathbf{r}$.

To implement this idea, we write the state of the two particles as

$$\left|\Psi_P(\mathbf{r})\right\rangle = \sum_M \psi_{M,P}(\mathbf{r})\left|M(\mathbf{r})\right\rangle \tag{27}$$

in which $\left|M(\mathbf{r})\right\rangle$ is the same transported spin basis as before, with the exchange sign (2) and (6). Singlevaluedness, that is

$$\left|\Psi_P(\mathbf{r})\right\rangle = \left|\Psi_{\overline{P}}(-\mathbf{r})\right\rangle \tag{28}$$

leads to the extended spin-statistics relation

$$\psi_{\overline{M},\overline{P}}(-\mathbf{r}) = (-1)^{2S}\psi_{M,P}(\mathbf{r}) \tag{29}$$

This is consistent with the requirement that the original spin-statistics relation must hold when the P state of both particles is the same, that is $P = \overline{P}$. (We remark that the whole argument can be rephrased in terms of singlevaluedness on a configuration space where $\mathbf{r}_1$, $\mathbf{r}_2$ are augmented by internal 'angular' variables conjugate to the discrete indices P.)

In the above argument, P has been treated differently from spin, notwithstanding the fact that the operators representing P (e.g. isospin) can have the same mathematical structure as angular momenta. The reason is that such mathematical resemblance conceals a physical difference: it is spin, and not any other property P, that is uniquely related to position, because of its connection (section 5) with galilean or Lorentz invariance.

An argument similar to that leading to (29) has been given by Anandan (1998) in the context of Kaluza-Klein theory.

The decision to regard the particles as identical, embodied in (28), needs further discussion. An alternative possibility would be to regard the different values p_1, p_2 as distinguishing the particles. It seems absurd to consider macroscopic objects such as apples and pears as identical particles in different states of quantum fruitiness (P). Nevertheless, it is possible to choose to do this - but the choice is inconsequential, because as is well known it leaves unconstrained the symmetry of the space-spin part of the state - the symmetry of the P part of the state can always be adjusted to satisfy (29). The extended spin-statistics relation has consequences only when superpositions of states with different p are meaningful, and the interactions are such that transitions can occur between them (so that the P physics is coherently entangled with the space-spin physics).

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