Abel Summability of Gap Series

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Poisson summation formula, gap series

We show how the Poisson summation formula can be used to investigate the Abel summability of gap series. We use the method to prove that the series $\Sigma_0^\infty(-1)^n x^{n^2}$ is Abel summable, but that the series $\Sigma_0^\infty(-1)^n x^{2^n}$ isn't. Hardy obtained essentially the same result as ours concerning the latter series in[1]. Our proof is shorter and more elementary. We give some numerical estimates for the size of the oscillation of the series $\Sigma_0^\infty(-1)^n x^{n^2}$ as $x \to 1_-$.

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Abstract We show how the Poisson summation formula can be used to investigate the Abel summability of gap series. We use the method to prove that the series $\sum_0^\infty (-1)^n x^{n^2}$ is Abel summable, but that the series $\sum_0^\infty (-1)^n x^{2^n}$ isn't. Hardy obtained essentially the same result as ours concerning the latter series in [1]. Our proof is shorter and more elementary. We give some numerical estimates for the size of the oscillation of the series $\sum_0^\infty (-1)^n x^{2^n}$ as $x \to 1_-$.

Definition The series $\sum_0^\infty a_n$ is Abel summable to A if the series $\sum_0^\infty a_n x^n$ converges for all $0 < x < 1$ and if

$$\lim_{x \to 1_-} \sum_0^\infty a_n x^n = A.$$

Abel's Theorem If $\sum_0^\infty a_n$ converges in the ordinary sense to S then it is Abel summable to S.

For a proof see [3] page 57.

Gap Series We shall be concerned with alternating gap series of the form

$$\sum_o^\infty (-1)^n x^{\alpha(n)}$$

where $\alpha(n)$ is an increasing sequence of positive integers. We call such a series Abel summable if

$$\lim_{x \to 1_-} \sum_o^\infty (-1)^n x^{\alpha(n)}$$

exists finite. For example if $\alpha(n) = n$ we obtain

$$\sum_0^\infty (-1)^n x^n = \frac{1}{1+x} \to \frac{1}{2}$$

as $x \to 1_-$.

Poisson Summation Formula If $f \in L^1(R)$ then we can define its Fourier transform to be

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{itu} f(t) dt.$$

We can also define the function

$$\phi(t) = \sum_{n=-\infty}^{\infty} f(t+n)$$

which $\in L^1[0,1]$, and has nth Fourier coefficient

$$c_n = \int_0^1 \phi(t) e^{-2\pi i n t} dt = \hat{f}(-2\pi n).$$

If the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$ of $\phi(t)$ converges to $\phi(t)$ at $t = 0$ then we obtain the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n).$$

The formula is valid, e.g., if $\phi(t)$ is continuous at $t = 0$, and the series $\sum_{n=-\infty}^{\infty} \hat{f}(2\pi n)$ converges. (See [2] page 129.)

Case $\alpha(n) = n$. Consider the function $f(t) = x^{|t|} = e^{-\lambda |t|}$ where $x = e^{-\lambda}$ ($\lambda > 0$). Its Fourier transform is

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{itu} e^{-\lambda |t|} dt = \frac{2\lambda}{\lambda^2 + u^2}.$$

The Poisson summation formula gives

$$\sum_{n=-\infty}^{\infty} x^{|n|} = \sum_{n=-\infty}^{\infty} \frac{2\lambda}{\lambda^2 + 4\pi^2 n^2}$$

for all $0 < x < 1$, equivalently $\lambda > 0$. Validity is assured since the series on the right hand side converges, and the series

$$\phi(t) = \sum_{n=-\infty}^{\infty} e^{-\lambda |t+n|}$$

converges uniformly over all $t \in R$.

It follows that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 + 4\pi^2 n^2} \to \infty$$

as $x \to 1_-$, equivalently $\lambda \to 0_+$, since the series on the right hand side converges uniformly over λ satisfying e.g. $0 < \lambda < 1$.

If we write instead $f(t) = (-1)^t x^{|t|} = e^{\pi i t} e^{-\lambda |t|}$ then we have

$$\hat{f}(u) = \frac{2\lambda}{\lambda^2 + (u + \pi)^2},$$

and therefore

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{|n|} = \sum_{n=-\infty}^{\infty} \frac{2\lambda}{\lambda^2 + (2n+1)^2 \pi^2},$$

which we can rewrite as

$$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2\lambda}{\lambda^2 + (2n+1)^2\pi^2}.$$

Hence, as we already know,

$$\sum_{n=0}^{\infty}(-1)^n x^n \to \frac{1}{2}$$

as $x \to 1_-$, $\lambda \to 0_+$.

Case $\alpha(n) = n^2$. Consider the function $f(t) = x^{t^2} = e^{-\lambda t^2}$ where $x = e^{-\lambda}$ $(\lambda > 0)$. The Fourier transform of $f(t)$ is

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{itu} e^{-\lambda t^2}\, dt = \sqrt{\frac{\pi}{\lambda}} e^{-u^2/4\lambda}.$$

The Poisson formula gives

$$\sum_{n=-\infty}^{\infty} x^{n^2} = \sqrt{\frac{\pi}{\lambda}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/\lambda},$$

validity being assured by the convergence of the series on the right hand side, and the uniform convergence over $t \in R$ of the series

$$\phi(t) = \sum_{n=-\infty}^{\infty} e^{-\lambda(t+n)^2}.$$

Working instead with $f(t) = e^{\pi i t} e^{-\lambda t^2}$ we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} = \sqrt{\frac{\pi}{\lambda}} \sum_{n=-\infty}^{\infty} e^{-(2n+1)^2\pi^2/4\lambda},$$

from which it follows that

$$\sum_{n=0}^{\infty}(-1)^n x^{n^2} = \frac{1}{2} + \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} e^{-(2n+1)^2\pi^2/4\lambda}$$

$$\to \frac{1}{2}$$

as $x \to 1_-$, $\lambda \to 0_+$ since

$$\begin{aligned}
\sqrt{\frac{\pi}{\lambda}}\sum_{n=0}^{\infty} e^{-(2n+1)^2\pi^2/4\lambda} &< \sqrt{\frac{\pi}{\lambda}}\sum_{n=0}^{\infty} e^{-n^2\pi^2/4\lambda} \\
&< \sqrt{\frac{\pi}{\lambda}}\sum_{n=0}^{\infty} e^{-n\pi^2/4\lambda} \\
&= \sqrt{\frac{\pi}{\lambda}}\frac{e^{-\pi^2/4\lambda}}{1-e^{-\pi^2/4\lambda}} \\
&\to 0
\end{aligned}$$

as $\lambda \to 0_+$.

<u>Case $\alpha(n) = 2^n$.</u> Consider the function $f(t) = e^{-e^{|t|}}$. Its Fourier transform is

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} e^{itu} e^{-e^{|t|}} dt \\
&= 2\mathrm{Re} \int_{0}^{\infty} e^{itu} e^{-e^{t}} dt \\
&= 2\mathrm{Re} \int_{1}^{\infty} s^{iu-1} e^{-s} ds,
\end{aligned}$$

putting $s = e^t$.

For $\mathrm{Re}\, a > 0$ we have

$$\int_1^{\infty} s^{a-1}e^{-s}ds = \Gamma(a) - \int_0^1 s^{a-1}e^{-s}ds.$$

Expanding e^{-s} in powers of s we have

$$\begin{aligned}
\int_0^1 s^{a-1}e^{-s}ds &= \int_0^1 s^{a-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}s^k ds \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^1 s^{a+k-1}ds \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{a+k} \\
&= \Delta(a).
\end{aligned}$$

Both $\Gamma(a)$, $\Delta(a)$ analytically continue to all a not an integer ≤ 0. Therefore for all real $u \neq 0$ we have

$$\hat{f}(u) = 2\mathrm{Re}\left(\Gamma(iu) - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{iu+k}\right).$$

If instead we write $f(t) = (-1)^t x^{e^{|t|}} = e^{\pi i t} e^{-\lambda e^{|t|}}$ where $x = e^{-\lambda}$ ($\lambda > 0$) then we have similarly

$$\hat{f}(u) = 2\mathrm{Re}\left(\frac{\Gamma(i(u+\pi))}{\lambda^{i(u+\pi)}} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\lambda^k}{i(u+\pi)+k}\right)$$

for all real $u \neq -\pi$.

Poisson's summation formula gives

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{e^{|n|}} = 2\mathrm{Re} \sum_{n=-\infty}^{\infty} \left(\frac{\Gamma((2n+1)\pi i)}{\lambda^{(2n+1)\pi i}} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\lambda^k}{(2n+1)\pi i + k}\right),$$

and hence

$$\sum_{n=0}^{\infty} (-1)^n x^{e^n} = \frac{x}{2} + 2\mathrm{Re} \sum_{n=0}^{\infty} \left(\frac{\Gamma((2n+1)\pi i)}{\lambda^{(2n+1)\pi i}} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\lambda^k}{(2n+1)\pi i + k}\right).$$

Observe firstly that

$$\begin{aligned}\mathrm{Re} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\lambda^k}{(2n+1)\pi i + k} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{k\lambda^k}{(2n+1)^2\pi^2 + k^2} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{(k-1)!} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 + k^2} \\ &\to 0\end{aligned}$$

as $\lambda \to 0_+$.

Observe secondly that if we write $\lambda = e^{-\mu}$ then

$$2\mathrm{Re} \sum_{n=0}^{\infty} \frac{\Gamma((2n+1)\pi i)}{\lambda^{(2n+1)\pi i}} = 2\mathrm{Re} \sum_{n=0}^{\infty} \Gamma((2n+1)\pi i) e^{(2n+1)\mu\pi i}$$

is 1-periodic in μ as $\mu \to \infty$, equivalently $\lambda \to 0_+$. Also the ratio of consecutive terms of

$$\sum_{n=0}^{\infty} |\Gamma((2n+1)\pi i)| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n+1)\sinh(2n+1)\pi^2}}$$

is $< e^{-\pi^2} = 5.17 \times 10^{-5}$. The first term of

$$2\mathrm{Re} \sum_{n=0}^{\infty} \frac{\Gamma((2n+1)\pi i)}{\lambda^{(2n+1)\pi i}}$$

oscillates with amplitude

$$\frac{2}{\sqrt{\sinh \pi^2}} = 2.03 \times 10^{-2}.$$

The sum of the rest of the terms is in modulus less than

$$\frac{2}{\sqrt{\sinh \pi^2}} \frac{1}{e^{\pi^2} - 1} = 1.05 \times 10^{-6}.$$

It follows that the series $\sum_{n=0}^{\infty} (-1)^n x^{e^n}$ oscillates as $x \to 1_-$ with asymptotic amplitude 2.03×10^{-2} to 3 significant figures.

Similar analysis applies for the series $\sum_{n=0}^{\infty} (-1)^n x^{2^n}$ except that the asymptotic amplitude of the oscillation is in this case

$$\frac{2}{\sqrt{(\log 2) \sinh(\pi^2 / \log 2)}} = 2.75 \times 10^{-3}$$

to 3 significant figures.

References [1] Hardy G.H., On Certain Oscillating series, Quar. Jour. of Math., Vol. 38 (1907), pages 146-167.

[2] Katznelson, Y., An Introduction to Harmonic Analysis, J. Wiley, 1968.

[3] Whittaker E.T. & Watson G.N., A Course of Modern Analysis, Cambridge, 1902.

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