Inner Products and Orthogonality in Colour Recording Filter Design

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We formalize ideas of orthogonality and inner products implicit in the development of a number of figures of merit (FOM, [1]) of colour recording filters. We show that, in negligible measurement noise, the data dependence of each FOM based on linear colour correction is equivalent to a choice of inner product (and hence of orthogonality). Further, we show that optimal sensors with respect to noise sensitivity are simply defined as orthogonal with respect to this inner product. We also develop the idea of a generalized Q-factor by generalizing the euclidean inner product to include all inner products. Simulations demonstrate the utility of our analytical results.

1 Introduction

Electronic image capture devices like scanners and digital cameras use charge-coupled device (CCD) or active pixel sensor (APS) technology for sensor fabrication. The colour capabilities of these devices come about from the colour filters that are used with the sensors, and there has been considerable interest and literature on the design of optimal colour filters from the point of view of colour fidelity [2, 3, 4, 5, 6, 7, 1] and robustness [8, 9, 6, 10]. There is a wide range of criteria for colour fidelity, varying in colorimetric accuracy, parameters and computational complexity. The common aspect of most of the criteria is that they may be expressed in terms of inner products, as we will show in section 3. The existing investigations into robustness of a recording filter set [8, 9, 6, 10] are based on specific colorimetric accuracy criteria.

This paper poses the problem of the accuracy of color recording filters in terms of inner products and orthogonality - exploiting the common aspect of most existing performance criteria. This allows a common framework for noise analysis. We define an inner product using data-dependence in the form of preferential weighting of errors in directions where most of the energy of the data set is concentrated. We show that Neugebauer's Q-factor can be extended using the generalized inner product. We illustrate the use of the generalized inner product, the induced generalized orthogonality and generalized Q-factor by addressing the problems of the characterization of colour filters with respect to colour fidelity and noise sensitivity.

The problem of colour fidelity has been addressed satisfactorily by a number of researchers, and we show that the inner product provides a single framework for similar approaches. We generalize the fundamental error, based on orthogonal directions in the HVSS, to other quadratic error measures in the target colour space. The FOMs of Sharma and Trussell [1] and Wolski et al [6] for negligible noise, Neugebauer [2], Vora and Trussell, (ν), [3] and Finlayson [7], are all choices of an inner product, and hence choices of 'orthogonality'. The purpose of this paper is to demonstrate the use of the idea of inner products in simplification of optimality expressions for colour recording filters. The rich mathematical results available for inner products may be used to produce other results in the future.

We address the problem of noise sensitivity in detail and show that optimal filters with respect to noise sensitivity are those that are orthonormal with respect to the generalized inner product. Vrhel and Trussell [4] show that optimal filters with respect to sensitivity to filter fabrication errors also satisfy the same criterion. We present the result here in a simpler form than they do, in terms of inner products and orthonormality.

The paper is organised as follows. We establish notation and background in section 2. In section 3, we define the tools we use in the rest of the paper - generalized inner products, orthogonality, projection operators and Q-factors. In section 4, we describe the use of the tools in studying colorimetric accuracy - in particular, we describe the specific translations from colour space and data statistics to inner products. In this section we also show how the measures of [2, 3, 6, 1] correspond to choices of inner products. We also show that these measures may be expressed as a weighted sum of generalized Q-factors of preferred directions in the target space. We present the use of our tools for noise analysis in section 5. Simulation results demonstrating the usefulness of the framework and the noise sensitivity results are in section 6. A summary of the major results of the paper is presented in section 7.

2 Preliminaries

The notation in this paper follows that of Trussell [11], Vora and Trussell [3, 9] and Sharma and Trussell [1]. Filter transmissivities, spectral reflectance functions, radiant illuminant spectral distributions, the CIE matching functions [12, pg.] and all other functions of wavelength are assumed to be represented by N samples in the visual range. The theoretical results presented here and the ideas used are independent of the sampling rate. The simulations have been performed for N = 31.

2.1 Notation

We list here the symbols used in the paper, they are also repeated where they are derived.

$\mathbf{f}$: reflective spectrum

$\mathbf{A} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$: matrix of the CIE matching functions

$\mathbf{M} = [\mathbf{m}_1\ \mathbf{m}_2 ....\mathbf{m}_r]$: matrix of r recording filters

$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_s]$: matrix of s target measurement filters

$\mathbf{t} = \mathbf{V}^T\mathbf{f}$, and $\hat{\mathbf{t}}$: s-stimulus values of $\mathbf{f}$ and their Linear Minimum Mean Square Error (LMMSE) estimates respectively

$\mathbf{h}$: combination of optical path, recording illuminant and sensor characteristic

$\mathbf{l}$: viewing illuminant

$\mathbf{H}$ and $\mathbf{L}$: diagonal matrices with $\mathbf{H}_{ii} = \mathbf{h}(i)$ and $\mathbf{L}_{ii} = \mathbf{l}(i)$ respectively

$\mathbf{M}_H$ and $\mathbf{A}_L$: matrix products $\mathbf{HM}$ (or 'effective recording system') and $\mathbf{LA}$ respectively

$\mathbf{A}_L^T\mathbf{f}$: CIE tristimulus vector of reflective spectrum $\mathbf{f}$ under viewing illuminant $\mathbf{l}$

$\mathbf{g} = \mathbf{M}_H^T\mathbf{f} + \mathbf{n}$: recorded noisy measurements

$\mathbf{B}$: colour correction matrix

$\mathbf{C}$: linear transformation of s-stimulus values for error metric

$\mathbf{D}$: preferred directions in target space, $\mathbf{D} = \mathbf{VC}^T$

$\mathbf{R}$, $\mathbf{R}_n$: correlation matrices of data and noise respectively

σ^2: noise variance

λ and ζ: eigenvalues of matrices $(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)\mathbf{R}_n^{-1}$ and $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$ respectively

κ and ω: condition numbers of matrices $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$ and $\mathbf{B}$ respectively

$J_{\mathcal{F}}$: Jacobian matrix for non-linear transformation from target space to uniform error space

$R(\mathbf{X})$: range space of matrix $\mathbf{X}$ - the span (set of linear combinations) of its column vectors

$\mathbf{X}^{-}$: pseudo-inverse of matrix $\mathbf{X}$

$<,>_e$, $<,>'$: euclidean and generalized inner products respectively

P_X, P'_X: projection operators onto $R(\mathbf{X})$ using the euclidean norm and the norm induced by the generalized inner product respectively

vec($\mathbf{X}$): matrix $\mathbf{X}$ stacked column by column as a vector

$\otimes$: Kronecker product

2.2 Problem Formulation and Background

The common problem in colour scanning, digital photography, and colour correction is the design of filters to obtain the values

$$\mathbf{t} = \mathbf{V}^T\mathbf{f} \tag{1}$$

where $\mathbf{f}$ is an N-vector representing the visual stimulus. The matrix $\mathbf{V}$ consists of s columns, $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_s]$. The columns of $\mathbf{V}$ represent the combined effect of the CIE matching functions and a viewing illuminant in the case of colour scanning and digital photography, or the combined effect of the CIE matching functions and many different viewing illuminants in the case of colour correction. The vector $\mathbf{t}$ may be referred to as the s-stimulus vector (s = 3 when $\mathbf{V} = \mathbf{A}_L$). This formulation allows the linear model ideas of [2, 13, 14, 15, 11, 3] to be extended to define colour fidelity criteria of sets of recording filters used for multi-band spectral measurements, even when the measurements are not those of the CIE tristimulus values [16]. The designed filters do not need to replicate the columns of $\mathbf{V}$, and it is sufficient to obtain measurements from which the values $\mathbf{V}^T\mathbf{f}$ may be determined through a linear transformation [13, 11, 3]. The properties of the linear transformation determine the noise amplification inherent in the procedure, and this is discussed in detail in section 5.

The output of the effective recording system represented by $\mathbf{M}_H$ is $\mathbf{g} = \mathbf{M}_H^T\mathbf{f} + \mathbf{n}$. The linear minimum mean square error (LMMSE) estimate of the s-stimulus values of zero-mean signal $\mathbf{f}$ in the presence of zero-mean signal-uncorrelated measurement noise is [4, 8]:

$$\hat{\mathbf{t}} = \mathbf{V}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H + \mathbf{R}_n)^-(\mathbf{M}_H^T\mathbf{f} + \mathbf{n}) \tag{2}$$

where $\mathbf{R} = E[\mathbf{f}\mathbf{f}^T]$ and $\mathbf{R}_n = E[\mathbf{n}\mathbf{n}^T]$ are the sample correlation matrices of the data and the noise respectively. The incorporation of non-zero signal and noise means does not change the basic results of the analysis. The correction matrix is the linear transformation used to obtain the estimate from the measurements:

$$\mathbf{B} = \mathbf{V}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H + \mathbf{R}_n)^- \tag{3}$$

The properties of the matrix $\mathbf{B}$ determine the noise amplification properties of the procedure of determining the target s-stimulus values using the effective recording system $\mathbf{M}_H$.

The error in estimating the s-stimulus values may be calculated in many ways. Linear models have been largely successful in explaining the colour responses of the sensor and filter combination [17, 18, 19], though errors perceived by the human visual system are far from linear. Commonly used error measures for the colour reproduction of patches include the mean-square error in a linear transformation of the CIE tristimulus space and the mean-square error in the perceptually uniform CIELAB space [12, pg. 166]. We discuss these errors in more detail in the rest of this section.

2.2.1 Quadratic Error Measures

A common instance where linear transformations of the tristimulus values are the target of measurements is when a colour is to be reproduced on an additive display, like a CRT monitor. While the mean-square error in a linear transformation of CIE tristimulus space is not a good approximation of perceptual error, it provides a numerical estimate of colorimetric accuracy and may be manipulated with the use of simple mathematics. Hence it is commonly used for rough optimality estimates. In general, euclidean distance in a linear transformation of the s-stimulus space corresponds to a weighted euclidean distance in the original space.

If $||.||_e$ represents the euclidean norm, the expression for the mean square value of the difference between a linear transformation, $\mathbf{C}$, of the required s-stimulus values $\mathbf{t}$ (equation (1)) and the same linear transformation of the estimated values $\hat{\mathbf{t}}$ (equation (2)) is the euclidean distance in the transformed space [16]:

$$E[||\mathbf{e}||_e^2] = Trace(\mathbf{C}\mathbf{V}^T\mathbf{R}\mathbf{V}\mathbf{C}^T - \mathbf{C}\mathbf{V}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H + \mathbf{R}_n)^{-}\mathbf{M}_H^T\mathbf{R}\mathbf{V}\mathbf{C}^T)$$

The euclidean distance in a space of linearly transformed errors may be thought of as a general form of a quadratic error measure because it is of the form $\sum \mathbf{e}_i w_{ij} \mathbf{e}_j$, where $\mathbf{e}$ symbolizes the error vector, and w_{ij} the weights.

When $\mathbf{C}$ is the identity, the error is measured as euclidean distance in the space of the s-stimulus values. Hence, when the s-stimulus values are the CIE tristimulus values, and $\mathbf{C} = \mathbf{I}$ the above error is the mean square tristimulus error. When $\mathbf{C}$ corresponds to determining P_V, i.e. when $\mathbf{C} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}$, the error is measured as the euclidean distance between fundamentals. Hence, when $\mathbf{C} = \mathbf{A}_L(\mathbf{A}_L^T\mathbf{A}_L)^{-1}$, the error is the euclidean distance in the HVISS as in [3]. Notice that the LMMSE estimate of the transformed tristimulus values is the same as the LMMSE estimate of tristimulus values with respect to the directions represented by $\mathbf{V}\mathbf{C}^T$. We will denote the matrix of preferred directions, $\mathbf{V}\mathbf{C}^T$, by the matrix $\mathbf{D}$, with columns (individual preferred directions) $\mathbf{d}_i$.

Proceeding as in [3, 16, 1], one may obtain a normalized measure based on euclidean distance in the transformed space:

$$\chi(\mathbf{D},\ \mathbf{R},\ \mathbf{R}_n,\ \mathbf{M}_H) = \frac{Trace\ \mathbf{D}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H + \mathbf{R}_n)^{-}\mathbf{M}_H^T\mathbf{R}\mathbf{D}}{Trace\ \mathbf{D}^T\mathbf{R}\mathbf{D}} \tag{4}$$

The above may be reduced to Neugebauer's Q-factor which is a measure of the quality of a single recording filter and negligible measurement noise, and is based on the fractional energy contained in the HVISS. The Q-factor of $\mathbf{x}$ is defined as [2]:

$$q(\mathbf{x}) = \frac{||P_V(\mathbf{x})||_e^2}{||\mathbf{x}||_e^2} \tag{5}$$

and substituting $\mathbf{R} = \sigma^2\mathbf{I}$, $\mathbf{R}_n = \mathbf{0}$, $\mathbf{C} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^-$, i.e $\mathbf{D} = P_V$, and $\mathbf{M}_H = \mathbf{Hm}$, in equation (4) gives,

$$q(\mathbf{Hm}) = \chi(P_V,\ \sigma^2\mathbf{I},\ \mathbf{0},\ \mathbf{Hm}) \tag{6}$$

The measure χ is, unfortunately, not good enough at predicting perceptual error [1]. Hence, Wolski et al [6] and Sharma and Trussell [1] suggest the use of Kronecker products to utilize locally linear approximations to more accurate, non-linear, perceptual error models. Their notation provides a more general analytical form than the error expressions discussed above, and its use is justified only when errors in spaces which are not linear transformations of the space spanned by $\mathbf{V}$ are required.

2.2.2 Accurate approximations of perceptual error

The average ΔE_{ab} error in CIE $L^*a^*b^*$ space [12, pg. 166] is a perceptual error which is not equivalent to euclidean distance in any linear transformation of CIE tristimulus space. This motivates the analysis of non-quadratic error measures. If $\mathcal{F}(\mathbf{t}(\mathbf{f}))$ and $\mathcal{F}(\hat{\mathbf{t}}(\mathbf{f}))$ represent the non-linear transformation of real and estimated s-stimulus values respectively, to a transformed space where the euclidean distance represents a valid error measure, then the error measure is $||\mathcal{F}(\mathbf{t}(\mathbf{f})) - \mathcal{F}(\hat{\mathbf{t}}(\mathbf{f}))||^2$. A locally linear approximation of $L^*a^*b^*$ error proposed by Wolski et al [6] allows the use of linear models to analyze non-quadratic error measures as follows.

If the error between estimated and real s-stimulus values is small, and the transformation $\mathcal{F}$ is differentiable with continuous first partial derivatives at both points, the error measure can be approximated by the first term of the Taylor series approximation [6, 1]. The linear transformation $\mathbf{B}$ that minimizes the approximated error measure is defined by [1]:

$$vec\mathbf{B} = [(\mathbf{M}_H^T \otimes \mathbf{I}_r)\mathbf{S}_f(\mathbf{M}_H \otimes \mathbf{I}_r) + \mathbf{S}_n]^{-1}(\mathbf{M}_H^T \otimes \mathbf{I}_r)\mathbf{S}_f vec(\mathbf{A}_L^T) \tag{7}$$

where:

$$\mathbf{S}_f = E\{(\mathbf{ff}^T \otimes (J_{\mathcal{F}}^T(\mathbf{t}(\mathbf{f})))\}$$

$$\mathbf{S}_n \;=\; \mathbf{R}_n \otimes E\{(J^T_{\mathcal{F}}(\mathbf{t}(\mathbf{f})))\}$$

As in equation (4), we obtain a normalized error measure which depends on both the preferred directions in the target space as well as the non-linear transformation from the target space to a perceptually uniform space.

$$\chi_{\mathcal{F}}(\mathbf{D}, \mathbf{S}_f, \mathbf{S}_n, \mathbf{M}_H)$$
$$= \frac{vec(\mathbf{D}^T)^T \mathbf{S}_f(\mathbf{M}_H \otimes \mathbf{I}_r)[(\mathbf{M}_H^T \otimes \mathbf{I}_r)\mathbf{S}_f(\mathbf{M}_H \otimes \mathbf{I}_r) \;+\; \mathbf{S}_n]^{-1}(\mathbf{M}_H^T \otimes \mathbf{I}_r)\mathbf{S}_f vec(\mathbf{D}^T)}{vec(\mathbf{D}^T)^T \mathbf{S}_f vec(\mathbf{D}^T)} \tag{8}$$

The main difference between expressions (4) and (8) is that the latter treats the entire matrix $\mathbf{D}$ as one vector in sN-space, unlike the former which treats each column of $\mathbf{D}$, i.e. each preferred direction, as a separate vector. Further, the latter treats each column of $\mathbf{M}_H$ as a vector in rN-space, with zeros padding the N-vector. Because of these differences, the latter allows the use of product terms of the form $\sum w_{ijkl}\mathbf{m}_l(i)\mathbf{m}_k(j)$ (elements of the matrix $\mathbf{M}_H^T \otimes \mathbf{I}_r)\mathbf{S}_f(\mathbf{M}_H \otimes \mathbf{I}_r)$) while the former allows only terms of the form $\sum w_{ij}\mathbf{m}_l(i)\mathbf{m}_k(j)$ (elements of the matrix $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$). Another difference is that the weights (elements of the matrices $\mathbf{R}$ and $\mathbf{S}_f$ for the former and latter respectively) depend only on the data and not on $\mathbf{M}_H$ or $\mathbf{V}$ in the former case, but depend on both the data and $\mathbf{V}$ in the latter. However, the two expressions are similar in that the weights in both do not depend on $\mathbf{M}_H$ and both can be expressed in terms of inner products - the former in rN-space or sN-space, the latter in N-space as we shall show in section 4.

In the following section, we generalize the euclidean inner product to include weighting of different directions based on the data statistics. This provides a common, simple framework for all colour spaces and data sets as we show in section 4. While [1] provides a common framework as well, and includes measurement noise which ours does not, our explicit use of the inner product allows simple expressions linking even the most complex-looking measures to Neugebauer's Q-factor.

3 Generalization of inner products, orthogonality and projection operators

Data dependence generally weights different directions differently in the N-space of reflectance functions and recording filters, and it is useful to define the following inner product which accounts for the weighting:

$$< \mathbf{x}, \mathbf{y} >' = \mathbf{x}^T \mathbf{R} \mathbf{y} \tag{9}$$

For equation (9) to define an inner product it is necessary and sufficient that $\mathbf{R}$ be positive definite. In particular, this implies that $\mathbf{R}$ be invertible. Clearly, the inner product is the euclidean inner product and induces the euclidean norm when $\mathbf{R} = \mathbf{I}$. Note that this inner product defines error measures in the 'parent' N-space and not in the lower dimensional $R(\mathbf{V})$.

3.1 Induced norm and projection operator

Consider the norm induced by this inner product,

$$||\mathbf{x}||'^2 = < \mathbf{x}, \mathbf{x} >' = \mathbf{x}^T \mathbf{R} \mathbf{x} \tag{10}$$

If P'_X denotes the projection operator onto the space $R(\mathbf{X})$ with respect to the inner product $<, >'$, i.e. $P'_X(\mathbf{x})$ is the vector in $R(\mathbf{X})$ closest to $\mathbf{x}$ with respect to the norm $||.||'$ of equation (10), it can be shown that,

$$P'_X = \mathbf{X}(\mathbf{X}^T \mathbf{R} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{R} \tag{11}$$

3.2 Generalized Q-factor

The generalization of inner products and projection operators induces a generalized Q-factor. The Q-factor with respect to the space $R(\mathbf{X})$ and the generalized inner product may be defined as a generalization of Neugebauer's Q-factor [2] (see equation (5)):

$$q'_X(\mathbf{x}) \;=\; \frac{||P'_X(\mathbf{x})||'^2}{||\mathbf{x}||'^2} \tag{12}$$

Let $\{\mathbf{y}_i\}_{i=1}^{\omega}$ be an orthonormal basis for $R(\mathbf{X})$ with respect to the inner product in (9) (i.e. $\mathbf{Y}^T\mathbf{R}\mathbf{Y} \;=\; \mathbf{I}$ or the $\mathbf{y}_i$ are '$\mathbf{R}$-orthonormal', and $R(\mathbf{Y}) \;=\; R(\mathbf{X})$, *i.e.* $\mathbf{Y} \;=\; \mathbf{X}\mathbf{\Gamma}$ for invertible $\mathbf{\Gamma}$). Then, $P'_X \;=\; \mathbf{Y}\mathbf{Y}^T\mathbf{R}$ and equation (12) is:

$$q'_X(\mathbf{x}) \;=\; \frac{\sum_{i=1}^{\omega} < \mathbf{x}, \mathbf{y}_i >'^2}{< \mathbf{x}, \mathbf{x} >'} \tag{13}$$

in terms of the generalized inner product.

3.3 Generalized inner product notation for matrices

The generalization of inner products above can be used to rewrite some useful matrix expressions:

$$\mathbf{X}^T\mathbf{R}\mathbf{Y} \;=\; [< \mathbf{x}_i, \mathbf{y}_j >']$$

where $\mathbf{x}_i$ and $\mathbf{y}_i$ are the i^{th} columns of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Further

$$Trace\ \mathbf{X}^T\mathbf{R}\mathbf{Y} \;=\; \sum_i < \mathbf{x}_i, \mathbf{y}_i >'$$

4 Generalized inner products and error measures

Generalized inner products may be used to simplify the expressions for all measures based on an affine colour correction procedure.

4.1 Quadratic error measures

The most general quadratic error measure (equation(4)) for negligible measurement noise may be simply expressed in terms of inner products as follows. Let $\mathbf{G} \;=\; [\mathbf{g}_1, \mathbf{g}_2, ...\mathbf{g}_\gamma]$

where $\{\mathbf{g}_i\}_{i=1}^{\gamma}$ is an $\mathbf{R}$-orthonormal basis for $R(\mathbf{M}_H)$. The numerator in equation (4) is $Trace\ \mathbf{D}^T\mathbf{R}\mathbf{G}\mathbf{G}^T\mathbf{R}\mathbf{D}$ and

$$\chi(\mathbf{D},\ \mathbf{R},\ \mathbf{0},\ \mathbf{M}_H)\ =\ \frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\gamma} < \mathbf{d}_i, \mathbf{g}_j >'^2}{\sum_{i=1}^{\alpha} < \mathbf{d}_i, \mathbf{d}_i >'}$$

One of the strengths of the measure ν was the fact that it generalized Neugebauer's Q-factor. An expression for ν is:

$$\nu(\mathbf{M}_H, \mathbf{V})\ =\ \frac{\sum_{i=1}^{i=\beta} q(\mathbf{o}_i)}{\alpha}$$

where $\{\mathbf{o}_i\}_{i=1}^{\beta}$ is an orthonormal basis for $R(\mathbf{M}_H)$ with respect to the euclidean inner product and α is the dimension of R($\mathbf{V}$).

The measure χ can be represented in terms of generalized Q-factors as follows.

$$\chi(\mathbf{D},\ \mathbf{R},\ \mathbf{0},\ \mathbf{M}_H)\ =\ \frac{\sum_{i=1}^{\alpha} \eta_i q'_{M_H}(\mathbf{d}_i)}{\sum_{i=1}^{\alpha} \eta_i}$$

where η_i is the energy in the i^{th} preferred direction in $R(\mathbf{V})$, $\mathbf{d}_i$. Thus the most general quadratic error measure consists of a weighted average of the generalized Q-factors of preferred directions in the target space.

4.2 Accurate approximations of non-linear (perceptual) error measures

To extend the ideas of the previous section to accurate approximations of non-linear error measures, the definitions of inner products, projection operators and Q-factors need to be slightly modified. The matrix $\mathbf{D}$ is replaced by the vector vec($\mathbf{D}$), the matrix $\mathbf{M}_H$ by the matrix $\mathbf{M}_H \otimes \mathbf{I}_r$, and the matrix $\mathbf{R}$ by the matrix $\mathbf{S}_f$. The expression of equation (8) for negligible noise may be represented in terms of inner products as follows. If

$$< \mathbf{x}, \mathbf{y} >^{\mathcal{F}}\ =\ \mathbf{x}^T\mathbf{S}_f\mathbf{y}$$

and $\{\mathbf{g}_i\}_{i=1}^{\gamma}$ is an $\mathbf{S}_f$-orthonormal basis for $R(\mathbf{M}_H \otimes \mathbf{I}_r)$ in rN-space, then

$$\chi_{\mathcal{F}}(\mathbf{D},\ \mathbf{S}_f,\ \mathbf{0},\ \mathbf{M}_H)\ =\ \frac{\sum_{j=1}^{\gamma}(< vec(\mathbf{D}), \mathbf{g}_j >^{\mathcal{F}})^2}{< vec(\mathbf{D}), vec(\mathbf{D}) >^{\mathcal{F}}}$$

In terms of generalized Q-factors,

$$\chi_{\mathcal{F}}(\mathbf{D},\ \mathbf{R},\ \mathbf{0},\ \mathbf{M}_H)\ =\ q^{\mathcal{F}}_{M_H \otimes I_r}(vec(\mathbf{D}))$$

In the next section we illustrate the use of the generalized inner product in noise analysis.

5 Noise analysis using generalized inner products

The analysis performed so far ignores noise and hence implies that any set of filters $\mathbf{M}$ which maximizes expressions (4) or (8) is a 'good' set of filters. However, it is clear that the correction (equation (2)) may unduly amplify measurement noise, especially when the inverse problem is ill-conditioned. It is well-known that the inclusion of noise statistics in any LMMSE makes the inverse problem better conditioned. We address the problem of conditioning in this section and derive filter design criteria to reduce noise amplification by the correction matrix. Vrhel and Trussell have addressed this problem while analyzing robustness of colour correction to errors in filter design [4] and while addressing optimality of filters with respect to noise performance [8]. One of the solutions we present is similar to their solution, and is presented here in terms of inner products and orthogonality.

Here, we think of noise as that component of the output that has considerable variation over patches, and has its origin in the measurement noise. The error in colour reproduction which would be constant across a patch is not thought of as noise. Another way of thinking about the noise is thinking of it as the variation of the error in colour reproduction, while ignoring the mean.

5.1 Worst-case Signal to Noise Ratio

The min-max method of [4] may be used to analyze the worst-case signal to noise ratio (SNR) as follows. For a specific reflective spectrum $\mathbf{f}$ and noise $\mathbf{n}$, the SNR after correction is lowest when $\mathbf{M}_H^T\mathbf{f}$ is an eigenvector of minimum eigenvalue of the correction matrix (equations (3) and (7)), and $\mathbf{n}$ is an eigenvector of maximum eigenvalue. This minimum SNR is:

$$SNR_{min} = (\frac{\rho_{min}}{\rho_{max}})^2 \times \frac{Trace\ \mathbf{M}_H^T\mathbf{f}\mathbf{f}^T\mathbf{M}_H}{Trace\ \mathbf{n}\mathbf{n}^T} = (\frac{\rho_{min}}{\rho_{max}})^2 \times \epsilon = \omega^2\epsilon$$

where ρ_{min} and ρ_{max} are minimum and maximum eigenvalues respectively of the correction matrix, ϵ is the SNR before correction, and ω is the condition number of the correction matrix (the ratio of maximum to minimum eigenvalue). A max-min approach of maximizing the minimum SNR leads to maximizing $\frac{\rho_{min}}{\rho_{max}}$ or minimizing ω. The optimal solution is when ω is unity, or the correction matrix is a multiple of the identity (the recording filters are identical scalar multiples of the preferred directions).

When the correction matrix is not a multiple of the identity, ω is a measure of the amount of noise amplification for a specific viewing illuminant $\mathbf{L}$ and a specific estimated value of $\mathbf{R}_n$. An increase in the estimated noise variance decreases ω and hence also the noise amplification, but at the cost of color saturation [20]. Hence ω is not an accurate predictor of the image quality from a specific filter set, though it is a predictor of noise amplification. Calculating the value of ω for $\mathbf{R}_n = \mathbf{0}$ is a means of estimating the noise amplification and color saturation trade-off, but it is useful only when the filters are to be evaluated for a fixed viewing illuminant, which is not always the case (digital camera and scanner output images may need to be rendered for many different viewing conditions). In the following section we show how we can get around this limitation.

5.2 Ratio of expected signal power to expected noise power

The min-max method of [4] may be used to analyze the ratio of expected signal power to expected noise power after correction (denoted $S_{exp}N_{exp}R$) as follows. The value of $S_{exp}N_{exp}R$ assuming signal-independent and signal-uncorrelated noise is:

$$S_{exp}N_{exp}R \;=\; \frac{E[Trace\ (\mathbf{B}\mathbf{M}_H^T\mathbf{f}\mathbf{f}^T\mathbf{M}_\mathbf{H}\mathbf{B}^T)]}{E[Trace\ (\mathbf{B}\mathbf{n}\mathbf{n}^T\mathbf{B}^T)]} \;=\; \frac{Trace\ (\mathbf{B}\mathbf{M}_H^T\mathbf{R}\mathbf{M}_\mathbf{H}\mathbf{B}^T)}{Trace\ (\mathbf{B}\mathbf{R}_n\mathbf{B}^T)}$$

where $E[.]$ represents the expectation operator.

From the theory of matrix inequalities,

$$\lambda_{min} \leq S_{exp}N_{exp}R \leq \lambda_{max} \tag{14}$$

where λ_{min} and λ_{max} are minimum and maximum eigenvalues respectively of $(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)\mathbf{R}_n^{-1}$.

With no more knowledge about the nature of the individual matrices, an optimal solution is one where $\lambda_{min} \;=\; \lambda_{max}$ or, equivalently,

$$(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)\mathbf{R}_n^{-1} \;=\; c\mathbf{I}$$

for a constant c. This implies that:

$$\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H \;=\; c\mathbf{R}_n$$

or:

$$< \mathbf{M}_{Hi}, \mathbf{M}_{Hj} >' = \; cE(\mathbf{n}_i\mathbf{n}_j) \tag{15}$$

Hence, the correlation matrix of optimal effective recording filters is a scalar multiple of the noise correlation matrix. In particular, orthogonal noise variables imply **R**-orthogonal optimal effective recording filters, and independent, identically distributed noise variables imply **R**-orthogonal optimal effective recording filters of equal norm (which may be thought of as **R**-ortho*normal* recording filters). The first known use of orthogonality in colour recording filters was in [21], though to date there has been no literature on reasons why orthogonality is important.

In the rest of this paper, we assume that the noise variables are orthogonal and istotropically distributed, i.e., $\mathbf{R}_n = \sigma^2\mathbf{I}$, which implies that:

$$\lambda = \frac{\zeta}{\sigma^2} \tag{16}$$

where ζ is an eigenvalue of $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$. Inequality (14) becomes:

$$\zeta_{min} \leq \sigma^2 \times S_{exp}N_{exp}R \leq \zeta_{max} \tag{17}$$

The ratio of expected signal power to expected noise power before color correction is

$$\frac{Trace\ \mathbf{M}_H^T\mathbf{R}\mathbf{M}_H}{Trace\ \mathbf{R}_n} = \frac{Trace\ \mathbf{M}_H^T\mathbf{R}\mathbf{M}_H}{r \times \sigma^2} = \frac{1}{r \times \sigma^2}\sum \zeta_i$$

where r is the number of recording filters, or the number of measurements. This implies that the factor by which the ratio of expected signal to expected noise changes after color correction is:

$$\varsigma \equiv \frac{S_{exp}N_{exp}R}{\frac{1}{r\times\sigma^2}\sum\zeta_i} = \frac{\sigma^2 \times S_{exp}N_{exp}R}{\frac{1}{r}\sum\zeta_i} \tag{18}$$

Using the simple algebraic inequality:

$$\zeta_{min} \leq \frac{1}{r}\sum\zeta_i \leq \zeta_{max}$$

and equations (17) and (18) we obtain bounds on the factor by which the ratio of expected signal to expected noise increases on colour correction:

$$\frac{\zeta_{min}}{\zeta_{max}} \leq \frac{\zeta_{min}}{\frac{\sum\zeta_i}{r}} \leq \varsigma \equiv \frac{\sigma^2 \times S_{exp}N_{exp}R}{\frac{\sum\zeta_i}{r}} \leq \frac{\zeta_{max}}{\frac{\sum\zeta_i}{r}} \leq \frac{\zeta_{max}}{\zeta_{min}}$$

The condition number of $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$ and its inverse provide upper and lower bounds respectively on ς. We refer to the condition number of $\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H$ as κ in the rest of this paper, and propose its use as a measure of filter orthogonality and noise sensitivity. A perfect value of κ is unity, and indicates an $\mathbf{R}$-orthonormal set of color filters. Larger values of κ indicate 'less orthonormal' filters and larger noise amplification in general.

The condition number of the color correction matrix (equations (3) and (7)), ω, is distinct from the value κ, and takes the viewing illuminant and the value of σ into consideration while evaluating the filters. An optimal value of the colour correction matrix corresponds to effective recording filters that are a scalar multiple of the preferred directions, which need not be $\mathbf{R}$-orthogonal. When there are many different viewing illuminants and hence no fixed set of preferred directions, however, orthogonality is a useful criterion for optimality. The value κ is to be used in cases where the set of illuminants is not known.

5.3 Combination of two optimality criteria

'Most optimal' colour recording filters with respect to both colour fidelity and noise sensitivity are those that lie along the preferred directions. If it is not possible to design filters that lie along the preferred directions (for example, it is not possible to design filters that mimic the NTSC phosphor matching functions, Figure 8, because they have negative values at some points), and in cases where the recording filters are to be designed for a number of different viewing conditions, the 'most optimal' recording filters are those that span most of the space $R(\mathbf{V})$ and are $\mathbf{R}$-ortho*normal*, assuming noise variables are independent, identically distributed, and signal-independent.

6 Experimental Results

The sets of spectral responses used here are representative of several that were used in experiments during this work. We used six sets of spectral responses that are used in the manufacture of consumer digital cameras. Figures 1-6 show the normalized spectral responses. Sets 1-5 are spectral responses of sensor arrays with RGB (Red, Green, Blue) filters, while Set 6 is the response of a sensor array with CMYG (Cyan, Magenta, Yellow, Green) filters.

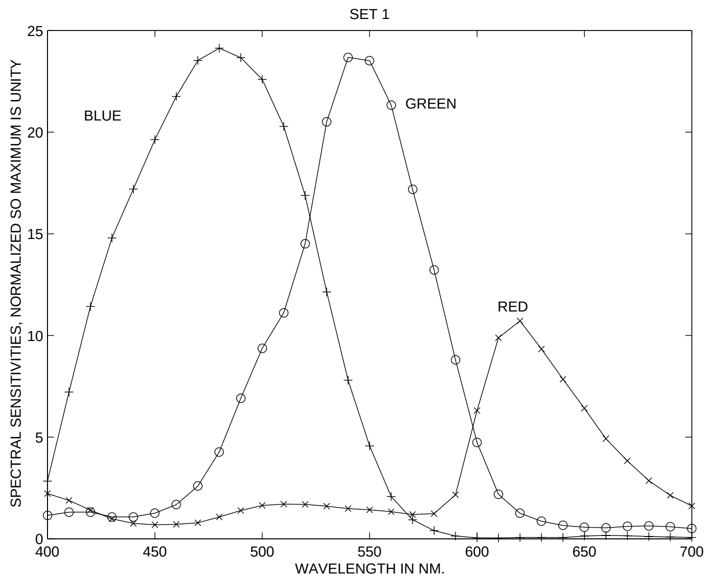

Figure 1: Set 1

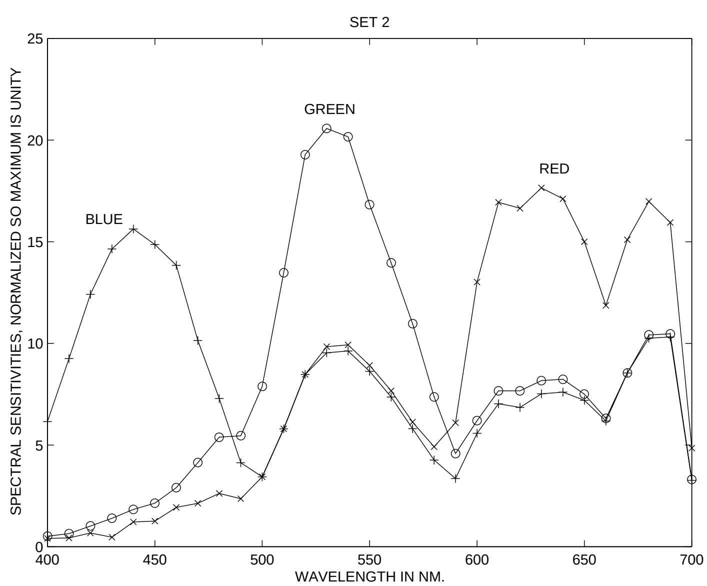

Figure 2: Set 2

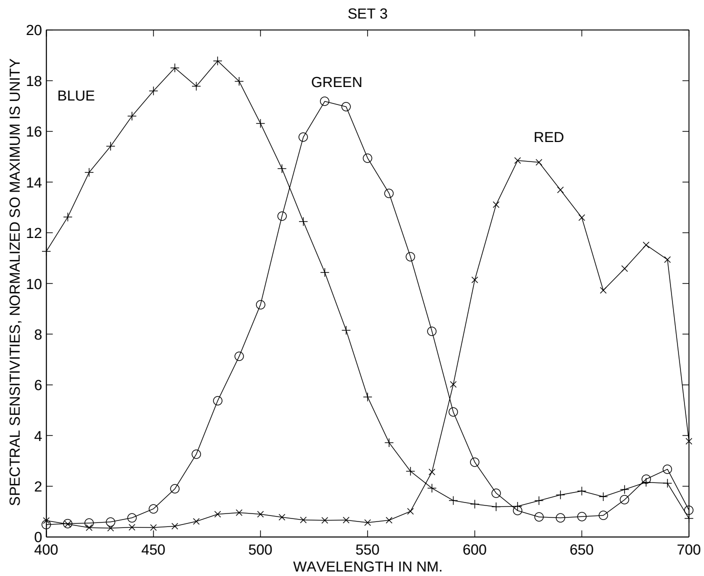

Figure 3: Set 3

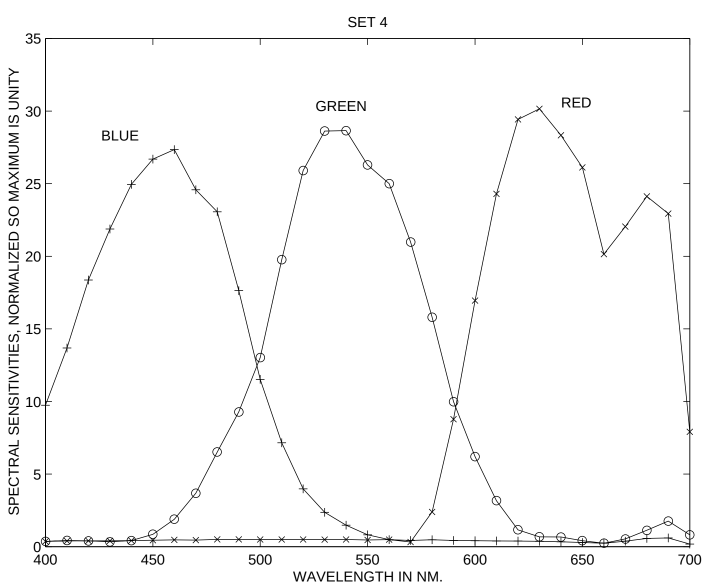

Figure 4: Set 4

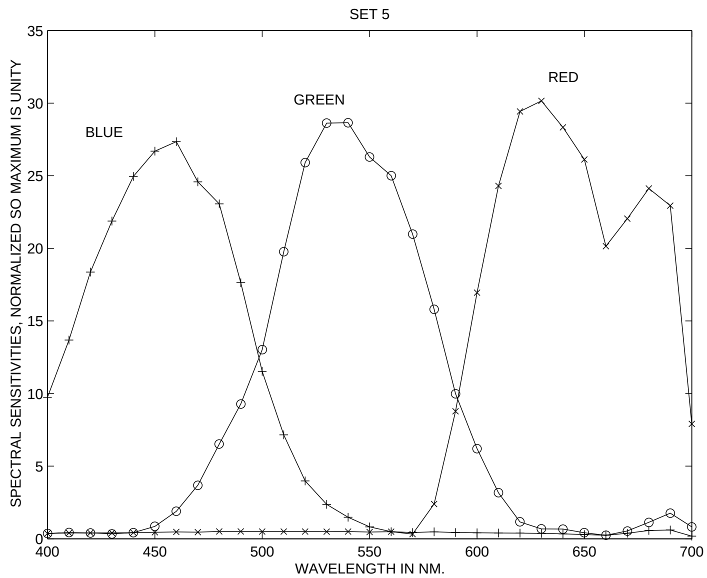

Figure 5: Set 5

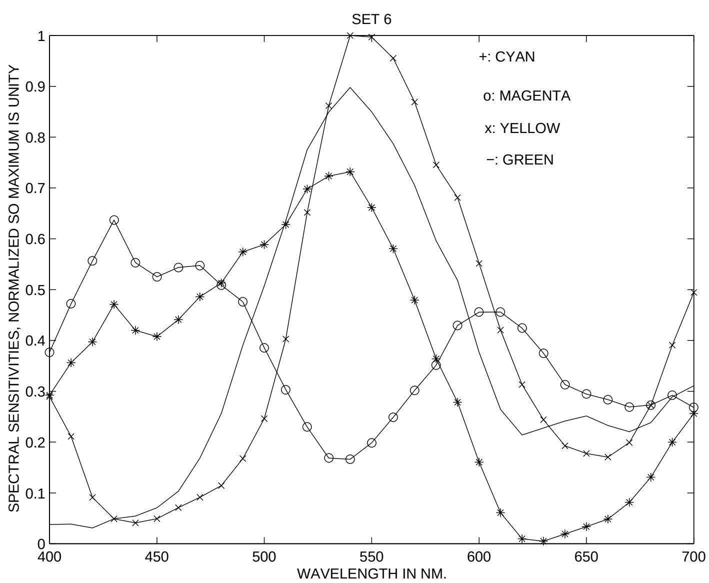

Figure 6: Set 6

For our simulations, we used four different illuminants - D65, a measured overcast daylight, a measured tungsten illuminant and a measured quartz illuminant. The normalized illuminants are plotted in Figure 7. The preferred target directions are NTSC phosphors [23] normalized so a flat spectrum gives equal R, G, and B values, in the corresponding illuminant. The NTSC phosphors are graphed in Figure 8.

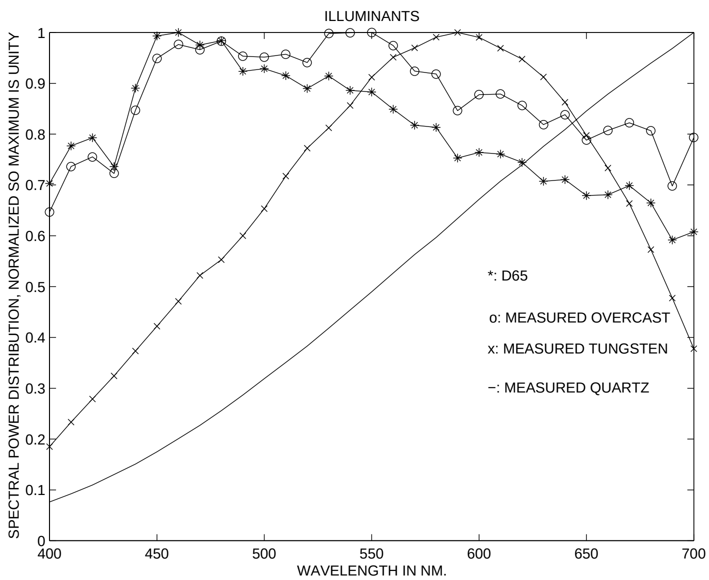

Figure 7: Illuminants

To test if our orthonormality measure was a valid predictor of noise performance, we performed the following simulations. For each set, combined with each illuminant as a recording illuminant and each illuminant as a viewing illuminant (sixteen combinations for each set) we calculated the LMMSE of tristimulus values wrt the NTSC phosphors for the Vrhel-Gershon-Iwan set assuming we knew the correlation matrix exactly. We calculated the colour correction matrix assuming four different values of noise variance (zero, estimated variance equal to the noise variance, estimated noise variance equal to twice and ten times the noise variance) and for 20 different realizations of a noise sequence with uniformly distributed noise corresponding to 8-bit quantization. We examined the values of different criteria considered in this paper (ω and κ) to determine their efficacy as predictors of ς.

Figure 9 shows a scatter plot of the value of ς vs. ω, the condition number of the corresponding colour correction matrix. ω predicts the value of ς well, as expected.

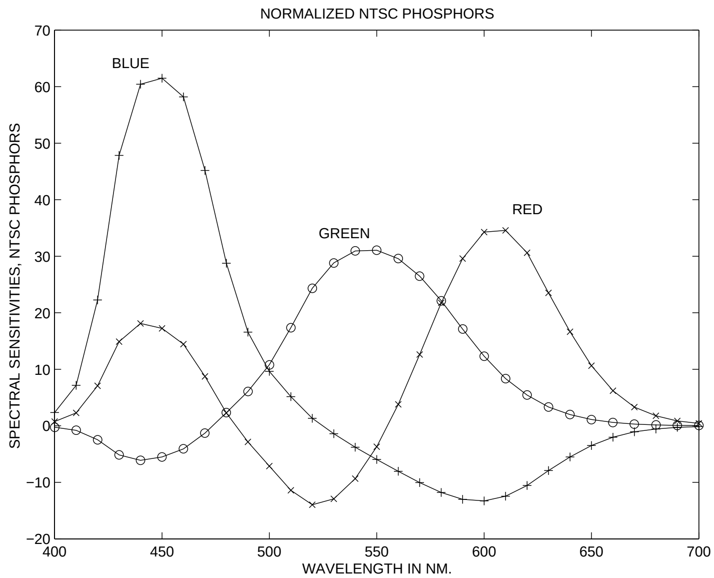

Figure 8: NTSC Phosphors, normalized

Figure 10 shows a scatter plot of the average value of ς over the different estimated noise variances vs. ω with estimated value of variance equal to zero. The plot is similar to the one in Figure 9, and average values of ς are well-predicted by values of ω for zero estimated noise variance.

Figure 11 shows a scatter plot of the average value of ς over the different estimated noise variances and the four different recording illuminants while fixing the viewing illuminant vs. the condition number of the colour correction matrix assuming zero noise variance and assuming that the viewing illuminant is identical to the recording illuminant.

Figure 12 shows a scatter plot of the average value of ς over the different estimated noise variances and the four different viewing illuminants while fixing the recording illuminant vs. the condition number of the colour correction matrix assuming zero noise variance and assuming that the viewing illuminant is identical to the recording illuminant.

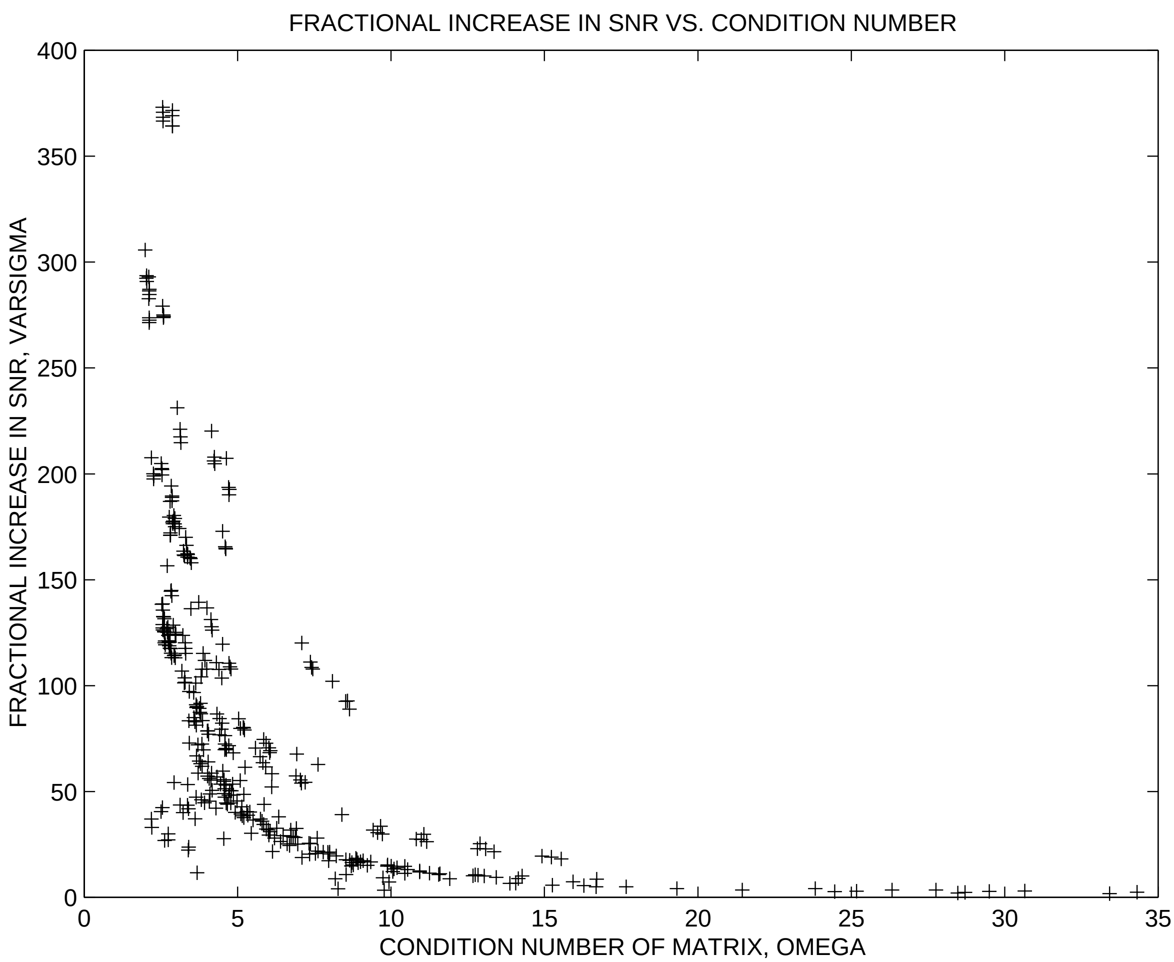

Figure 9: ς vs. ω

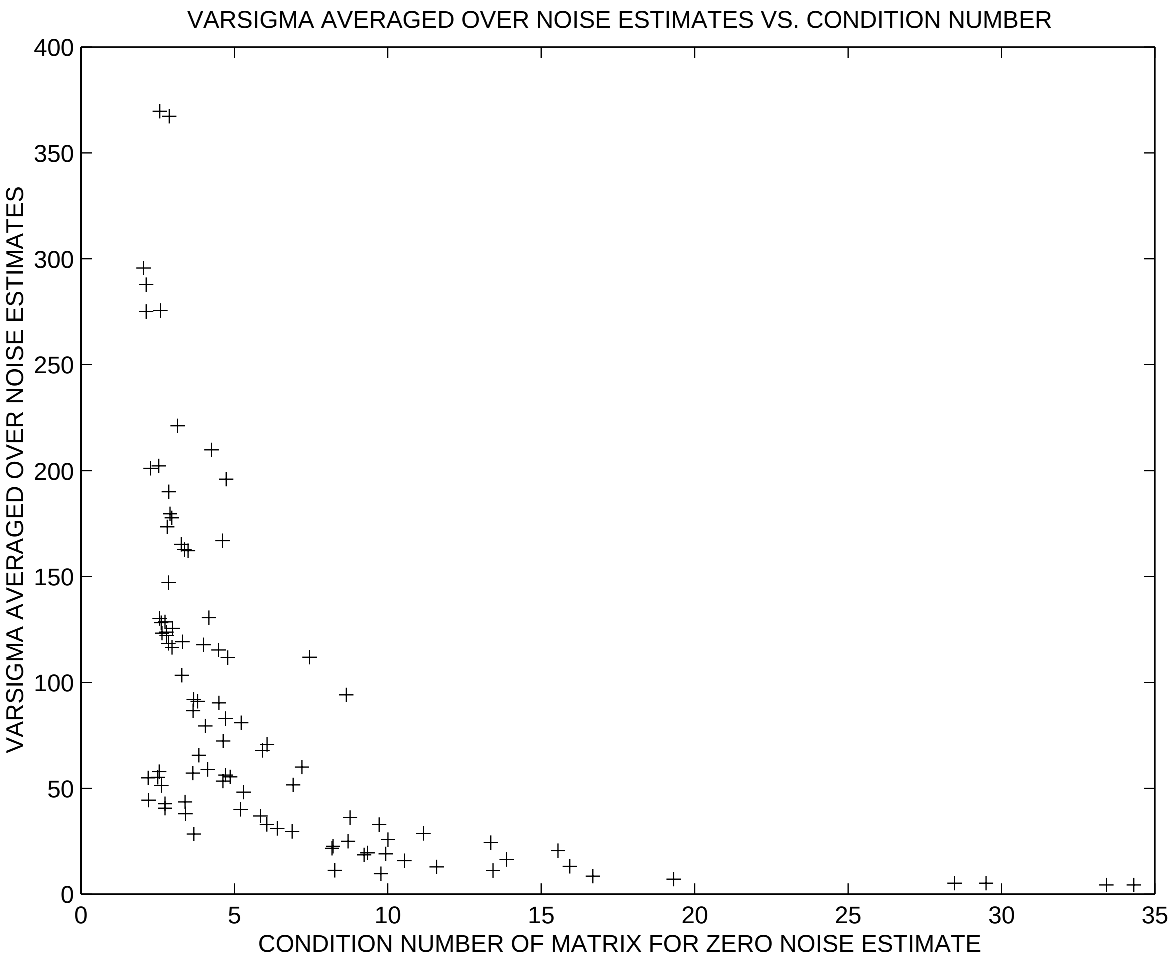

Figure 10: Average ς over different noise estimates vs. ω, noise estimate = 0

Figures 11 and 12 show very similar graphs. While the value of ς decreases in general with respect to the value of ω, it is not monotonic, and the value of ω does not seem to predict well the value of ς when averaged over many different viewing or recording illuminants. The data points for Set 6 are plotted as asterisks (*) and they lie well in the middle of the plots.

Figures 13 and 14 show the same data as in Figure 12 vs. the natural logarithm of the value of κ, for $E[\mathbf{ff}]^T = \mathbf{I}$ and $E[\mathbf{ff}]^T = \mathbf{R}$ respectively. Figure 14 shows a clear monotonic decreasing relationship between ς averaged over many viewing illuminants and the value of κ for $E[\mathbf{ff}]^T = \mathbf{R}$. The points that do not fit as well in this graph as most of the points are points associated with Set 6, and are shown with asterisks (*) as in Figures 11 and 12. Figures 13 and 14 demonstrate quite clearly that κ, a measure of filter orthonormality, is at least as good a measure of noise sensitivity as ω, though it depends upon fewer variables (it is independent of the viewing illuminant and of the estimated noise variance).

Figures 15 and 16 show the average of ς over the different noise estimates as well as over both recording and viewing illuminants, vs. the natural logarithm κ for $E[\mathbf{ff}]^T = \mathbf{I}$ and $E[\mathbf{ff}]^T = \mathbf{R}$ respectively. κ is clearly a good predictor of average performance over different viewing and recording illuminants, especially when the data set is known. The sets with high values of κ either have high overlap among the spectral response functions of each channel, (Sets 2 and 6) or have very different channel gains as well as fairly high overlap (Set 5).

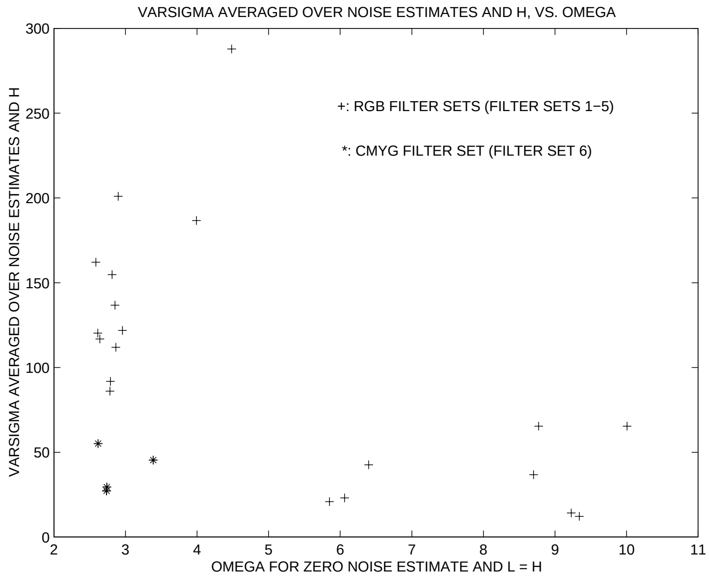

Figure 11: ς averaged over noise estimates and different recording illuminants, vs. ω, Viewing and Recording Illuminants identical

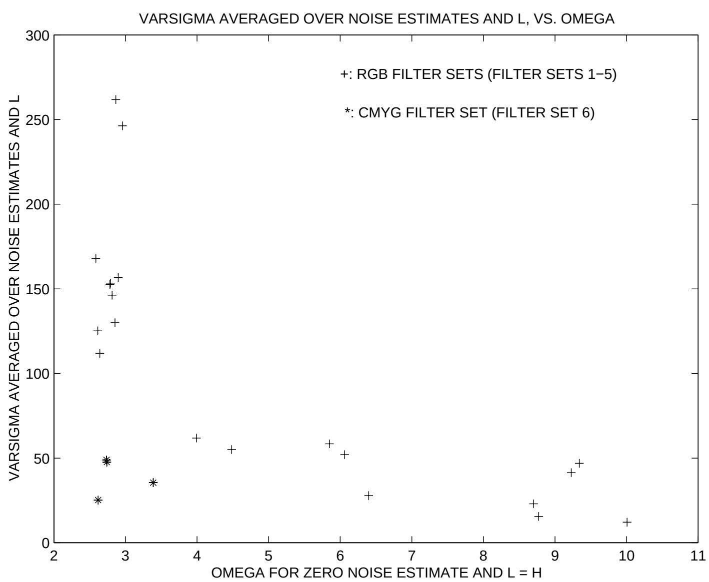

Figure 12: ς averaged over noise estimates and different viewing illuminants, vs. ω, Viewing and Recording Illuminants identical

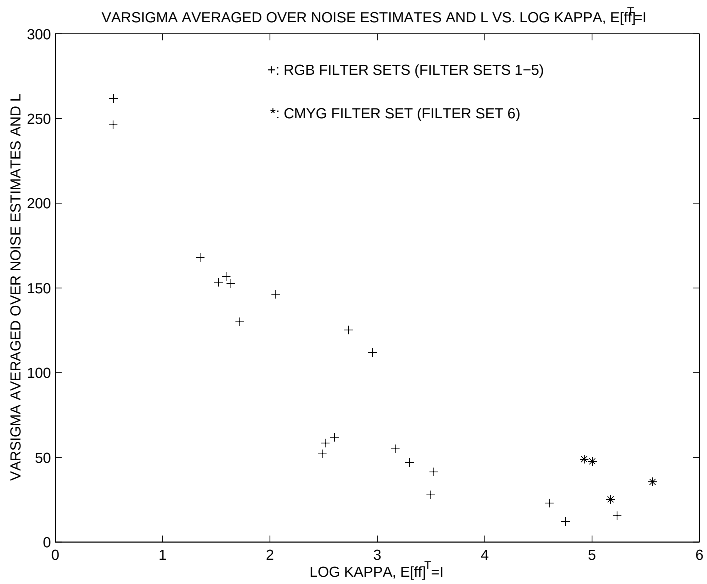

Figure 13: ς averaged over noise estimates and different viewing illuminants, vs. κ for $E\mathbf{f}\mathbf{f}^T = \mathbf{I}$

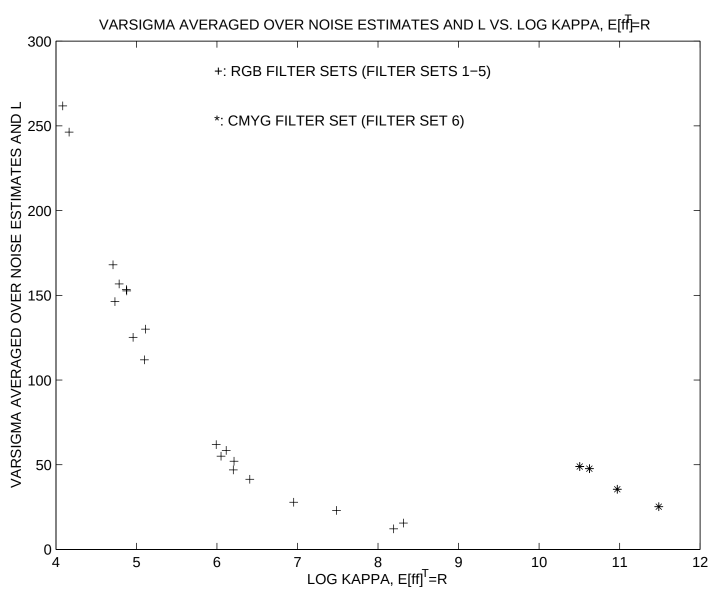

Figure 14: ς averaged over noise estimates and different viewing illuminants vs. κ for $E\mathbf{f}\mathbf{f}^T = \mathbf{R}$

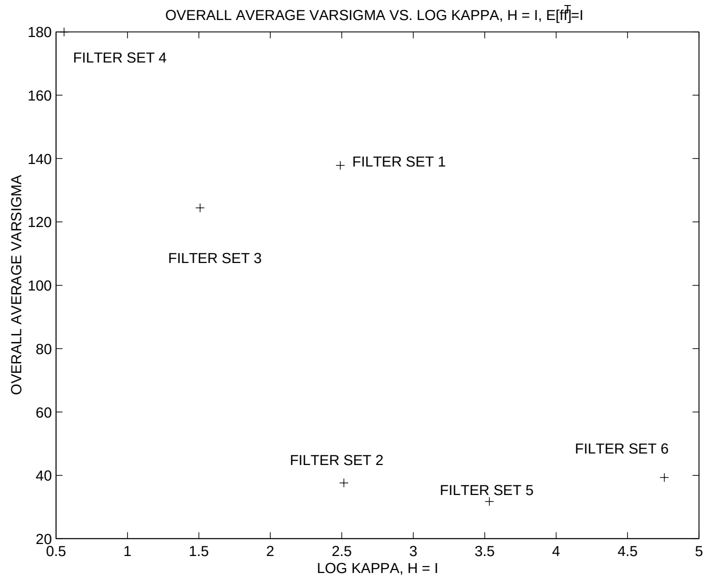

Figure 15: Average ς over all conditions vs. Orthonormality Criterion without recording illuminant, $E\mathbf{f}\mathbf{f}^T = \mathbf{I}$

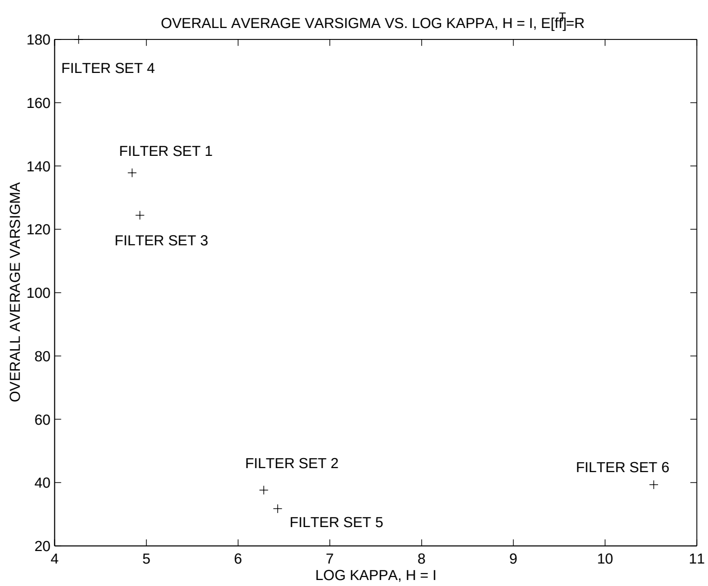

Figure 16: Average ς over all conditions vs. Orthonormality Criterion without recording illuminant, $E\mathbf{f}\mathbf{f}^T = \mathbf{R}$

7 Conclusions

We have built on the linear model ideas of [2, 13, 14, 15, 11, 3] to demonstrate how the use of inner products and orthogonality can simplify the procedure of colour recording filter design. We show how definitions of inner products and orthogonality are influenced by the choice of error measures for colour recording filters, and we use these ideas to develop a generalized Q-factor as an extension of Neugebauer's Q-factor [2]. We also use these ideas to show that optimal recording filters with respect to noise sensitivity are orthonormal filters when the noise variables are independent and identically distributed. Lastly, we present simulations to support our claims.

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