# *H=xp* and the Riemann Zeros

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We speculate on the possibility that the imaginary parts of the nontrivial zeros of the Riemann zeta function are the eigenvalues of the quantum operator corresponding to the classical Hamiltonian *H=xp*



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**Abstract**

We speculate on the possibility that the imaginary parts of the nontrivial zeros of the Riemann zeta function are the eigenvalues of the quantum operator corresponding to the classical Hamiltonian $H=xp$.

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We start by listing and briefly commenting on the properties of the Riemann operator that are suggested by the quantum analogy (see also [5-7]). We will call the operator $H$.

a. $H$ has a classical counterpart (the 'Riemann dynamics'), corresponding to a hamiltonian flow, or a symplectic transformation, on a phase space. This is based on a formal resemblance between the Von Mangoldt expansion [2] for the logarithm of the Euler product for $\zeta(1/2+iE)$ and the semiclassical expansion [8, 9] of quantum traces as sums over classical periodic orbits, and also on statistical evidence (see property b below).

b. The Riemann dynamics is chaotic, that is unstable and bounded. This is based on the observation that the local statistics of the $E_n$ are those of the eigenvalues of random matrices [10-14], and the connection of random-matrix statistics with the quantum mechanics of classically chaotic motion [6, 15-17]. Long-range correlations, between distant $E_n$, differ from those predicted by random-matrix theory [17, 18], and the differences are characteristic of quantum systems that have classical counterparts.

c. The Riemann dynamics does not have time-reversal symmetry. This is because the statistics of the $E_n$ are locally those of the gaussian unitary ensemble of complex hermitian random matrices [10, 19, 20], rather than the gaussian orthogonal ensemble of real matrices (which corresponds to systems with time-reversal symmetry). Related to this is the recent discovery [21, 22] of modified statistics of the low zeros for the ensemble of Dirichlet L-functions, associated with a symplectic structure.

d. The Riemann dynamics is homogenously unstable. This is suggested by the fact that the instability (Lyapunov) exponents of the periodic orbits are all unity, which follows from the exponential decay of the terms in the Von Mangoldt formula: $q^{-m/2}=\exp(-T_{m,q}/2)$, where $T_{m,q}$ is the orbit period defined in (3).

e. The classical periodic orbits of the Riemann dynamics have periods that are independent of energy $E$, and given by multiples of logarithms of prime numbers, that is

$$T_{m,q} = m \log q \quad (m = 1,2,\ldots;\ q \text{ prime}) \tag{3}$$

and the associated actions are

$$S_{m,q} = Em \log q \tag{4}$$

This follows from the form of the oscillatory terms in the analogy with the semiclassical trace formula. In terms of symbolic dynamics, the Riemann dynamics is peculiar, and resembles Chinese: each primitive orbit is labelled by its own symbol (the prime $q$) in contrast to the usual situation where periodic orbits can be represented as words made of letters in a finite alphabet.

f. The Maslov phases associated with the orbits are also peculiar: they are all $\pi$. This follows [5] from the negative signs of the terms in the Von Mangoldt formula. The result appears paradoxical in view of the relation between these phases and the winding numbers of the stable and unstable manifolds associated with periodic orbits [23], but finds an explanation in the scheme of Connes[3].

g. The Riemann dynamics possesses complex periodic orbits (instantons) whose periods are

$$T_{\text{complex},m} = im\pi \tag{5}$$

This is suggested by the small exponentials arising in the large-$E$ asymptotics of $\zeta(1/2+iE)$, associated with the high orders of the Riemann-Siegel expansion [24] and the high orders of the Stirling series for the gamma functions representing the smooth part of the counting function for the zeros [25].

h. For the Riemann operator, leading-order semiclassical mechanics is exact: $\zeta(1/2+iE)$ is a product over classical periodic orbits without quantum corrections (as in the case of the Selberg trace formula [26] for geodesic motion on surfaces of constant negative curvature).

i. The Riemann dynamics is quasi-one-dimensional. There are two indications of this. First, the number of zeros less than $E$ increases as $E\log E$ (see (9) below); for a $d$-dimensional scaling system, with energy parameter $\alpha(E)$ proportional to $1/\hbar$, the number of energy levels increases as $\alpha(E)^d$. Second, the presence of the factor $q^{-m/2}$ in the Von Mangoldt formula, rather than the determinant in the more general Gutzwiller formula, suggests that there is a single expanding direction and no contracting direction.

j. The functional equation for $\zeta(s)$ [2] resembles the corresponding relation - a consequence of hermiticity - for the quantum spectral determinant [7].

We note immediately that the system (2) represents the simplest form of instability, because it has a hyperbolic point at $x=0$, $p=0$. Hamilton's equations, and their solutions, are

$$\dot{x} = x, \quad \text{i.e. } x(t) = x(0)\exp(t); \quad \dot{p} = -p, \quad \text{i.e. } p(t) = p(0)\exp(-t) \tag{6}$$

Thus classical evolution is simply dilation in $x$ (that is, multiplication) and contraction in $p$, and the stretching exponent is unity, so that the instability is indeed homogeneous as required. In addition, $xp$ does not possess time-reversal symmetry, because it is not invariant under $p\rightarrow -p$; more fundamentally, reversal of velocity $\dot{x}$ for fixed $x$ does not lead to retracing of the orbit, for the simple reason that $\dot{x}$ is tied to $x$ and so cannot be reversed independently. Furthermore, dynamics generated by $xp$ is semiclassically exact.

## 2. SEMICLASSICAL LEVEL COUNTING

For any classically bound hamiltonian $H_{cl}(x, p)$ in one dimension, the number of quantum levels with energy less than $E$, the counting function, is

$$N(E) = A(E)/h + \ldots \tag{7}$$

where ... denotes higher-order terms in Planck's constant $\hbar=h/2\pi$ and $A(E)$ is the phase-space area under the contour $H_{cl}(x, p)=E$. With (1) there is the immediate problem that the classical motion is not bound, so that $A$ is infinite. Therefore the system must be regularized. The simplest regularization is to truncate $x$ and $p$ by extending the Planck cell with sides $l_x$, $l_p$ and area $h=l_x l_p$ as in figure 1, so that $A$ becomes the finite area indicated, which depends on $\hbar$. This makes the system quantum-mechanically quasi-one-dimensional. We cannot justify the regularization procedure, but note the analogy between this phase-space regularization and the fact that the hyperbola billiard in two dimensions is classically unbound but has a discrete quantum spectrum [27-29]. Thus

$$N(E) = \frac{1}{h}\left[ E \int_{l_x}^{E/l_p} \frac{dx}{x} - l_p\left(\frac{E}{l_p} - l_x\right)\right] + \ldots = \frac{E}{h}\left(\log\left(\frac{E}{h}\right) - 1\right) + 1 + \ldots \tag{8}$$

The constant (sub-leading) term should be modified by the Maslov phase. To guess this, we note that for a closed phase-space contour which turns by $-2\pi$, the extra term in the counting function is $+1/2$ (cf.the harmonic oscillator with frequency $\omega$, for which $N(E)=\mathrm{Int}(E/\hbar\omega+1/2)$). For (1) the turn is $+\pi/2$, so the extra term should be $-1/8$. Choosing units such that $\hbar=1$, (equivalent to replacing $E$ by $\hbar E$), we now obtain

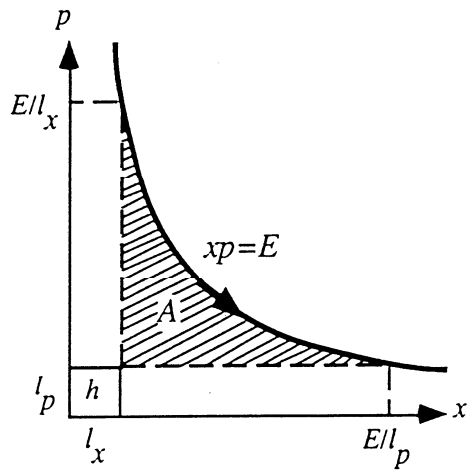


**Figure 1.** Phase space for $H_{\rm cl}=xp$, with cutoffs $l_x$ and $l_p$ for semiclassical regularization.

$$N(E)=\frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right)-1\right)+\tfrac{7}{8}+\ldots \tag{9}$$

This is precisely the asymptotic form of the smoothed counting function for the Riemann zeros, namely

$$N_{\rm sm}(E)=\theta(E)/\pi+1 \tag{10}$$

where

$$\theta(E)=-\frac{E}{2}\log\pi+\operatorname{Im}\log\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\mathrm{i}E\right) \tag{11}$$

correct to terms that do not vanish as $E\to\infty$. This is unlikely to be a coincidence.

## 3. CONFIGURATION AND MOMENTUM EIGENFUNCTIONS

The simplest formally hermitian operator corresponding to (1) is

$$H=\tfrac{1}{2}(xp+px)=-\mathrm{i}\hbar\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\tfrac{1}{2}\right) \tag{12}$$

The formal eigenfunctions, satisfing

$$H\psi_E(x)=E\psi_E(x) \tag{13}$$

are

$$\psi_E(x)=\frac{A}{x^{1/2-\mathrm{i}E/\hbar}} \tag{14}$$

We note the appearance of the power $x^{-s}$ appearing in the Dirichlet series for $\zeta(s)$ (as integer$^{-s}$) and the Euler product (as prime$^{-s}$).

The corresponding momentum eigenfunction is

$$\phi_E(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \mathrm{d}x\, \psi_E(x) \exp(-\mathrm{i}px/\hbar) \tag{15}$$

To evaluate this, we must choose a continuation of $\psi_E(x)$ across the singularity at $x$=0. The simplest choice is that the eigenfunctions are even. Then

$$\begin{aligned} \phi_E(p) &= \frac{A}{|p|^{1/2+\mathrm{i}E/\hbar}\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{|u|^{1/2-\mathrm{i}E/\hbar}} \exp(-\mathrm{i}u) \\ &= \frac{A}{|p|^{1/2+\mathrm{i}E/\hbar}} \left(\frac{h}{\pi}\right)^{\mathrm{i}E/\pi} \frac{\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}E}{2\hbar}\right)}{\Gamma\left(\frac{1}{4}-\frac{\mathrm{i}E}{2\hbar}\right)} \end{aligned} \tag{16}$$

where the reflection and duplication formulas for the gamma function have been used. Noting the similarity with (11), and writing $x$ and $p$ in terms of the sides of the Planck cell, we find

$$\begin{aligned} \psi_E(x) &= \frac{\exp\{-\mathrm{i}\theta(E/\hbar)\}}{\sqrt{l_x}|x/l_x|^{1/2-\mathrm{i}E/\hbar}} \\ \phi_E(p) &= \frac{\exp\{\mathrm{i}\theta(E/\hbar)\}}{\sqrt{l_p}|p/l_p|^{1/2+\mathrm{i}E/\hbar}} = \sqrt{\frac{l_x}{l_p}}\left[\psi_E\left(pl_x/l_p\right)\right]^* \end{aligned} \tag{17}$$

Henceforth we set $l_x$=$l_p$=$\sqrt{(2\pi)}$, i.e. $\hbar$=1.

The meaning of this symmetry is that position and momentum eigenfunctions are each other's time-reverse (cf. figure 1): thus we have a physical interpretation of the function $\theta(E)$ at the heart of the functional equation for $\zeta(s)$ [30], which states that the function

$$Z(E) \equiv \exp\{\mathrm{i}\theta(E)\}\zeta(1/2+\mathrm{i}E) \tag{18}$$

is even, and from which it follows that $Z(E)$ is real when $E$ is real.

If the hamiltonian had not been symmetrized to make it formally hermitian, we would not have obtained the results (14) and (17), containing the same combination 1/2+i$E$ as occurs in $\zeta(s)$ on the critical line.

Equation (17) is a special case of a more general relation between the position and momentum eigenfunctions, obtained by allowing the multipliers $A$ in (14) to be different for positive and negative $x$. The relation is

$$\begin{aligned} \psi_E(x) &= \frac{\exp\{-\mathrm{i}\theta(E)\}}{|x/2\pi|^{1/2-\mathrm{i}E}}\left[A_+\Theta(x) + A_-\Theta(-x)\right] \\ \phi_E(p) &= \frac{\exp\{\mathrm{i}\theta(E)\}}{|p/2\pi|^{1/2+\mathrm{i}E}}\left[B_+\Theta(x) + B_-\Theta(-x)\right] \end{aligned} \tag{19}$$

where $\Theta$ denotes the unit step function, and the $x$ and $p$ multipliers are related by

$$\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = \mathsf{M}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \tag{20a}$$

where M is the unitary matrix

$$\mathsf{M} = \frac{(\exp(E\pi) - \mathrm{i})}{2\cosh(E\pi)} \begin{pmatrix} 1 & \mathrm{i}\exp(-E\pi) \\ \mathrm{i}\exp(-E\pi) & 1 \end{pmatrix} \tag{20b}$$

The unitarity of M implies

$$|A_+|^2 + |A_-|^2 = |B_+|^2 + |B_-|^2 \tag{21}$$

- a relation that can be interpreted in terms of phase-space currents: the total $x$ current flowing out from the origin equals the total $p$ current flowing into the origin (figure 2a). These currents $J_x$ and $J_p$ are the expectation values of the local velocity operators:

$$\begin{aligned} J_x(x) &= \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x' \psi^*(x') \left[ \delta(x - x') \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \delta(x - x') \right] \psi(x') \\ &= x|\psi(x)|^2 = 2\pi \left[ |A_+|^2 \Theta(x) - |A_-|^2 \Theta(x) \right] \end{aligned} \tag{22a}$$

and similarly

$$J_p(p) = 2\pi \left[ -|B_+|^2 \Theta(p) + |B_-|^2 \Theta(p) \right] \tag{22b}$$

Of course, the hamiltonian $xp$ is simply a canonically rotated form of the upturned harmonic oscillator $p^2$-$x^2$, which is in turn a complexified version of the usual harmonic oscillator $p^2$+$x^2$. These connections have been noted before. Nonnemacher and Voros[31] calculate the Wigner function corresponding to $xp$, in a study of eigenstates near hyperbolic points. Bhaduri et al [32] and Khare [33] show that the density of scattering states of the second-order operator $p^2$-$x^2$ resembles $\mathrm{d}\theta(E)/\mathrm{d}E$ (the difference is a constant); Armitage[34] studies the fourth-order combinations $(p^2 \pm x^2)^2$; and Okubo[35] studies the two-dimensional hamiltonian $p_x^2$-$x^2$-$p_y^2$+$y^2$. The first-order operator $xp$ is the simplest representative of this class, with the monomials (14) avoiding the complications of the parabolic cylinder eigenfunctions of $p^2$-$x^2$. Indeed, it is possible give a very simple derivation of transmission and reflection from the potential -$x^2$, using a quantum canonical transformation of the states (14) with appropriate connections across the singularity at $x$=0.

## 4. $x$ AND $p$ CONNECTIONS

It would be desirable to replace the semiclassical regularization of $xp$ in (section 2) with a quantum boundary condition that would generate a discrete spectrum in a natural way. We do not know how to do this, but offer some remarks.

It is likely that $x$ and -$x$ should be identified, and also $p$ and -$p$, as in (17). This is suggested by a consideration of the complex periodic orbits of $xp$. With imaginary time the orbits (6) are periodic (as in an ordinary, rather than an inverted, harmonic oscillator), but the periods are wrong: $2i\pi m$, rather than $i\pi m$ as required by property g in section 1. Note however that after odd multiples of the time $i\pi$, $x$ evolves to -$x$ and $p$ to -$p$, so that identification of $\pm x$ and $\pm p$, as shown in figure 2, produces the required complex periods.

Even after these identifications, the system remains open. Ways to close it, and thereby force the spectrum to be discrete, are suggested by the *symmetries* of $xp$. Using these, we will try to incorporate the fact that the eigenstates of a hermitian operator with

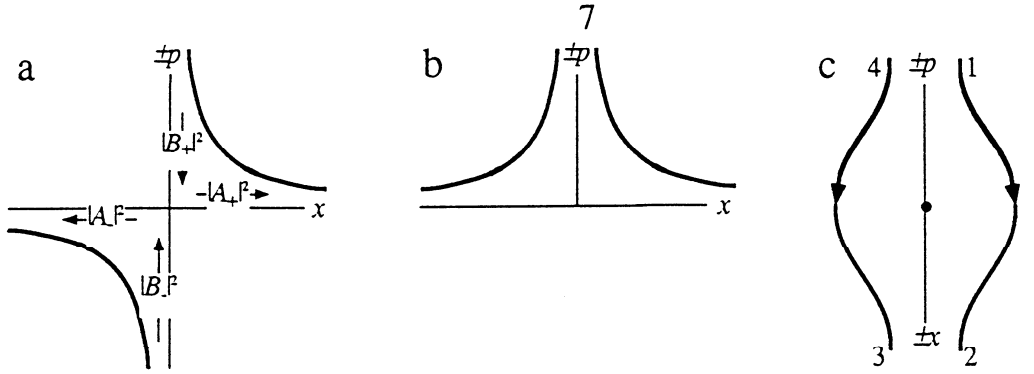


**Figure 2.** (a) phase space for $H_{Cl}=xp$, showing positive-energy contours; (b) with $p$ and $-p$ identified; (c) with $x$ and $-x$ identified.

symmetry can be written as superpositions of solutions of the eigenequation acted on by operations in the symmetry group, with each solution in the superposition multiplied by the appropriate group character.

An obvious symmetry of (2) is that $xp$ is invariant under *dilations*:

$$x \to Kx, \quad p \to p/K \tag{23}$$

From (6), $K$ corresponds to evolution after time $\log K$. This implies that the operator (11), corresponding to $xp$, generates dilations, in the same way that the momentum operator generates translations; the following sequence of transformations makes this obvious:

$$\begin{aligned} f(Kx) &= f\left(\exp\{\log K + \log x\}\right) = \exp\left\{(\log K)\frac{\mathrm{d}}{\mathrm{d}\log x}\right\} f(x) \\ &= \exp\left\{(\log K)x\frac{\mathrm{d}}{\mathrm{d}x}\right\} f(x) = K^{x\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = \frac{1}{K^{\frac{1}{2}-iH}} f(x) \end{aligned} \tag{24}$$

It is tempting to choose the integer dilations $K=m$, corresponding to evolution times $\log m$, and the characters unity, and write

$$\psi_E(x) \to \sum_{m=1}^{\infty} \psi_E(mx) = \frac{\text{constant}}{|x|^{1/2-iE}} \times \sum_{m=1}^{\infty} \frac{1}{m^{1/2-iE}} = \frac{\text{constant}}{|x|^{1/2-iE}} \zeta\left(\tfrac{1}{2} - iE\right) \tag{25}$$

A requirement that this must vanish would, if interpreted as an eigencondition, yield the Riemann zeros $E_n$ as eigenvalues. However, we see no reason to impose this requirement, and moreover the set of dilations $K=m$ does not form a group (the inverse multiplications $1/m$ are missing). Even worse, putting $E=E_n$ in (25) destroys the 'eigenfunction' by making it vanish for all $x$.

Another possibility, closely related to the ideas of Connes[3], is to use not all integers but the group of integers under multiplication (mod $k$). This would have two advantages. First, the group involves only integer and not fractional dilations. Second, it opens the possibility that the group characters [36] can appear as multipliers in the Dirichlet series for $\zeta$, thereby yielding the zeros of the different Dirichlet $L$-functions (which are all conjectured to have zeros in the line $\mathrm{Re}s=1/2$) as eigenvalues of different self-adjoint extensions of $xp$.

Another way to close the system $xp$ could be to connect the asymptotic positions with the asymptotic momenta. Then the current flowing out at $x=\pm\infty$ would be re-injected at $p=\pm\infty$. We envisage two such connections. Referring to figure 2c, we could connect 1 with 2 and 3 with 4, thus preserving the separation of the original quadrants (opposite in figure 2a)

and yielding a phase space with cylindrical topology; or we can connect 1 with 3 and 2 with 4, thereby connecting the quadrants (as does the matrix M in (20)) and yielding a phase space with Möbius topology.

A way to accomplish this connection is suggested by the fact that the dilations $K$ under which $xp$ is invariant need not be constant but can be any function of $xp$. The choice $K=h/(xp)$ yields the canonical transformation

$$x \to x_1 = \frac{h}{p}, \quad p \to p_1 = \frac{xp^2}{h} \tag{26}$$

Because of the $h$-dependence, we call this *quantum exchange* (the simpler canonical exchange $x \to p$, $p \to -x$ does not leave $xp$ invariant). Under quantum exchange, the hyperbolas $xp=E$ are of course invariant curves; $E=h$ is a curve of fixed points, with points on the curves $E<h$ mapping towards increasing $x$, and points on the curves $E>h$ mapping towards decreasing $x$. To see the corresponding transformation of quantum states, we represent these in Hilbert space as kets $|\psi\rangle$, and employ the notations

$$\begin{aligned} &\langle x|\psi\rangle \equiv \psi(x), \qquad \langle p|\psi\rangle \equiv \phi(p), \\ &\langle x_1|\psi\rangle \equiv \psi_1(x_1), \quad \langle p_1|\psi\rangle \equiv \phi_1(p_1) \end{aligned} \tag{27}$$

Then the quantum implementation of exchange is

$$\psi_1(x_1) = \frac{\sqrt{h}}{|x_1|}\phi_1\left(\frac{h}{|x_1|}\right) \tag{28}$$

(obviously, this would preserve normalization of the state).

Superposition of states related by this exchange operation gives, after using (17)

$$\psi_E(x) \to \psi_E(x) + \frac{\sqrt{h}}{x}\phi_E\left(\frac{h}{x}\right) = \frac{2\cos\{\theta(E)\}}{\left|x/\sqrt{h}\right|^{1/2-iE}} \tag{29}$$

If we could argue that this should vanish, the resulting 'quantization condition' would be the vanishing of the first term of the main sum of the Riemann-Siegel formula [2]. This would give zeros with the correct density, and it is tempting to regard it as arising from some hamiltonian operator, and then seek to generate the true Riemann operator from a series of corrections.

However, this hope is unlikely to be realised, because (29) possesses complex zeros and so cannot be associated naively with a hermitian operator. To demonstrate the existence of these zeros off the critical line, we write (29) in the following form, which follows from (11):

$$\begin{aligned} &g(s) = f(s) + f(1-s) = 0, \\ &\text{where } f(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \end{aligned} \tag{30}$$

This has zeros for $s$ real, that is $E$ imaginary, at

a b

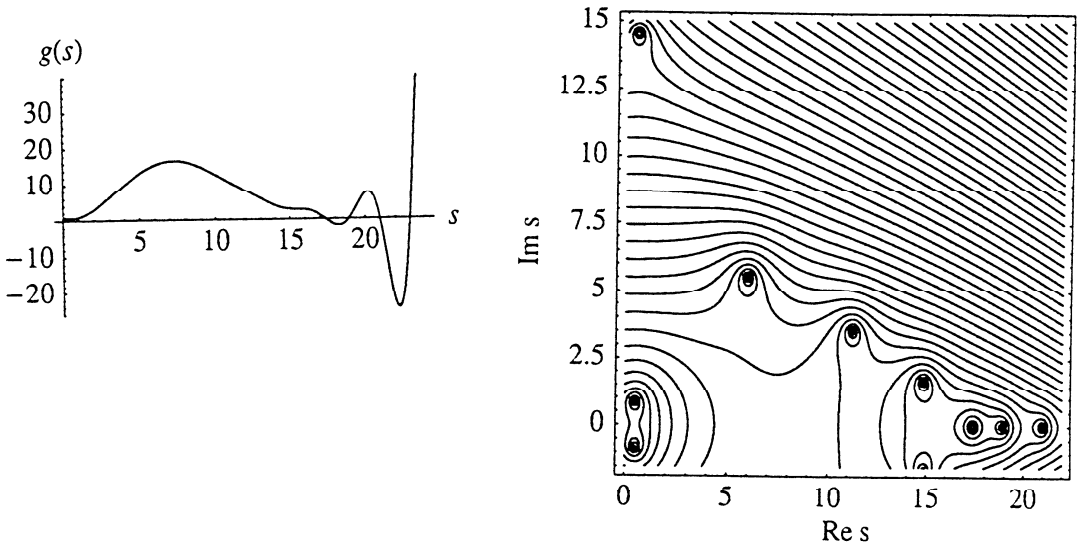


**Figure 3.** The function $g(s)$, whose zeros are the same as the first term of the Riemann-Siegel main sum, (a) on the real $s$ axis, (b) contours of $|g(s)|$, showing zeros enclosed by loops.

$$s \approx 2m+1+\frac{(-1)^m}{\sqrt{\pi m}}\left(\frac{\pi e}{m}\right)^{2m}, \quad (m=8,9,\ldots) \tag{31}$$

The first few are illustrated in figure 3a. There are also at least three zeros, shown in figure 3b, between the real axis and the critical line. And visible in figure 3b are zeros of (29) that are on the line but do not correspond to Riemann zeros; these lie near $s$=1/2±0.82i. Similar arguments establish the existence of zeros off the line when more terms of the Riemann-Siegel main sum are added to (29). It follows that the vanishing of (29) is not a boundary condition corresponding to a hermitian operator.

Combining the two symmetries - integer dilation and quantum exchange - suggests the 'boundary condition'

$$\sum_{m=1}^{\infty}\psi_E(mx)+\frac{\sqrt{h}}{x}\sum_{k=1}^{\infty}\phi_E(mh/x)=\frac{2}{\left|x/\sqrt{h}\right|^{1/2-iE}}Z(E)=0 \tag{32}$$

Using (24), this can be put into the intriguing form (with operators temporarily denoted by carats for clarity)

$$\langle x|\hat{x}^{\frac{1}{2}}\zeta\left(\tfrac{1}{2}-i\hat{H}\right)|\psi_E\rangle+\langle p|\hat{p}^{\frac{1}{2}}\zeta\left(\tfrac{1}{2}+i\hat{H}\right)|\psi_E\rangle=0 \quad (xp=h) \tag{33}$$

These conditions do generate the Riemann zeros, but we see no way to interpret either of them geometrically.

## 5. GAUSS MAP AS A BOUNDARY CONDITION?

The relations (25) and (32) are multiplicative: they involve formal eigenfunctions of $xp$ at multiples of any given $x$ and the associated momentum $h/x$. A different relation, combining multiplication with addition, connects values of $x$ related by the Gauss map that generates continued fractions. This involves the generalized transfer operator [37], and the requirement that this operator has eigenvalue unity [38]. The eigencondition corresponding to this map is

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)^s} f_s\left(\frac{1}{n+x}\right) = f_s(x) \tag{34}$$

This was introduced[38] as a quantum map giving discrete eigenvalues associated with the modular domain. The natural exponent is then $s=1+iE$, with $E$ real, so that factors in the sum are 'semiclassical' complexified square roots of the jacobians in the corresponding 'classical' transfer operator, which would have $s=2$. However, the Riemann zeros follow from the different association $s=1/2+iE$, with $E$ real, This is semiclassically mysterious because the factors in (34) now correspond to 1/4 powers of the classical jacobians. The argument, explained to us by Bogomolny (personal communication) is as follows.

Define

$$h_s(x) \equiv f_s(x-1) \tag{35}$$

and seek a formal eigenfunction of (34) in the form

$$h_s(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(mx+k)^s} \tag{36}$$

where $s=1/2+iE$. The condition (34) becomes

$$\begin{aligned} h_s(x) &= \sum_{n=1}^{\infty} \frac{1}{(n+x-1)^s} h_s\left(\frac{n+x}{n+x-1}\right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\left[x(m+k)+n(m+k)-k\right]^s} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\left[m(x+n)\right]^s} \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=0}^{l-1} \frac{1}{\left[xl+nl-k\right]^s} - \zeta(s) \sum_{n=1}^{\infty} \frac{1}{(x+n)^s} \\ &= h_s(x) - \zeta(s) \sum_{n=1}^{\infty} \frac{1}{(x+n)^s} \end{aligned} \tag{37}$$

where the last equality follows after noticing that the sums over $n$ and $k$ can be conflated into a single sum over the variable $nl-k$. Obviously the condition is satisfied whenever $1/2+iE$ is a Riemann zero.

It might seem that the eigenfunctions disappear at the Riemann zeros even without the condition (34), because the summation in (36) can be taken over multiples of coprime $(m,k)$ pairs and $\zeta(s)$ extracted as a factor:

$$h_s(x) = \sum_{l=1}^{\infty} \sum_{(m,k)=1} \frac{1}{\left[l(mx+k)\right]^s} = \zeta(s) \sum_{(m,k)=1} \frac{1}{(mx+k)^s} \tag{38}$$

If this were a valid objection, the solution (36) would be empty. But it is not valid, because (36) is a formal expression that does not converge when $E$ is real. It can be analytically continued onto the critical line, for example by

$$h_s(x) = \frac{1}{\Gamma(s)\left[\exp(2\pi i s)-1\right]} \int_C dt \frac{t^{s-1}}{\left[\exp(t)-1\right]\left[\exp(xt)-1\right]} \tag{39}$$

where C is a loop starting and ending at $t=+\infty$, encircling the origin positively and enclosing no other poles. The integral, when evaluated numerically, does not vanish at the Riemann zeros (figure 4).

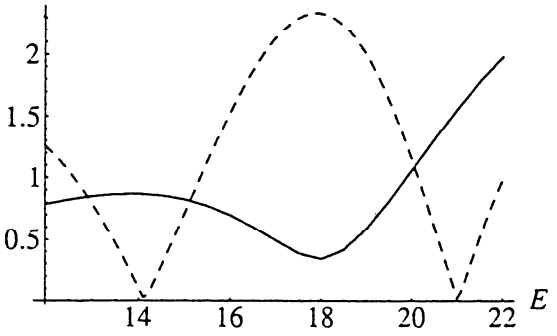


**Figure 4.** Absolute values of $\zeta(s)$ (dashed curve) and the Gauss map eigenfunction $h_S(0.5)$ (computed from the integral (35)) (full curve) on the critical line $s=1/2+iE$.

Now, $h_S(x)$ in (36) can be regarded as a sum of eigenfunctions (14) of $xp$, evaluated at positions $x+k/m$ that differ by rational numbers. Therefore the condition (34) might be interpretable as a boundary condition, relating the eigenfunction at each such position to its pre-images under the Gauss map. We do not know how to pursue this suggestion.

## 6. Concluding remarks

We have presented several tantalizing connections between $xp$ and $\zeta(s)$. However, it is clear that more is required to transform our hints and guesses into an unambiguous and satisfactory construction of the Riemann operator. There are two principal unsolved problems.

First, the space on which $xp$ acts is not known. Somehow the plane must be sewn up into a region that makes the dynamics bound, at least quantally. We have speculated that this might involve connecting $x$ and $p$, or relating multiples of $x$ or rational translations of $x$ (to see how complicated this can get, compare the space obtained by identifying $x$ with $nx$ for real $x$ and all integers $n$ with the familiar circle obtained by identifying $x$ with $n+x$ ). Perhaps

the required space is a quantum graph [39, 40], with $xp$ acting on bonds between vertices (one difficulty is that $xp$ does not sit naturally on a general graph).

Second, we do not know how to associate the primes with the periodic orbits of the Riemann dynamics.

In terms of the properties listed in the introduction, $xp$ is consistent with a, part of b ($xp$ dynamics is unstable but not bound), c, d, g, h i and j. Concerning e, the appearance of times that are logarithms of integers begins to be plausible in view of the association between dilation and evolution, but primes do not appear in any obvious way. We have no explanation of f.

There are probably more connections between $xp$ and $\zeta(s)$. Our hope is that in writing this paper we will stimulate others to uncover them.

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