Measurement of Time-of-Arrival in Quantum Mechanics

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It is argued that the time-of-arrival cannot be precisely defined and measured in quantum mechanics. By constructing explicit toy models of a measurement, we show that for a free particle it cannot be measured more accurately than $\Delta t_A \sim \hbar/E_k$, where E_k is the initial kinetic energy of the particle. With a better accuracy, particles reflect off the measuring device, and the resulting probability distribution becomes distorted. It is shown that the measurements considered here do not correspond to a measurement of the time-of-arrival operator.

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I. INTRODUCTION

Consider a beam of free particles, upon which a measurement is performed to determine the time-of-arrival to $x = x_A$. The time-of-arrival can be recorded by a clock situated at $x = x_A$ which switches off when the particle reaches it. In classical mechanics we could, in principle, achieve this with the smallest non-vanishing interaction between the particle and the clock, and hence measure the time-of-arrival with arbitrary accuracy.

In classical mechanics there is also another indirect method to measure the time-of-arrival. First invert the equation of motion of the particle and obtain the time in terms of the location and momentum, $T_A(x(t), p(t), x_A)$. This function can be determined at *any time* t, either by a simultaneous measurement of $x(t)$ and $p(t)$ and evaluation of T_A, or by a direct coupling to $T_A(x(t), p(t), x_A)$.

These two different methods, namely, the direct measurement, and indirect measurement, are classically equivalent. They give rise to the same classical time-of-arrival. They are not equivalent however, in quantum mechanics [1].

In quantum mechanics the corresponding operator $\mathbf{T_A}(\mathbf{x}(t), \mathbf{p}(t), x_A)$, if well defined, can in principle be measured to any accuracy. On the other hand, we shall argue that a direct measurement cannot determine the time-of-arrival of free particles with accuracy better then

$$\Delta t_A > \hbar / E_k, \tag{1}$$

where E_k is the initial kinetic energy of the particle. The basic reason is that, unlike a classical mechanical clock, in quantum mechanics the uncertainty in the clock's energy grows when its accuracy improves [2]. We find that particles with initial kinetic energy E_k bounce off without switching off a clock if this clock is set to record the time-of-arrival with accuracy better than in Eq. (1). (The occurrence of a similar phenomenon is well known in optics as an impedance miss-match which causes reflection in wave guides.) Furthermore, for the small fraction of the ensemble that does manage to turn off the clock, the resulting probability distribution becomes distorted. A detailed discussion of direct time-of-arrival measurements is given in Section II.

Still, one can imagine an indirect determination of arrival time as described above, by a measurement of some regularized time-of-arrival operator $\mathbf{T_A}(\mathbf{x}(t), \mathbf{p}(t), x_A)$ [3]. An obvious requirement of $\mathbf{T_A}$ is that it is a constant of motion; i.e., the time-of-arrival cannot change in time. As we shall show in Section III., a Hermitian time-of-arrival operator, with a continuous spectrum, can satisfy this requirement only for systems with an unbounded Hamiltonian. This difficulty can however by circumvented by "projecting out" the singularity at $p = 0$ and by using only measurements of $\mathbf{T_A}$ which do not cause a "shift" of the energy towards the ground state. Nevertheless, unlike the classical case, in quantum mechanics the result of such a measurement may have nothing to do with the time-of-arrival to $x = x_A$. As is argued in Section IV, $\mathbf{T_A}$ does not correspond to the result obtained by the direct measurement discussed in Section II. We conclude in Section V. by a discussion of the main results. An explicit calculation of the clock's final probability distribution is given in the Appendix.

II. MEASUREMENT OF TIME-OF-ARRIVAL

In this section we consider toy models of a measurement of time-of-arrival. To begin with, assume that a beam of particles interacts with a detector that is located at $x = 0$ and is coupled to a clock. Initially, as the beam is prepared, the clock is set to show $t = 0$. Our purpose is to design a particular set-up such that as a particle crosses the point $x = 0$ the detector stops the clock. Since the masses of the particle detector and the clock are unlimited we can ignore the uncertainty in the position of the measurement device. We shall consider four models. The first model describes a direct interaction of the particle with the clock. In the second model, the particle is detected by a two-level detector, which turns the clock off. To avoid the reflection due to "impedance miss-match", we look next at the possibility of boosting the energy of the particle in order to turn off the clock. We shall also consider the case of a "smeared" interaction, and conclude with a general discussion.

A. Measurement with a clock

The simplest model which describes a direct interaction of a particle and a clock [2,4], without additional "detector" degrees of freedom, is described by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{P_x}^2 + \theta(-\mathbf{x})\mathbf{P_y}. \tag{2}$$

Here, the particle's motion is confined to one spatial dimension, x, and $\theta(x)$ is a step function. The clock's Hamiltonian is represented by $\mathbf{P_y}$, and the time is recorded on the conjugate variable $\mathbf{y}$. [1]

The equations of motion read:

$$\dot{\mathbf{x}} = \mathbf{P_x}/m, \qquad \dot{\mathbf{P}}_\mathbf{x} = -\mathbf{P_y}\delta(\mathbf{x}) \tag{3}$$

$$\dot{\mathbf{y}} = \theta(\mathbf{x}), \qquad \dot{\mathbf{P}}_\mathbf{y} = 0. \tag{4}$$

At $t \to \infty$ the clock shows the time of arrival:

$$\mathbf{y}_\infty = \mathbf{y}(t_0) + \int_{t_0}^{\infty} \theta(-\mathbf{x}(t))dt \tag{5}$$

A crucial difference between the classical and the quantum case, can be noted from Equation (3). In the classical case the back-reaction can be made negligible small by choosing $P_y \to 0$. In this case, the particle follows the undisturbed solution, $x(t) = x(t_0) + \frac{p_x}{m}(t - t_0)$. If initial we set $y(t_0) = t_0$ and $x(t_0) < 0$ the clock finally reads:

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[1] We have represented here the ideal clock by a Hamiltonian $H_{clock} = \mathbf{P_y}$ that is linear in the momentum. This linear Hamiltonian can be obtained approximately for a free particle with $H = \mathbf{P}_y^2/2M$. For a given duration t we can approximate $H \simeq \frac{\langle \mathbf{P}_y \rangle}{M} P_y + const.$ by letting the mass be sufficiently large. One could also consider a Larmor clock with a bounded Hamiltonian $H_{clock} = \omega\mathbf{J}$ [2,4]. The Hilbert space is spanned by 2j+1 vectors where j is a natural number, and the clock's resolution can be made arbitrarily fine by increasing j.

$$y_\infty = y(t_0) + \int_{t_0}^{\infty} \theta[-x(t_0) - \frac{p_x}{m}(t - t_0)]dt = -\frac{mx(t_0)}{p_x}. \tag{6}$$

The classical time-of-arrival is $t_A = y_\infty = -mx(t_0)/p_x$. The same result would have been obtained by measuring the classical variable $-mx_0/p_x = -mx(t)/p_x + (t - t_0)$, at arbitrary time t. Consequently, the continuous and the indirect measurements alluded to in Section I, are classically equivalent.

On the other hand, in quantum mechanics the uncertainty relation dictates a strong back-reaction, i.e. in the limit of $\Delta y = \Delta t_A \to 0$, p_y in (3) must have a large uncertainty, and the state of the particle must be strongly affected by the act of measuring. Therefore, the two classically equivalent measurements become inequivalent in quantum mechanics.

Before we proceed to examine the continuous measurement process in more detail, we note that a more symmetric formulation of the above measurement exists in which knowledge of the direction from which particles are arriving is not needed. We can consider

$$H = \frac{1}{2m}\mathbf{P_x}^2 + \theta(-\mathbf{x})\mathbf{P}_{\mathbf{y}_1} + \theta(\mathbf{x})\mathbf{P}_{\mathbf{y}_2}. \tag{7}$$

As before, the particle's motion is confined to one spatial dimension, x. Two clocks are represented by $\mathbf{P}_{\mathbf{y}_1}$ and $\mathbf{P}_{\mathbf{y}_2}$, and time is recorded on the conjugate variables $\mathbf{y}_1$ and $\mathbf{y}_2$, respectively.

The first clock operates only when the particle is located at $x < 0$ and the second clock at $x > 0$. For example, if we start with a beam of particle at $x < 0$, a measurement at $t \to \infty$ of $\mathbf{y}_1$ gives the time-of-arrival. Alternatively we could measure $t - \mathbf{y}_2$. As a check we have $\mathbf{y}_1 + \mathbf{y}_2 = t$. It is harder to determine the time-of-arrival if the particle arrives from both directions. If however it is known that initially $|x| < L$, we can measure $\mathbf{y}_1$ and $\mathbf{y}_2$ after $t >> L/v$. The time-of-arrival will then be given by $t_A = \min(\mathbf{y}_1, \mathbf{y}_2)$.

Let us examine this system in more detail. For simplicity we shall consider the case of only one clock and a particle initially at $x < 0$, which travels towards the clock at $x = 0$. The eigenstates of the Hamiltonian are

$$\phi_{kp}(x,y,t) = \begin{cases} (e^{ikx} + A_R e^{-ikx}) e^{ipy - i\omega(t)} & x < 0 \\ A_T e^{iqx + ipy - i\omega(t)} & x \geq 0 \end{cases} \tag{8}$$

where k and p, are the momentum of the particle and the clock, respectively, and $\omega(t) = \frac{k^2 t}{2m} + pt$. Continuity of ϕ_{kp} requires that

$$A_T = \frac{2k}{k+q}$$
$$A_R = \frac{k-q}{k+q}, \tag{9}$$

where $q = \sqrt{k^2 + 2mp} = \sqrt{2m(E_k + p)}$.

The solution of the Schrödinger equation is

$$\psi(x,y,t) = N \int_{-\infty}^{\infty} dk \int_0^{\infty} dp f(p) g(k) \phi_{kp}(x,y,t), \tag{10}$$

where N is a normalization constant and $f(p)$ and $g(k)$ are some distributions. For example, with

$$f(p) = e^{-\Delta_y{}^2 (p-p_0)^2}$$
$$g(k) = e^{-\Delta_x{}^2 (k-k_o)^2 + ikx_0}. \tag{11}$$

and $x_0 > 0$, the particle is initially localized on the left ($x < 0$) and the clock (with probability close to 1) runs. The normalization in eq. (10) is thus $N^2 = \frac{\Delta x \Delta y}{2\pi^3}$. By choosing $p_0 \approx 1/\Delta_y$, we can now set the the clock's energy in the range $0 < p < 2/\Delta_y$.

Let us first show that in the stationary point approximation the clock's final wave function is indeed centered around the classical time-of-arrival. Thus we assume that Δ_y and Δ_x are large such that $f(p)$ and $g(k)$ are sufficiently peaked. For $x > 0$, the integrand in (10) has an imaginary phase

$$\theta = qx + kx_o + py - \frac{k^2 t}{2m} - pt. \tag{12}$$

$\frac{d\theta}{dk} = 0$ implies

$$x_{peak}(p) = -\frac{q(k_0)}{k_0} x_o + \frac{q(k_0) t}{m}, \tag{13}$$

and $\frac{d\theta}{dp} = 0$ gives

$$y_{peak}(k) = t - \frac{mx}{q_0}. \tag{14}$$

Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_0}. \tag{15}$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \tag{16}$$

In the case of inaccurate measurements with a small back-reaction on the particle $A_T \simeq 1$. The clocks density matrix is then found (see Appendix) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \tag{17}$$

where the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$. As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $1/dE = \frac{m}{kdk}$ and is given by $\frac{dx(y)\,m}{k_o}$ where $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave function is spreading as time increases, so that at later y, the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the back-reaction causes a small disturbance to the particle, the clock records the time-of-arrival. What happens when we wish to make more accurate measurements? Consider the exact transition probability $T = \frac{q}{k}|A_T|^2$, which also determines the probability to stop the clock. The latter is given by

$$\sqrt{\frac{E_k + p}{E_k}} \left[\frac{2\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}} \right]^2. \tag{18}$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \Delta t_A > \hbar. \tag{19}$$

Here Δt_A stands for the initial uncertainty in position of the dial $\mathbf{y}$ of the clock, and is interpreted as the accuracy of the clock. $\bar{E}_k$ can be taken as the typical initial kinetic energy of the particle.

In measurements with accuracy better then $\hbar/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \Delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction which has been detected depends on the accuracy Δt_A and can become distorted with increased accuracy. This observation becomes apparent in the following simple example. Consider an initial wave packet that is composed of a superposition of two Gaussians centered around $k = k_1$ and $k = k_2 >> k_1$. Let the classical time-of-arrival of the two gaussians be t_1 and t_2 respectively. When the inequality (19) is satisfied, two peaks around t_1 and t_2 will show up in the final probability distribution. On the other hand, for $\frac{2m}{k_1^2} > \Delta t_A > \frac{2m}{k_2^2}$, the time-of-arrival of the less energetic peak will contribute less to the distribution in y, because it is less likely to trigger the clock. Thus, the peak at t_1 will be suppressed. Clearly, when the precision is finer than $\hbar/\bar{E}_k$ we shall obtain a distribution which is considerably different from that obtained for the case $\Delta t_A > \hbar/\bar{E}_k$ when the two peaks contribute equally.

B. Two-level detector with a clock

A more realistic set-up for a time-of-arrival measurement is one that also includes a particle detector which switches the clock off as the particle arrives. We shall describe the particle detector (the "trigger") as a two-level spin degree of freedom. The particle will flip the state of the trigger from "on" to "off", ie. from $\uparrow_z$ to $\downarrow_z$. First let us consider a model for the detector without including the clock:

$$H_{particle+detector} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}). \tag{20}$$

The particle interacts with the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or with a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \to \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and width $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the "on" state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \quad \to \quad \frac{1}{\sqrt{2}}\Big[|\psi_R\rangle|\uparrow_x\rangle + |\psi_T\rangle|\downarrow_x\rangle\Big], \tag{21}$$

where ψ_R and ψ_T are the reflected and transmitted wave functions of the particle, respectively.

The latter equation can be rewritten as

$$\frac{1}{2}|\uparrow_z\rangle(|\psi_R\rangle + |\psi_T\rangle) + \frac{1}{2}|\downarrow_z\rangle(|\psi_R\rangle - |\psi_T\rangle) \tag{22}$$

Since $\uparrow_z$ denotes the "on" state of the trigger, and $\downarrow_z$ denotes the "off" state, we have flipped the trigger from the "on" state to the "off" state with probability 1/2. By increasing the number of detectors, this probability can be made as close as we like to one. To see this, consider N spins as N triggers and set the Hamiltonian to be

$$\mathbf{P}_{\mathbf{x}}{}^2/2m + (\alpha/2)\Pi_n(1+\sigma_x^{(n)})\delta(\mathbf{x}). \tag{23}$$

We will say that the particle has been detected if at least one of the spin has flipped. One can verify that in this case the probability that at least one spin has flipped is now $1 - 2^{-N}$.

So far we have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also worth noting that the net energy exchange between the trigger and the particle is zero, ie. the particle's energy is unchanged.

Let us proceed and couple the trigger to a clock. The total Hamiltonian is now given by

$$H_{detector+clock} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1+\sigma_z)\mathbf{P}_{\mathbf{y}}. \tag{24}$$

Since we can have $\alpha >> P_y$ it would seem that the triggering mechanism need not be affected by the clock. If the final wave function includes a non-vanishing amplitude of $\downarrow_z$, the clock will be turned off and the time-of-arrival recorded. However, the exact solution shows that this is not the case. Consider for example an initial state of an incoming wave from the left and the spin in the $\uparrow_z$ state.

The eigenstates of the Hamiltonian in the basis of σ_z are

$$\Psi_L(x) = \begin{pmatrix} e^{ik_\uparrow x} + \phi_{L\uparrow} e^{-ik_\uparrow x} \\ \phi_{L\downarrow} e^{-ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{25}$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{ik_\uparrow x} \\ \phi_{R\downarrow} e^{ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{26}$$

for $x > 0$. Here $k_\uparrow = \sqrt{2m(E-p)} = \sqrt{2mE_k}$ and $k_\downarrow = \sqrt{2mE} = \sqrt{2m(E_k + p)}$.

Matching conditions at $x = 0$ yields

$$\phi_{R\uparrow} = \frac{\frac{2k_\uparrow}{m\alpha} - \frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m\alpha} - (1 + \frac{k_\uparrow}{k_\downarrow})} \tag{27}$$

$$\phi_{R\downarrow} = \frac{k_\uparrow}{k_\downarrow}((\phi_{R\uparrow} - 1) = \frac{\frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m\alpha} - (1 + \frac{k_\uparrow}{k_\downarrow})}, \tag{28}$$

and

$$\phi_{L\downarrow} = \phi_{R\downarrow}, \tag{29}$$

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1. \tag{30}$$

We find that in the limit $\alpha \to \infty$ the transmitted amplitude is

$$\phi_{R\downarrow} = -\phi_{R\uparrow} = \frac{\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}}. \tag{31}$$

Precisely as in the previous section, the transition probability decays like $\sqrt{E_k/p}$. ¿From eqs. (29,30) we get that $\phi_{L\downarrow} \to 0$, and $\phi_{L\uparrow} \to 1$ as the accuracy of the clock increases. Hence the particle is mostly reflected back and the spin remains in the $\uparrow_z$ state; i.e., the clock remains in the "on" state.

The present model gives rise to the same difficulty as the previous model. Without the clock, we can flip the trigger spin by means of a localized interaction, but when we couple the trigger to the clock, the probability to flip the spin and turn the clock off decreases gradually to zero when the clock's precision is improved.

C. Local amplification of Kinetic Energy

The difficultly with the previous examples seems to be that the particle's kinetic energy is not sufficiently large, and energy can not be exchanged with the clock. To overcome this difficulty one can imagine introducing a "pre-booster" device just before the particle arrives to the clock. If it could boost the particle's kinetic energy arbitrarily high, without distorting the incoming probability distribution (i.e. amplifying all wave components k with the same probability), and at an arbitrary short distance from the clock, then the time-of-arrival could be measured to arbitrary accuracy. Thus, an equivalent problem is: can we boost the energy of a particle by using only localized (time independent) interactions?

Let us consider the following toy model of an energy booster described by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^2 + \alpha\sigma_x\delta(\mathbf{x}) + \frac{W}{2}\theta(\mathbf{x})(1+\sigma_z) + \frac{1}{2}[V_1\theta(-\mathbf{x}) - V_2\theta(\mathbf{x})](1-\sigma_z). \tag{32}$$

Here, α, W, V_1 and V_2 are positive constants. Let us consider an incoming wave packet propagating from left to right. The role of the term $\alpha\sigma_x\delta(x)$ is to flip the spin $\uparrow_z$ to $\downarrow_z$. The $\uparrow_z$ component of the wave function is damped out exponential by the W term for $x > 0$. The $\downarrow_z$ component is damped out for $x < 0$ by the term V_1, but increases its kinetic energy for $x > 0$ by V_2. As we shall see, for a given momentum k, one can chose the four free parameters above such that the wave is transmitted through the booster with probability 1, while the gain in energy V_2 can be made arbitrarily large. On the other hand, the potential barrier W can be made arbitrarily large. The last requirement means that the unamplified

($\uparrow_z$) component, decays for $x > 0$ on arbitrary short scales, which allows us to locate the booster arbitrarily close to the clock, while preventing destructive interference between the amplified and unamplified transmitted waves.

The eigenstates of (32), in the basis of σ_z, are given by

$$\Psi_L(x) = \begin{pmatrix} e^{ikx} + \phi_{L\uparrow} e^{-ikx} \\ \phi_{L\downarrow} e^{qx} \end{pmatrix} \tag{33}$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{-\lambda x} \\ \phi_{R\downarrow} e^{ik'x} \end{pmatrix} \tag{34}$$

for $x > 0$, where $k^2 = V_1 - q^2 = -\lambda^2 + W = -V_2 + k'^2$. Matching conditions at $x = 0$ we find

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1 = \frac{k'k + q\lambda + i(kq - k'\lambda) - \alpha^2}{k'k - q\lambda + i(k'\lambda + kq) + \alpha^2}, \tag{35}$$

$$\phi_{R\downarrow} = \phi_{L\downarrow} = \frac{\alpha}{ik' - q}\big(1 + \phi_{L\uparrow}\big). \tag{36}$$

For a given k, W and V_2 (or given k, λ and k') we still are free to chose α and V_1 (or q). We now demand that

$$\alpha = k'k + q\lambda, \qquad q = \lambda\frac{k'}{k}. \tag{37}$$

With this choice we obtain:

$$J_{L\uparrow} = 0, \qquad J_{R\downarrow} = \frac{k'}{k}|\phi_{R\downarrow}|^2 = 1. \tag{38}$$

Therefore, the wave has been fully transmitted and the spin has flipped with probability 1.

So far we have considered an incoming wave with fixed momentum k. For a general incoming wave packet only a part of the wave will be transmitted and amplified. Furthermore one can verify that the amplified transmitted wave has a different form than the original wave function since different momenta have been amplified with different probabilities. Thus, in general, although amplification is possible and indeed will lead to a much higher rate of detection, it will give rise to a distorted probability distribution for the time-of-arrival.

There is however one limiting case in which the method does seem to succeed. Consider a narrow wave peaked around k with a width δk. To first order in δk, the probability $T_\downarrow$ that the particle is successfully boosted is given by

$$J_{R\downarrow} \simeq 1 + \frac{2\delta k}{k}. \tag{39}$$

Therefore in the special case that $\frac{dk}{k} << 1$, the transmition probability is still close to one. If in this case we known in advance the value of k up to $\Delta k << k$, we can indeed use the booster to improve the bound (19) on the accuracy.

The reason why this seems to work in this limiting case is as follows. The probability of flipping the particle's spin depends on how long it spends in the magnetic field described by the α term in (32). If however, we know beforehand, how long the particle will be in this field, then we can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $\Delta k/k << 1$ is thus equivalent to having a small uncertainty in the "interaction time" with this field. It must be emphasized however, that these measurements cannot be used for general wave functions, and that even in the special case above, one still requires some prior information of the incoming wave function.

D. Gradual triggering of the clock

In order to avoid the reflection found in the previous two models, we shall now replace the sharp step-function interaction between the clock and particle by a more gradual transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon << 1, \tag{40}$$

where $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2). \tag{41}$$

Solving the equation for the potential with a given ϵ we obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2}\frac{1}{x^2}. \tag{42}$$

Now we observe that any particle with $E \geq E_0$ also satisfies the WKB condition (40) above for the *same* potential V_ϵ. Furthermore $p_y V_\epsilon$ also satisfies the condition for any $p_y > 1$.

These considerations suggest that we should replace the Hamiltonian in eq. (7) with

$$H = \mathbf{P_x}^2/2m + V(x)\mathbf{P_y}, \tag{43}$$

where

$$V(x) = \begin{cases} -\frac{x_A^2}{x^2} & x < x_A \\ -1 & x \geq x_A. \end{cases} \tag{44}$$

Here $x_A^{-2} = 2m\epsilon^2$.

Thus this model describes a gradual triggering *on* of the clock which takes place when the particles propagates from $x \to -\infty$ towards $x = x_A$. In this case the arrival time is approximately given by $t - \mathbf{y}$ where $t = t_f - t_i$. Since without limiting the accuracy of the clock we can demand that $p_y >> 1$, the reflection amplitude off the potential step is exponentially small for *any* initial kinetic energy E_k.

The problem is however that the final value of $t - \mathbf{y}$ does not always correspond to the time-of-arrival since it contains errors due to the affect of the potential $V(x)$ on the particle which we shall now proceed to examine.

In the following we shall ignore ordering problems and solve for the classical equations of motion for (43). We have

$$y(t_f) - y(t_i) = \int_{t_i}^{t_f} V(\mathbf{x}(t'))dt', \tag{45}$$

which can be decomposed to

$$y(t_f) - y(t_i) = (t_i - t_0) + (t_f - t_i), + \int_{t_i}^{t_0} V(x(t'))dt' = A + B + C \tag{46}$$

where

$$A = \frac{1}{\sqrt{2mE}}\left[\sqrt{x_A^2 + p_y x_A^2/E} - \sqrt{x_i^2 + p_y x_A^2/E},\right] \tag{47}$$

is the time that the particle travels from x_i to x_A in the potential $p_y V(x)$, B is the total time, and

$$C = -\frac{x_A}{\sqrt{2mp_y}}\left[\log \frac{1+\sqrt{1+\frac{E}{p_y}}}{1+\sqrt{1+\frac{Ex_i^2}{p_y x_A^2}}} + \log \frac{x_i}{x_A}\right]. \tag{48}$$

The last term C, corresponds to an error due to the imperfection of the clock, i.e. the motion of the clock prior to arrival to x_A. By making p_y large we can minimize the error from this term to $\sim (x_A \log p_y/\sqrt{2mp_y})$.

Inspecting equation (46) we see that by measuring $y_f - y_i$ and then subtracting $B = t_f - t_i$ (which is measured by another clock) we can determine the time $t_0 - t_i$, which is the time-of-arrival for a particle in a potential $p_y V(x)$, up to the correction C. However this time reflects the motion in the presence of an external (unknown) potential, while we are interested in the time-of-arrival for a free particle.

Nevertheless, if $p_y/E << 1$ we obtain

$$-A = \frac{x_A - x_i}{\sqrt{2mE}} + O\left(\frac{p_y}{E}\right). \tag{49}$$

The time-of-arrival can hence be measured provided that $E_k \Delta t >> \hbar$. On the other hand, when the detector's accuracy is $\Delta t < \hbar/E$, the particle still triggers the clock. However the measured quantity, A, no longer correspond to the time-of-arrival. Again, we see that when we ask for too much accuracy, the particle is strongly disturbed and the result has nothing to do with the time-of-arrival of a free particle.

E. General considerations

We have examined several models for a measurement of time-of-arrival and found a limitation,

$$\Delta t_A > \hbar/\bar{E}_k, \tag{50}$$

on the accuracy that t_A can be measured. Is this limitation a general feature of quantum mechanics?

First we should notice that eq. (50) does not seem to follow from the uncertainty principle. Unlike the uncertainty principle, whose origin is kinematic, (50) follows from the nature of the *dynamic* evolution of the system during a measurement. Furthermore here we are considering a restriction on the measurement of a *single* quantity.

While it is difficult to provide a general proof, in the following we shall indicate why (50) is expected to hold under more general circumstances.

Let us examine the basic features that gave rise to (50). In the toy models considered in sections IIA. and IIB., the clock and the particle had to exchange energy $p_y \sim 1/\Delta t_A$. The final kinetic energy of the particle is larger by p_y. As a result, the effective interaction by which the clock switchs off, looks from the point of view of the particle like a step function potential. This led to "non-detection" when (50) was violated.

Can we avoid this energy exchange between the particle and the clock? Let us try to deliver this energy to some other system without modifying the energy of the particle. For example consider the following Hamiltonian for a clock with a reservoir:

$$H = \frac{\mathbf{P_x}^2}{2m} + \theta(-\mathbf{x})H_c + H_{res} + V_{res}\theta(\mathbf{x}) \tag{51}$$

The idea is that when the clock stops, it dumps its energy into the reservoir, which may include many other degrees of freedom, instead of delivering it to the particle. In this model, the particle is coupled directly to the clock and reservoir, however we could as well use the idea of section IIB. above. In this case:

$$H = \frac{\mathbf{P_x}^2}{2m} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1+\sigma_z)H_c + H_{res} + \frac{1}{2}(1-\sigma_z)V_{res}. \tag{52}$$

The particle detector has the role of providing a coupling between the clock and reservoir.

Now we notice that in order to transfer the clock's energy to the reservoir without affecting the free particle, we must also prepare the clock and reservoir in an initial state that satisfies the condition

$$H_c - V_{res} = 0 \tag{53}$$

However this condition does not commute with the clock time variable $\mathbf{y}$. We can measure initially $\mathbf{y}-\mathbf{R}$, where R is a collective degree of freedom of the reservoir such that $[\mathbf{R}, V_{res}] = i$, but in this case we shall not gain information on the time-of-arrival y since R is unknown. We therefore see that in the case of a sharp transition, i.e. for a localized interaction with the particle, one cannot avoid a shift in the particle's energy. The "non-triggering" (or reflection) effect cannot be avoided.

We have also seen that the idea of boosting the particle "just before" it reaches the detector, fails in the general case. What happens in this case is that while the detection rate increase, one generally destroys the initial information stored in the incoming wave packet. Thus although higher accuracy measurements are now possible, they do not reflect directly the time-of-arrival of the initial wave packet.

Finally we note that in reality, measurements usually involve some type of cascade effect, which lead to signal amplification and finally allows a macroscopic clock to be triggered. A typical example of this type would be the photo-multiplier where an initialy small energy is amplified gradually and finally detected. Precisely this type of process occurs also in the model of section IID. In this case the particle gains energy gradually by "rolling down" a smooth step function. It hence always triggers the clock. The basic problem with such a detector is that when (50) is violated, the "back reaction" of the detector on the particle, during the gradual detection, becomes large. The relation between the final record to the quantity we wanted to measure is lost.

III. CONDITIONS ON A TIME-OF-ARRIVAL OPERATOR

As discussed in the introduction, although a direct measurement of the time-of-arrival may not be possible, one can still try to observe it indirectly by measuring some operator $\mathbf{T_A}(\mathbf{p}, \mathbf{x}, x_A)$. In the next two sections we shall examine this operator and its relation to the continuous measurements described in the previous sections. First in this section we show that an exact time-of-arrival operator cannot exist for systems with bounded Hamiltonian.

In the following we shall use the Heisenberg representation.

To begin with, let us start with the assumption that the time-of-arrival is described, as other observables in quantum mechanics, by a Hermitian operator $\mathbf{T_A}$.

$$\mathbf{T_A}(t)|t_A\rangle_t = t_A|t_A\rangle_t \tag{54}$$

Here the subscript $\rangle_t$ denotes the time dependence of the eigenkets, and $\mathbf{T_A}$ may depend explicitly on time. Hence for example, the probability distribution for the time-of-arrival for the state

$$|\psi\rangle = \int g(t'_A)|t'_A\rangle dt'_A \tag{55}$$

will be given by $prob(t_A) = |g(t_A)|^2$. We shall now also assume that the spectrum of $\mathbf{T_A}$ is continuous and unbounded: $-\infty < t_A < \infty$.

Should $\mathbf{T_A}$ correspond to time-of-arrival it must satisfy the following obvious condition. $\mathbf{T_A}$ must be a constant of motion:

$$\frac{d\mathbf{T_A}}{dt} = \frac{\partial \mathbf{T_A}}{\partial t} + \frac{1}{i\hbar}[\mathbf{T_A}, H] = 0. \tag{56}$$

That is, the time-of-arrival cannot change in time.

Since the spectrum $\mathbf{T_A}$ is identical to that of time we may substitute $t = t_A$ in Eq. (54) and get

$$\mathbf{T_A}(t)|t\rangle_t = t|t\rangle_t \tag{57}$$

The eigenket $|t\rangle_t$ can be expanded as

$$|t\rangle_t = \int g(x', t)|x'\rangle_t dx' \tag{58}$$

where the function $g(x,t)$ is expected to be localized around the point of arrival $x = x_A$, i.e. since the particle must be found at the location of arrival at the time-of-arrival with certainty, we expect $g(x', t) \to \delta(x - x_A)$. However we shall not use this localization requirement.

We next assume the $g(x,t)$ has a Fourier transform with respect to time

$$g(x',t) = \int d\omega \tilde{g}_\omega(x) e^{i\omega t} \tag{59}$$

Eq.. (41) then yields

$$\int d\omega e^{i\omega t}(\mathbf{T_A}(t) - t) \int dx' \tilde{g}_\omega(x')|x'\rangle_t = 0 \tag{60}$$

hence

$$(\mathbf{T_A}(t) - t) \int dx' \tilde{g}_\omega(x')|x'\rangle_t \equiv (\mathbf{T_A}(t) - t)|\psi_\omega\rangle_t = 0 \quad \forall\omega \tag{61}$$

or

$$U^\dagger(t)\mathbf{T_A}(t)U(t)|\psi_\omega\rangle_0 = t|\psi_\omega\rangle_0 \tag{62}$$

Since (62) is true for any t we get [2]

$$\mathbf{T_A}(t) = t + \mathbf{f}(t) \tag{63}$$

where $\mathbf{f}(t)$ satisfies

$$\mathbf{f}(t)|\psi_\omega\rangle_t = 0 \quad \forall\omega \tag{64}$$

If $\mathbf{f}$ is does not depend explicitly on time, i.e., $\partial_t\mathbf{f} = 0$ we get by substituting into (56)

$$[\mathbf{f}, H] = i\hbar \tag{65}$$

Hence $\mathbf{f}$ is a generator of energy translations, i.e. it is the "time operator" of the system whose Hamiltonian is H. It is well know that such an operator can exist only if the Hamiltonian is unbounded from above and below [5].

Finally we shall prove that $\partial_t\mathbf{f} = 0$. By expanding $\mathbf{f}(t)$ as a Fourier integral

$$\mathbf{f}(t) = \int \mathbf{f}_{\tilde{\omega}} e^{i\tilde{\omega}t} d\tilde{\omega} \tag{66}$$

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[2]For example if $H = \mathbf{p}^2/2m$ and ordering problems are ignored we find $\mathbf{T_A}(t) = m\frac{x_0 - \mathbf{x}(t)}{p(t)} + t$.

and substituting into (56) we get

$$[\mathbf{f}_{\tilde{\omega}}, H] = i(\delta(\tilde{\omega}) - \tilde{\omega}) \tag{67}$$

Thus

$$[(\mathbf{f}_{\tilde{\omega}} + \mathbf{f}_{-\tilde{\omega}}), H] = 0 \quad \forall \tilde{\omega} \neq 0 \tag{68}$$

are constants of motion. On the other hand, from (64) we get

$$(\mathbf{f}_{\tilde{\omega}} + \mathbf{f}_{-\tilde{\omega}})|\psi\rangle_t = 0 \quad \forall \omega, \tilde{\omega} \tag{69}$$

and since by (52) $(\mathbf{f}_{\tilde{\omega}} + \mathbf{f}_{-\tilde{\omega}})$ also commutes with $U(t) = \exp(-iHt)$ we obtain

$$(\mathbf{f}_{\tilde{\omega}} + \mathbf{f}_{-\tilde{\omega}})U(t_0)|\psi_{\tilde{\omega}}\rangle_t = 0 \tag{70}$$

for all t_0 and $\tilde{\omega} \neq 0$. Since $U(t_0)|\psi_{\tilde{\omega}}\rangle_t$ span the whole Hilbert space we get that

$$\mathbf{f}_{\tilde{\omega}} + \mathbf{f}_{-\tilde{\omega}} = 0 \quad \forall \tilde{\omega} \neq 0 \tag{71}$$

Thus $\mathbf{f}$ cannot depend explicitly on time.

IV. MEASURING THE TIME-OF-ARRIVAL OPERATOR VS. CONTINUOUS MEASUREMENTS

Although formally there cannot exist a time-of-arrival operator $\mathbf{T_A}$, it may be possible to approximate $\mathbf{T_A}$ to arbitrary accuracy [3]. Kinematically, one expects that the time-of-arrival operator for a free particle arriving at the location $x_A = 0$ might be given by

$$\mathbf{T_A} = -\frac{m}{2}\frac{1}{\sqrt{\mathbf{p}}}\mathbf{x}(0)\frac{1}{\sqrt{\mathbf{p}}}. \tag{72}$$

The choice for the time operator is clearly not unique. An equally valid choice is $-m(\frac{1}{\mathbf{p}}\mathbf{x}+\mathbf{x}\frac{1}{p}$, etc. Furthermore, since $\mathbf{T_A}$ is ill defined at p=0, it's eigenvalues

$$\langle k|T^{\pm}\rangle = \theta(\pm k)\sqrt{\frac{k}{2\pi m}}e^{i\frac{Tk^2}{2m}} \tag{73}$$

are not orthogonal.

$$\langle T|T'\rangle = \delta(T-T') - \frac{i}{\pi(T-T')}. \tag{74}$$

Thus, $\mathbf{T_A}$ is not Hermitian. We can however, define the regularized Hermitian operator $\mathbf{T_A}' = OTO$ where $O = 1 - |p=0><p=0|$. It's eigenvalues are complete and orthogonal, and it circumvents the proof given above, because it satisfies $[\mathbf{T_A}', H] = i\hbar O$ i.e. it is not conjugate to H at $p=0$. Although $\mathbf{T_A}$ is not always the shift operator of the energy, the measurement can be carried out in such a way that this will not be of consequence. To see this, consider the interaction Hamiltonian

$$H_{meas} = \delta(t)\mathbf{q}\mathbf{T_A}', \tag{75}$$

which modifies the initial wave function $\psi \to \exp(-iqT')\psi$. We need to demand that $\mathbf{T_A}'$ acts as a shifts operator of the energy of ψ during the measurement. Therefore we need that $q > -E_{min}$, where E_{min} is the minimal energy in the energy distribution of ψ. In this way, the measurement does not shift the energy down to $E=0$ where $\mathbf{T_A}'$ is no longer conjugate to H. The value of $\mathbf{T_A}'$ is recorded on the conjugate of q – call it P_q. Now the uncertainty is given by $dT'_A = d(P_q) = 1/dq$, thus naively from $dq = 1/dT'_A < E_{min}$, we get $E_{min}dT' > 1$. However here, the average $\langle q\rangle$ was taken to be zero. There is no reason not to take $\langle q\rangle$ to be much larger than E_{min}, so that $\langle q\rangle - dq >> -E_{min}$. If we do so, the measurement increases the energy of ψ and $\mathbf{T_A}'$ is always conjugate to H. The limitation on the accuracy is in this case $dT'_A > 1/\langle q\rangle$ which can be made as small as we like.

Nevertheless, there are still problems with this time-of-arrival operator. One finds that at the time of arrival, the eigenstates of $\mathbf{T_A}$, $\langle x|T(t_A)\rangle$ are not delta functions $\delta(x)$ but are proportional to $x^{-3/2}$. This means that at the time of arrival, the particle has a probability of not being found at the origin. Furthermore, if $\mathbf{P}_0$ is the projector onto $x=0$, one finds that

$$\langle\psi[\mathbf{T_A}, \mathbf{P}_0]|\psi\rangle = -\frac{i}{2}Re\{\psi(x=0)\int dk\psi^*(k)\frac{m}{k^2}\}. \tag{76}$$

A measurement of the time-of-arrival operator is not equivalent to monitoring the point-of-arrival. Furthermore, if one measures a time-of-arrival operator at a time t' before the particle arrives, then one needs to know the full Hamiltonian from time t' until t_A. Even if one knows the full Hamiltonian, and can find an approximate time-of-arrival operator, one has to have faith that the Hamiltonian will not be perturbed after the measurement has been made. On the other hand, the continuous measurements we have described can be used with any Hamiltonian.

Finally, how does the resulting measurement of a time-of-arrival operator compare with that of a continuous measurement? ¿From the discussion in Section IIA, it should be clear that in the limit of high precision, continuous measurements respond very differently in comparison to the time operator. At the limit of $dt_A \to 0$ all the particles bounce back from the detector. Such a behavior does not occur for the time of arrival operator. Nevertheless, one may still hope that since the eigenstates of $\mathbf{T_A}$ have an infinitely spread in energy, they do trigger a clock even if $dt_A \to 0$. For the type of models we have been considering, we can show however that this will not happen.

Let us assume that the interaction of one eigenstate of $\mathbf{T_A}$ with the clock (of, say, Section IIA) evolves as

$$|t_A\rangle|y = t_0\rangle \;\rightarrow\; |\chi(t_A)\rangle|y = t_A\rangle + |\chi'(t_A)\rangle|y = t\rangle. \tag{77}$$

Here, $|y = t_0\rangle$ denotes an initial state of the clock with $dt_A \to 0$, $|\chi(t_A)\rangle$ denotes the final state of the particle if the clock has stopped, and $|\chi'(t_A)\rangle$ the final state of the particle if the clock has not stopped.

Since the eigenstates of $\mathbf{T_A}$ form a complete set, we can express any state of the particle as $|\psi\rangle = \int dt_A C(t_A)|t_A\rangle$. We then obtain :

$$\int dt_A C(t_A)|t_A\rangle|y = t_0\rangle \;\rightarrow\; \int dt_A C(t_A)|\chi(t_A)\rangle|y = t_A\rangle + \Big(\int dt_A C(t_A)|\chi'(t_A)\rangle\Big)|y = t\rangle. \tag{78}$$

The final probability to measure the time-of-arrival is hence $\int dt_a |C(t_A)\chi(t_A)|^2$. On the other hand we found that for a general wave function ψ, in the limit of $dt_A \to 0$, the probability for detection vanishes. Since the states of the clock, $|y = t_A\rangle$, are orthogonal in this limit, this implies that $\chi(t_A) = 0$ in eq. (77) for all t_A. Therefore, the eigenstates of $\mathbf{T_A}$ will not trigger the clock.

V. CONCLUSION

We have examined various models for the measurement of time-of-arrival, t_A, and found a basic limitation on the accuracy that t_A can be determined reliably: $\Delta t_A > \hbar/\bar{E}_k$. This limitation is quit different in origin from that due to the uncertainty principle; here it applies to a *single* quantity. Furthermore, unlike the kinematic nature of the uncertainty principle, in our case the limitation is essentially dynamical in its origin; it arises when the time-of-arrival is measured by means of a continuous interaction between the measuring device and the particle.

We would also like to stress, that continuous measurements, differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. Operationally one performs here two completely different measurements. While the time-of-arrival operator is a formally constructed operator which can be measured by an impulsive von-Neumann interaction, it seems that continuous measurements are much more closer to actual experimental set-ups. Furthermore, we have seen that the result of these two measurements do not need to agree, in particular in the high accuracy limit, continuous measurements give rise to entirely different behavior. This suggests that as in the case of the problem of finding a "time operator" [6] for closed quantum systems, the time-of-arrival operator has a somewhat limited physical meaning.

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APPENDIX A

Using the simple model (2) of Section IIA, we now calculate the probability distribution of a clock which measures the time-of-arrival of a Gaussian wave packet. We will perform the calculation in the limits when the clock is extremely accurate and extremely inaccurate. The wave function of the clock and particle is given by (10) and the distributions are both Gaussians given by (11). In the inaccurate limit, when $p_o << k$, $A_T \sim 1$. We trace over the position of the particle on the condition that the clock was triggered, ie. $x > 0$.

$$\begin{aligned}\rho(y,y)_{x>0} &= \int dx|\psi(x>0,y,t)|^2 \\ &\simeq N^2\int_{-\infty}^{\infty} dk\,dk' \int_0^{\infty} dp\,dp'\,dx g(k)g^*(k')f(p)f^*(p')e^{i(q-q')x+i(p-p')y-\frac{i(q^2-q'^2)t}{2m}}. \end{aligned} \quad \text{(A1)}$$

After a sufficiently long time, ie. $t >> t_A$ the wave function has no support on the negative x-axis, and if $p_o > 1/\Delta y$, then it will not have support in negative p. We can thus integrate p and x over the entire axis. Integrating over x gives a delta-function in q. We can then integrate over p' to give

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2}{m}\int dk\,dk'\,dp\sqrt{k^2+2mp}\,g(k)g^*(k')f(p)f^*(p+\frac{k^2-k'^2}{2m})e^{i(k'^2-k^2)\frac{y}{2m}}, \quad \text{(A2)}$$

where we have used the fact that $\delta(f(z)) = \frac{\delta(z-z_o)}{f'(z=z_o)}$ when $f(z_o) = 0$. The square root term varies little in comparison with the exponential terms and can be replaced by its average value $\sqrt{k_o^2+2mp_o} \simeq k_o$. Integrating over p gives

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2 k_o}{m}\sqrt{\frac{\pi}{2\Delta y^2}}\int dk\,dk'\,e^{\frac{-\Delta y^2}{8m^2}(k+k')^2(k-k')^2}g(k)g^*(k')e^{i(k'^2-k^2)\frac{y}{2m}}. \quad \text{(A3)}$$

Since $\Delta y\,k >> 1$, for a wave packet peaked around k_o we can approximate the argument of the first exponential by $\frac{-\Delta y^2 k_o^2}{2m^2}(k-k')^2$. This allows us to integrate over k and k'

$$\rho(y,y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}}e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \quad \text{(A4)}$$

where the width is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$. As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial width Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $1/dE = \frac{m}{kdk}$ and is given by $\frac{dx(y)m}{k_o}$ where $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the width in x arises because the wave packet is spreading as time increases, so that at later y, the wave packet is wider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles with larger mass.

When the clock is extremely accurate ie. $p_o >> k_o$ we have $A_T \sim k\sqrt{\frac{2}{mp}}$.

$$\rho(y,y)_{x>0} \simeq \frac{2N^2}{m}\int_{-\infty}^{\infty} dk\, dk' \int_0^{\infty} dp\, dp'\, dx \frac{kk'}{\sqrt{pp'}} g(k)g^*(k')f(p)f^*(p')e^{i(q-q')x+i(p-p')y-\frac{i(q^2-q'^2)t}{2m}}$$
$$\simeq \frac{4\pi N^2}{m}\int dk\, dk'\, dp \frac{kk'}{m}\sqrt{\frac{k^2+2mp}{p(p+\frac{k^2-k'^2}{2m})}} g(k)g^*(k')f(p)f^*(p+\frac{k^2-k'^2}{2m})e^{i(k'^2-k^2)\frac{y}{2m}}. \quad \text{(A5)}$$

Since $p_o >> k_o$, we can approximate this integral as

$$\rho(y,y)_{x>0} \simeq \frac{A}{m} \mid \int dk\, k\, g(k)e^{-i\frac{k^2 y}{2m}}\mid^2, \quad \text{(A6)}$$

where $A \equiv 4\pi\sqrt{\frac{2}{m}}N^2 \int \frac{dp}{\sqrt{p}}|f(p)|^2$. We can approximate p by p_o to take it outside the integrand, giving

$$A \simeq \sqrt{\frac{\pi}{mp_o}\frac{2\Delta x}{\pi^2}}. \quad \text{(A7)}$$

The final integration over k yields

$$\rho(y,y)_{>0} \simeq 4\sqrt{\frac{k_o^2}{2mp_o}\frac{\tilde{\gamma}(t_c)}{\tilde{\gamma}(y)}}\frac{1}{\sqrt{2\pi\tilde{\gamma}(y)}}e^{-\frac{(y-t_c)^2}{2\tilde{\gamma}(y)}}, \quad \text{(A8)}$$

where the width $\tilde{\gamma}(y) = \Delta x^2 + (\frac{y}{2k_o\Delta x})^2$ is independent of Δy because the kinematic spread in the time-of-arrival $1/dE$ is much larger than the spread in the position of the clock. In this limit we see two additional factors. The amplitude decays like $\sqrt{E_o/p_o}$ so that improved accuracy decreases our chances of detecting the particle. Also, there is a minor correction of $\frac{\tilde{\gamma}(t_c)}{\tilde{\gamma}(y)}$. More energetic particles with faster arrival times are more likely to trigger the clock.

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