Geometric Phases, Reduction and Lie-Poisson Structure for the Resonant Three-Wave Interaction

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Hamiltonian Lie-Poisson structures of the three-wave equations associated with the Lie algebras *su* (3) and *su* (2,1) are derived and shown to be compatible. Poisson reduction is performed using the method of invariants and geometric phases associated with the reconstruction are calculated. These results have important implications for nonlinear-wave applications in, for instance, nonlinear optics. Some of the general structures presented in the latter part of this paper are implicit in the literature; our purpose is to put the three-wave interaction in the modern setting of geometric mechanics and to explore some new things, such as explicit geometric phase formulas, as well as some old things, such as integrability, in this context.

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1 Introduction

Resonant wave interactions permit the exchange of wave action or energy among nonlinear modes in a variety of physical systems. For instance the three-wave interaction occurs when the wavenumbers and frequencies of three nonlinear waves satisfy either the matching conditions $\mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}_3$, and $\omega_1 = \omega_2 - \omega_3$ for a decay process or the matching conditions $\mathbf{k}_1 = -\mathbf{k}_2 - \mathbf{k}_3$, and $\omega_1 = -\omega_2 - \omega_3$ for an explosive process. The three-wave equations describe the resonant quadratic nonlinear interaction of three waves and are obtained as amplitude equations in an asymptotic reduction of primitive equations in optics, fluid dynamics and plasma physics.

The purely quadratic three-wave system of ordinary differential equations is

$$\frac{dq_1}{dt} = is_1\gamma_1 q_2\bar{q}_3 \ , \tag{1}$$
$$\frac{dq_2}{dt} = is_2\gamma_2 q_1 q_3 \ , \tag{2}$$
$$\frac{dq_3}{dt} = is_3\gamma_3 \bar{q}_1 q_2 \ . \tag{3}$$

Here, the γ_i are nonzero real numbers such that $\gamma_1 + \gamma_2 + \gamma_3 = 0$, (s_1, s_2, s_3) is either $(1,1,1)$ or $(-1,1,1)$. Each $q_i \in \mathbb{C}$, so this is a system of ordinary differential equations on $\mathbb{C}^3$. The choice of signs, determined by (s_1, s_2, s_3) distinguishes between the *decay interaction* which has bounded solutions in time and the *explosive interaction* which has solutions that blow up in finite time. These two systems are represented below as Lie-Poisson systems for the groups SU(3) and SU(2,1).

Decay Interaction. This is obtained by choosing $(s_1, s_2, s_3) = (1,1,1)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, -2)$. After changing variables to $Q_1 = \sqrt{2}q_1, Q_2 = \sqrt{2}q_2$ and $Q_3 = \bar{q}_3$, this system takes the standard form

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$$\frac{dQ_1}{dt} = iQ_2Q_3 \ , \tag{4}$$
$$\frac{dQ_2}{dt} = iQ_1\bar{Q}_3 \ , \tag{5}$$
$$\frac{dQ_3}{dt} = iQ_1\bar{Q}_2 \ , \tag{6}$$

and models the dynamics of three resonantly coupled positive-energy waves. All solutions in $\mathbb{C}^3$ remain uniformly bounded.

Explosive Interaction. This is obtained by choosing $(s_1, s_2, s_3) = (-1, 1, 1)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, -2)$. After changing variables to $Q_1 = \sqrt{2}\bar{q}_1, Q_2 = \sqrt{2}q_2$ and $Q_3 = \bar{q}_3$, this system takes the standard form

$$\frac{dQ_1}{dt} = i\bar{Q}_2\bar{Q}_3 \ , \tag{7}$$
$$\frac{dQ_2}{dt} = i\bar{Q}_1\bar{Q}_3 \ , \tag{8}$$
$$\frac{dQ_3}{dt} = i\bar{Q}_1\bar{Q}_2 \ , \tag{9}$$

and models the dynamics of three resonantly coupled negative-energy waves. Solutions in $\mathbb{C}^3$ can blow up in finite time. (see Zakharov and Manakov [1] or Ablowitz and Segur [2]).

Basic wave interactions of this kind are fundamental in the understanding and analysis of a variety of phenomena including patterns, symmetry induced instabilities, the Benjamin-Feir instability and many others. The three-wave equations are closely related to the equations governing coupled harmonic oscillators, tops, the rigid body and even the interaction of light with a two-level atom. This is understood by realizing that the three-wave equations are the complex equations for a resonant three degree of freedom Hamiltonian system. They contain the Euler equations (see for instance Guckenheimer and Mahalov [3]) associated with $SO(3)$ as a real subspace of $SU(3)$. The Maxwell-Bloch equations are also contained in the three-wave equations (see David and Holm [4] for details). Some general references to literature on the integrable three-wave equations is found in Whitham [5] Ablowitz and Haberman [6], Kaup [7–9], Zakharov and Manakov [10], Ablowitz and Segur [2], Newell [11], and Ablowitz and Clarkson [12].

The integrable Hamiltonian structure of the three-wave equations is of course well known; we explore it from a somewhat novel point of view in what follows. As we will show, these equations posses a Lie-Poisson structure in addition to the canonical Hamiltonian structure. The three-wave decay system is Lie-Poisson for the Lie algebra $\mathfrak{su}(3)$, and the explosive three-wave system is Lie-Poisson for $\mathfrak{su}(2,1)$. Using the method of translation of the argument, two compatible Hamiltonian structures are obtained. One is the canonical Hamiltonian structure embedded in $\mathfrak{su}(3)$ or $\mathfrak{su}(2,1)$; it has a cubic Hamiltonian. The other is non-canonical having a standard left invariant Lie-Poisson bracket; it has a quadratic Hamiltonian. These two Poisson brackets lead to a recursion relation that is expressed in terms of Lie brackets. This recursion relation is the same one that is found using the Lax pair approach. We will show that the λ-representation used by Manakov follows directly from this.

Solutions for the three-wave equations and other similar systems are well known. In our approach below, they are reduced and integrated using a pair of S^1 actions, the canonical Hamiltonian structure and the technique of invariants. In solving the reconstruction problem, phase formulas analogous to those obtained for the rigid body [13] are obtained. These formulas give the value of the phase shifts that accompany the periodic exchange of wave action in resonant wave systems (see for instance [14–18]).

Though the main development is restricted to the three-wave system, we remark that as in the case of the n-component Euler equations [19], all of the basic results described below generalize to n-wave systems (see Kummer [20] for a treatment of reduction for the n-degree of freedom Hamiltonian with resonances). The structure of the n-wave interaction is related to the family of Lie algebras in $\mathfrak{su}(n)$ or $\mathfrak{su}(p,q)$.

The general picture developed here is useful for many other purposes, including polarization control (building on work of Holm, David and Tratnik [21]) and perturbations of Hamiltonian normal forms (see work of Knobloch, Mahalov and Marsden [22], Kirk, Marsden and Silber [23], and Haller and Wiggins [24]). They have also been used to analyze quasi-phase-matched second harmonic generation [25] and cascaded nonlinearities [18] in nonlinear optics.

2 The Canonical Hamiltonian Structure.

A γ_i-weighted canonical Poisson bracket on $\mathbb{C}^3$ is used. This bracket has the real and imaginary parts of each complex dynamical variable q_i as conjugate variables. The Hamiltonian for the three-wave equations is cubic in this setting.

The Canonical Symplectic and Poisson Structure. Writing $q_k = x_k + iy_k$ and treating x_k and y_k as conjugate variables, the (scaled) canonical Poisson bracket is given by

$$\{F, G\} = \sum_{k=1}^{3} s_k\gamma_k \left(\frac{\partial F}{\partial x_k}\frac{\partial G}{\partial y_k} - \frac{\partial G}{\partial x_k}\frac{\partial F}{\partial y_k} \right) . \tag{10}$$

In standard matrix notation this is

$$\{F, K\} = (\nabla F)\, \mathbb{J}\, (\nabla K) , \tag{11}$$

where the gradients are standard gradients in $\mathbb{R}^6$ (with the variables ordered as $(x_1, x_2, x_3, y_1, y_2, y_3)$) and where

$$\mathbb{J} = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix} \tag{12}$$

and Γ is the 3×3 matrix with $s_k\gamma_k$ on the diagonal and zeros elsewhere.

This bracket may be written in complex notation as

$$\{F, G\} = -2i \sum_{k=1}^{3} s_k\gamma_k \left(\frac{\partial F}{\partial q^k}\frac{\partial G}{\partial \bar{q}_k} - \frac{\partial G}{\partial q^k}\frac{\partial F}{\partial \bar{q}_k} \right) . \tag{13}$$

The corresponding symplectic structure is written as follows:

$$\Omega((z_1, z_2, z_3), (w_1, w_2, w_3)) = -\sum_{k=1}^{3} \frac{1}{s_k\gamma_k} \mathrm{Im}(z_k\bar{w}_k) . \tag{14}$$

The Hamiltonian. The Hamiltonian for the three-wave interaction is

$$H_3 = -\frac{1}{2}\left(\bar{q}_1 q_2 \bar{q}_3 + q_1 \bar{q}_2 q_3\right) . \tag{15}$$

Hamilton's equations are

$$\frac{dq_k}{dt} = \{q_k, H\} , \tag{16}$$

and it is straightforward to check that Hamilton's equations are given in complex notation by

$$\frac{dq_k}{dt} = -2is_k\gamma_k \frac{\partial H}{\partial \bar{q}_k} . \tag{17}$$

One checks that Hamilton's equations with $H = H_3$ coincide with (1).

Therefore, the following standard result holds.

Proposition 1 *With the preceding Hamiltonian H_3 and the symplectic or equivalently the Poisson structure given above, Hamilton's equations are given by the three-wave equations* (1).

Integrals of Motion. In addition to H_3 itself, one identifies the following constants of motion,

$$K_1 = \frac{|q_1|^2}{s_1\gamma_1} + \frac{|q_2|^2}{s_2\gamma_2} , \tag{18}$$

$$K_2 = \frac{|q_2|^2}{s_2\gamma_2} + \frac{|q_3|^2}{s_3\gamma_3} , \tag{19}$$

$$K_3 = \frac{|q_1|^2}{s_1\gamma_1} - \frac{|q_3|^2}{s_3\gamma_3} . \tag{20}$$

These are often referred to as the *Manley-Rowe relations.* The Hamiltonian with any two of the K_j are checked to be a complete and independent set of conserved quantities in involution (the K_j clearly give only two independent invariants since $K_1 - K_2 = K_3$). In the sense of Liouville-Arnold this system is therefore integrable.

To integrate the three-wave equations one typically makes use of the Hamiltonian, H_3, plus two of the integrals, K_j, to reduce the system to quadratures. This procedure is usually carried out using the transformation $q_j = \sqrt{\rho_j} \exp i\phi_j$. Below a different approach is described. It appears to be less cumbersome, and it provides considerable geometric insight.

First observe that

Proposition 2 *The vector function (K_1, K_2, K_3) is the momentum map for the following symplectic action of $T^3 = S^1 \times S^1 \times S^1$:*

$$(q_1, q_2, q_3) \to (q_1 \exp(i\gamma), q_2 \exp(i\gamma), q_3) , \tag{21}$$

$$(q_1, q_2, q_3) \to (q_1, q_2 \exp(i\gamma), q_3 \exp(i\gamma)) , \tag{22}$$

$$(q_1, q_2, q_3) \to (q_1 \exp(i\gamma), q_2, q_3 \exp(-i\gamma)) . \tag{23}$$

Any combination of two of these actions is generated by the third reflecting the fact that the K_j are linearly dependent. Another way of saying this is that the group action by T^3 is really captured by the action of T^2.

3 Poisson Reduction

In this section symplectic and Poisson reduction are performed on the three-wave Hamiltonian system using the S^1 symmetries associated with the momentum maps K_k. In terms of Poisson reduction, the process is to replace $\mathbb{C}^3 \to \mathbb{C}^3/T^2$. The symplectic leaves in this reduction are obtained using the method of invariants.

Invariant coordinates for three-wave reduction. Invariants for the T^2 action are:

$$X + iY = q_1\bar{q}_2q_3 \ , \tag{24}$$

$$Z_1 = |q_1|^2 - |q_2|^2 \ , \tag{25}$$

$$Z_2 = |q_2|^2 - |q_3|^2 \ . \tag{26}$$

These quantities provide coordinates for the four-dimensional orbit space $\mathbb{C}^3/T^2$. The coordinates, X, Y, Z_1 and Z_2 are Hopf-like variables (see, e.g., [26]) and they generalize the well known Stokes parameters (see, e.g., [27]).

Reduced three-wave surfaces. The following identity holds for these invariants and the conserved quantities:

$$X^2 + Y^2 = \kappa_4(s_2\gamma_2K_1 + Z_1)(s_3\gamma_3K_2 + Z_2)(s_2\gamma_2K_2 - Z_2) \ , \tag{27}$$

where $\kappa_4 = (s_1\gamma_1s_2\gamma_2s_3\gamma_3)/(s_1\gamma_1 + s_2\gamma_2)(s_2\gamma_2 + s_3\gamma_3)^2$. Trajectories in these reduced coordinates lie on the set defined by this relation. Using the conservation laws K_k and the definitions of Z_1 and Z_2, a second relation between K_1, K_2 and Z_1, Z_2 is identified and either of the coordinates Z_j is removed reducing the number of real dimensions to three. In $\mathbb{R}^3$ the reduced trajectories lie on the invariant set:

$$X^2 + Y^2 = \kappa_3(\delta - Z_2)(s_3\gamma_3K_2 + Z_2)(s_2\gamma_2K_2 - Z_2) \ , \tag{28}$$

where $\kappa_3 = (s_1\gamma_1s_2\gamma_2s_3\gamma_3)/(s_2\gamma_2 + s_3\gamma_3)^3$ and $\delta = s_2\gamma_2K_1 + s_3\gamma_3(K_1 - K_2)$. This relation defines a two dimensional (perhaps singular) surface in (X, Y, Z_2) space, with Z_1 determined by the values of these invariants and the conserved quantities (so it may also be thought of as a surface in (X, Y, Z_1, Z_2) as well). The relations between the invariants and the conserved quantities may imply inequalities for, say, Z_2; these may imply that the corresponding surface is compact. A sample of one of these surfaces is plotted in Fig. 1. These surfaces will be called the *three-wave surfaces* below.

Reduced three-wave equations. Any trajectory of the original equations defines a curve on each three-wave surface, in which the K_j are set to constants. These three-wave surfaces are the symplectic leaves in the four-dimensional Poisson space having coordinates (X, Y, Z_1, Z_2).

The original equations define a dynamical system in the Poisson reduced space and on the symplectic leaves as well. Using these new coordinates the Poisson bracket and the Hamiltonian are reduced directly. The reduced Hamiltonian is

$$H_r = -X \ . \tag{29}$$

With the reduced Poisson brackets in (X, Y, Z_2), H_r produces the reduced equations of motion

$$\frac{dX}{dt} = 0 \ , \tag{30}$$

$$\frac{dY}{dt} = \frac{\partial \phi}{\partial Z_2} \ , \tag{31}$$

$$\frac{dZ_2}{dt} = -2(s_2\gamma_2 + s_3\gamma_3)Y \ , \tag{32}$$

where the dynamical invariant ϕ is defined by

$$\begin{aligned} \phi = (s_2\gamma_2 + s_3\gamma_3) \Big[(X^2 + Y^2) \\ - \ \kappa_3(\delta - Z_2)(s_3\gamma_3 K_2 + Z_2)(s_2\gamma_2 K_2 - Z_2) \Big] \ . \end{aligned} \tag{33}$$

Following Kummer [28,20], the reduced equations may be written as $\dot{F} = \{F, H_r\}$ for the Poisson bracket

$$\{F, G\} = \nabla\phi \cdot (\nabla F \times \nabla G) \ . \tag{34}$$

The trajectories on the reduced surfaces are also obtained by slicing the surface with the planes H_r = Constant. The Poisson structure on $\mathbb{C}^3$ drops to a Poisson structure on (X, Y, Z_1, Z_2)-space and this in turn induces the Poisson structure above. Correspondingly, the symplectic structure drops to one on each three-wave surface – this is an example of the general procedure of symplectic reduction (MMW reduction [29]). Notice however, that the three-wave surfaces may have singularities – this is because the group action is not free; this is one aspect of singular reduction. Also, from the geometry, it is clear that a homoclinic orbit passes through such a singular point–these are cut out by the plane $H_r = 0$ when $H_r = -X$. Using ϕ and H_r, (30)–(32) are reduced to quadratures to obtain explicit solutions.

4 Three-wave phase formulas

The goal of this section is to reconstruct the original system on $\mathbb{C}^3$. The technique used generalizes that used by Richard Montgomery [13] in his derivation of the phase formula for the rigid body, and provides a clear geometrical picture for phase shifts in resonant wave interactions. The general theory is found in Marsden, Montgomery and Ratiu [30].

For most initial data on the reduced phase space, the amplitudes of the three interacting waves evolve along closed orbits. As they evolve their phases shift. When viewed in $\mathbb{R}^6$ the wave system is quasiperiodic. The main result here will be phase formulas obtained by reconstructing the dynamics in $\mathbb{C}^3$ associated with the orbits in $\mathbb{R}^3$. These formulae provide the phase shifts associated with each of the three amplitudes in the wave system as they complete one closed orbit. They have a dynamical part that is associate with the period of the reduced orbit and a geometric part associated with the symplectic area enclosed by that orbit.

Let $M \subset \mathbb{C}^3$ be the manifold defined by setting the conserved quantities to specific values (a level set of the momentum map). Construct a closed curve C on M in two pieces as shown in Fig. 2. The first portion c_0 is the dynamical trajectory on M joining the two points P_0 and P_1. It covers a closed curve in the reduced space. The invariants define the reduction map $W : \mathbb{C}^3 \to \mathbb{R}^3$ so that the curve c_0 projects onto a closed trajectory in the base space or three-wave surface m under W as shown in Fig. 2. In M introduce three curves c_1, c_2, and c_3 any pair of which complete the curve C by connecting the points P_0 and P_1 as follows: use phase rotations in each factor (the curve c_1 goes with the first S^1 etc.) so that $c_1 \cup c_2$ closes the curve, as shown in Fig. 2; $c_1 \cup c_3$ and $c_2 \cup c_3$ also close the curve. The segment of C associated with the group curves is drawn perpendicular to the base space m to stress that it corresponds to the S^1 symmetries that generate motion "orthogonal" to that on the base space.

Now let Θ be a canonical one form on $\mathbb{C}^3$, a scaling of the Poincaré one form $\frac{1}{2}(p_k dq^k - q^k dp_k)$. Thus, $d\Theta = -\Omega$ is the symplectic form, and

$$\langle \Theta(q_1, q_2, q_3), (v_1, v_2, v_3) \rangle = -\sum_{k=1}^{3} \frac{1}{s_k \gamma_k} \mathrm{Im}(q_k \bar{v}_k) \ . \tag{35}$$

Proposition 3 *Using Stokes theorem in connection with C on $\mathbb{C}^3$ produces the system of integral equations,*

$$\int_{c_0} \Theta + \int_{c_1} \Theta + \int_{c_2} \Theta = \int_{S_3} d\Theta \ , \tag{36}$$

$$\int_{c_0} \Theta + \int_{c_1} \Theta + \int_{c_3} \Theta = \int_{S_2} d\Theta \; , \tag{37}$$

$$\int_{c_0} \Theta + \int_{c_2} \Theta + \int_{c_3} \Theta = \int_{S_1} d\Theta \; , \tag{38}$$

where S_k are surfaces that project to the cap Σ shown in the figure.

A more careful argument–as in holonomy theorems–shows that the existence of these surfaces is not necessary.

Next evaluate the integrals associated with the group curves in $\mathbb{C}^3$ to extract the phase shifts associated with these actions and obtain

$$\int_{c_k} \Theta = -K_k \phi_k \; , \quad k = 1, 2, 3 \; , \tag{39}$$

where K_k is the (constant) conserved quantity and ϕ_k is the phase shift associated with the k^{th} S^1 action. This formula is particularly simple because the K_k are homogeneous of degree 2 in the q_k.

On the dynamic trajectory c_0 in $\mathbb{C}^3$ the line integral is

$$\int_{c_0} \Theta = \int_{c_0} \langle \Theta, X_H \rangle \, dt = \frac{3}{2} HT \; ,$$

where H is the constant energy of the trajectory and T is the period of the orbit on the three-wave surface. The factor $3/2$ appears because H is homogeneous of degree 3 in the q_k.

Now using the symplectic form and noting that

$$\int_{\tilde{\Sigma}} \Omega = \int_{\Sigma} \Omega_m \; ,$$

where Ω_m is the reduced symplectic form, and find

$$K_1\phi_1 + K_2\phi_2 = \left[A(\Sigma) + \frac{3}{2} HT \right] \; , \tag{40}$$

$$K_1\phi_1 + K_3\phi_3 = \left[A(\Sigma) + \frac{3}{2} HT \right] \; , \tag{41}$$

$$K_2\phi_2 + K_3\phi_3 = \left[A(\Sigma) + \frac{3}{2} HT \right] \; , \tag{42}$$

where $A(\Sigma)$ is used to denote the symplectic area of the surface Σ enclosed by the orbit on the three-wave surface. Solve these equations to obtain the phase shifts.

Theorem 4 *The phase shifts in $\mathbb{C}^3$ induced by S^1 actions over periodic orbits on the three-wave surfaces are*

$$\phi_1 = \frac{1}{2K_1}\left[A(\Sigma) + \frac{3}{2}HT\right] , \tag{43}$$

$$\phi_2 = \frac{1}{2K_2}\left[A(\Sigma) + \frac{3}{2}HT\right] , \tag{44}$$

$$\phi_3 = \frac{1}{2K_3}\left[A(\Sigma) + \frac{3}{2}HT\right] . \tag{45}$$

These actions of the momentum map for one period, T, on the base space, m, produce shifts in the initial data on $\mathbb{C}^3$ so that

$$q_1(T) = q_1(0)\exp[i(\phi_1 + \phi_3)] , \tag{46}$$

$$q_2(T) = q_2(0)\exp[i(\phi_1 + \phi_2)] , \tag{47}$$

$$q_3(T) = q_3(0)\exp[i(\phi_2 - \phi_3)] . \tag{48}$$

Proof The proof follows directly from the discussion above. ■

These formulas show that as the values of the actions of the three waves complete one cycle, the phases of the waves are shifted by the amounts calculated. These shifts include a dynamic part that scales linearly with the period and a geometric part that corresponds to non-uniform jumps in the phases of the waves. These jumps are known to occur at the point where the flow of energy or wave action reverses (see for instance Refs [14–18]). Three-wave phase formulas for non-homogeneous Hamiltonians were derived in [18].

Notice that for the fixed point set $3 \leftrightarrow 1$, which corresponds to second harmonic generation, $K_2 = K_1$ and $Z_2 = -Z_1$. With this restriction, a double root is introduced into the cubic in Z_2 on the right-hand side of (33). All of the phase formulas reduce readily in this case to $\phi_1 = \phi_2 = [A(\Sigma) + 3HT/2]/(2K_1)$ and $\phi_3 = 0$. Over a period then, $q_1(T) = q_1(0)\exp[i\phi_1]$ and $q_2(T) = q_2(0)\exp[i2\phi_1]$.

5 The Lie-Poisson Structure

In this section the three-wave equations are written on the dual of the Lie algebra of the group $SU(3)$ or $SU(2,1)$ using a Lie-Poisson structure. This Lie-Poisson system of equations has a quadratic rather than cubic Hamiltonian.

The compatibility of the canonical and Lie-Poisson structures is discussed in the next section.

The Lie-Poisson description is obtained by recasting (1) as a differential equation in $\mathfrak{su}(3)^*$, the dual of the Lie algebra of $SU(3)$, for the decay interaction and a differential equation in $\mathfrak{su}(2,1)^*$, the dual of the Lie algebra of $SU(2,1)$, for the explosive interaction.

Map to the dual of the Lie algebra. Define a map $U : \mathbb{C}^3 \to \mathfrak{su}(3)^*$ as follows. Identify $\mathfrak{su}(3)$ with $\mathfrak{su}(3)^*$ using the standard Killing form:

$$\langle A, B\rangle = \mathrm{Tr}(AB) \ . \tag{49}$$

Thus, $\mathfrak{su}(3)^* \cong \mathfrak{su}(3)$ is concretely realized as the space of complex skew Hermitian matrices with zero trace. Map $q = (q_1, q_2, q_3)$ to the matrix

$$U = \begin{pmatrix} 0 & q_1 & q_2 \\ -m_1\bar{q}_1 & 0 & q_3 \\ -m_2\bar{q}_2 & -m_3\bar{q}_3 & 0 \end{pmatrix} , \tag{50}$$

where $(m_1, m_2, m_3) = (1,1,1)$ for $\mathfrak{su}(3)$. The standard Killing form is also used to pair $\mathfrak{su}(2,1)$ with $\mathfrak{su}(2,1)^*$. While the resulting inner product remains nondegenerate in this case, it does become Lorenzian. The map $U : \mathbb{C}^3 \to \mathfrak{su}(2,1)^*$ is also defined by (50) taking $(m_1, m_2, m_3) = (1,1,-1)$.

The quadratic Hamiltonian. The quadratic Lie-Poisson Hamiltonian is $H_2 = -\mathrm{Tr}(UQ_1)/2$, where

$$Q_1 = \begin{pmatrix} 0 & \alpha_1 q_1 & \alpha_2 q_2 \\ -m_1\alpha_1\bar{q}_1 & 0 & \alpha_3 q_3 \\ -m_2\alpha_2\bar{q}_2 & -m_3\alpha_3\bar{q}_3 & 0 \end{pmatrix} . \tag{51}$$

The explicit form of this quadratic Hamiltonian is

$$H_2 = i\sum_{k=1}^{3} m_k\alpha_k|q_k|^2 ,$$

where the α_k are purely imaginary and are related to the γ_k in the following way:

$$s_1\gamma_1 = -im_3(\alpha_2 - \alpha_3) \ ,$$
$$s_2\gamma_2 = -im_2(\alpha_3 - \alpha_1) \ ,$$
$$s_3\gamma_3 = -im_1(\alpha_1 - \alpha_2) \ .$$

By relating $\mathbb{C}^3$ and $\mathfrak{su}(3)^*$ using the map above, $(m_1, m_2, m_3) = (1,1,1)$ implies that $\sum_{k=1}^{3}\gamma_k = 0$ and $(s_1, s_2, s_3) = (1,1,1)$ for the decay interaction. Relating $\mathbb{C}^3$ and $\mathfrak{su}(2,1)^*$ using the map above, $(m_1, m_2, m_3) = (1,1,-1)$ implies that $\sum_{k=1}^{3}\gamma_k = 0$ and $(s_1, s_2, s_3) = (-1,1,1)$ for the explosive interaction. In both cases $\mathbb{I}m(\alpha_k) > 0$ for $k = 1,2,3$ and $|\alpha_2| > |\alpha_3| > |\alpha_1|$.

The Lie-Poisson bracket. The equation

$$\frac{dU}{dt} = -[U, Q_1] \tag{52}$$

is equivalent to the three-wave system, where $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket and in this context is equivalent to standard commutation of matrices. As we show below, $Q_1 = -\delta H_2/\delta U$, so we can write $dU/dt = [\delta H_2/\delta U, U]$. The general theory of Lie-Poisson structures is used to construct the Lie-Poisson bracket

$$\{f,k\}_1(U) = -\left\langle U, \left[\frac{\delta f}{\delta U}, \frac{\delta k}{\delta U}\right]\right\rangle \ , \tag{53}$$

where for $\mathfrak{g} = \mathfrak{su}(3)$ or $\mathfrak{su}(2,1)$, $f, k : \mathfrak{g}^* \to \mathbb{R}$, $\delta f/\delta U, \delta f/\delta U \in \mathfrak{g}$, and $U \in \mathfrak{g}^*$.

Theorem 5 *With these definitions, the three-wave decay equations are Lie-Poisson equations on* $\mathfrak{su}(3)^*$ *for* $(s_1, s_2, s_3) = (1,1,1)$ *and* $(m_1, m_2, m_3) = (1,1,1)$*; the explosive three-wave equations are Lie-Poisson equations on* $\mathfrak{su}(2,1)^*$ *for* $(s_1, s_2, s_3) = (-1,1,1)$ *and* $(m_1, m_2, m_3) = (1,1,-1)$.

Proof Let $F : \mathfrak{g}^* \to \mathbb{R}$, then with the definitions above, $dF/dt = \{F, H_2\}$, or

$$\left\langle \frac{\delta F}{\delta U}, \frac{dU}{dt}\right\rangle = -\left\langle U, \left[\frac{\delta F}{\delta U}, \frac{\delta H_2}{\delta U}\right]\right\rangle \ , \tag{54}$$

where $\langle , \rangle$ is defined by the trace as above. Now, $DH_2(U)\cdot V = -\mathrm{Tr}(VQ_1(U))/2 - \mathrm{Tr}(UQ_1(V))/2$ for $V \in \mathfrak{g}^*$. We claim that Q_1 is a symmetric linear function of U. In fact, one can check directly that $Q_1(U)_{i,j} = c_{i,j}U_{i,j}$ (no sum), where $c_{i,j}$ is a symmetric matrix. Thus,

$$\mathrm{Tr}(UQ_1(V)) = \mathrm{Tr}\left(\sum_j U_{k,j}c_{j,k}V_{j,k}\right) = \sum_{j,k} U_{k,j}c_{j,k}V_{j,k} = \mathrm{Tr}(Q_1(U)V) \ .$$

Hence, $DH_2(U) \cdot V = -\mathrm{Tr}(VQ_1(U))$ and so $\delta H_2/\delta U = -Q_1(U)$. Using this fact, write

$$\left\langle \frac{\delta F}{\delta U}, \frac{dU}{dt} \right\rangle = - \left\langle \frac{\delta F}{\delta U}, [U, Q_1] \right\rangle , \tag{55}$$

to obtain

$$\frac{dU}{dt} = - [U, Q_1] . \tag{56}$$

It is checked that these indeed are the three-wave equations. ■

6 Connections between the two Hamiltonian structures

The three-wave equations have now been expressed using both the well known canonical Hamiltonian structure and the Lie-Poisson structure. In this section the relationship between them is discussed. A recursion relation is also produced and it is shown to be the same one obtained using the Lax approach.

Second Hamiltonian structure. Modify the Lie-Poisson bracket for the three-wave equations as follows:

$$\{f, k\}_0 (U) = - \left\langle U_0, \left[\frac{\delta f}{\delta U}, \frac{\delta k}{\delta U} \right] \right\rangle , \tag{57}$$

in which the first matrix is frozen at U_0, where $U_0 \in \mathfrak{su}(3)^*$ is independent of t and is to be specified. Taking $\delta f/\delta U$ and $\delta k/\delta U$ at U, this new bracket produces the equations of motion,

$$\frac{dU}{dt} = \left[U_0, \frac{\delta k}{\delta U} \right] . \tag{58}$$

By choosing $U_0 = iA$ and $k \propto H_3$, so that $\delta k/\delta U = Q_2$, where Q_2 is quadratic in the q_i, we arrive at the three-wave equations. In this way the scaled canonical Hamiltonian structure is obtained directly from the Lie-Poisson bracket. Compatibility follows since this is a "translation of the argument" of the Lie-Poisson bracket, where $\{,\} = \{,\}_1(U) + \xi\{,\}_0(U_0)$ for an arbitrary real constant ξ. Both $\{,\}_1$ and $\{,\}_0$ are Poisson Brackets and the Lie-Poisson bracket with a shifted argument is also a Poisson bracket [31,32]. The two three-wave brackets are therefore compatible.

Recursion relation. Having obtained the Lie-Poisson structure and the compatibility of the two Poisson brackets the recursion relation for the three-wave equations are found. Equate the two Poisson brackets and write

$$\left\langle U_0, \left[\frac{\delta f}{\delta U}, \left(\frac{\delta k}{\delta U}\right)_{j+1}\right]\right\rangle = \left\langle U, \left[\frac{\delta f}{\delta U}, \left(\frac{\delta k}{\delta U}\right)_{j}\right]\right\rangle . \tag{59}$$

For this relation to hold the Lie brackets,

$$\left[\left(\frac{\delta k}{\delta U}\right)_{j+1}, U_0\right] = \left[\left(\frac{\delta k}{\delta U}\right)_{j}, U\right] , \tag{60}$$

must also be equal. This is exactly the recursion relation obtained using the Lax approach. For the three-wave system it is invertible, and a complete set of $(\delta k/\delta U)_j$ is constructed.

The Lax equations. To demonstrate the connection with the Lax approach let $D, P, Q \in \mathfrak{su}(3)$ for the decay case and $D, P, Q \in \mathfrak{su}(2,1)$ for the explosive case, and write

$$\xi D = [P, D] , \tag{61}$$

$$\frac{dD}{dt} = [Q, D] . \tag{62}$$

Compatibility of these two equations leads to

$$\frac{dP}{dt} + [P, Q] = 0 . \tag{63}$$

Let $P = i\xi A + U$ and $Q^{(N)} = \sum_{j=1}^{N} Q_j \xi^{N-j}$, where $A, U, Q_j \in \mathfrak{su}(3)$ for the decay case and $A, U, Q_j \in \mathfrak{su}(2,1)$ for the explosive case. Define A to be $A = \mathrm{diag}(\beta_1, \beta_2, \beta_3)$ with $\sum_{k=1}^{3} \beta_k = 0$. The Q_j are general elements of the Lie algebra. As before U maps $\mathbb{C}^3$ into $SU(3)$ or $SU(2,1)$. With this definition for P (63) becomes

$$\frac{dU}{dt} + i\xi[A, Q^{(N)}] + [U, Q^{(N)}] = 0 . \tag{64}$$

Now using the series for $Q^{(N)}$, the coefficients of powers of ξ yield

$$\frac{dU}{dt} + [U, Q_N] = 0 , \ldots \tag{65}$$

$$i[A, Q_j] + [U, Q_{j-1}] = 0 \ , \ldots \tag{66}$$
$$[A, Q_0] = 0 \ . \tag{67}$$

The first equation is the integrable three-wave system. The second is the recursion relation. The final equation constrains the Q_j so that $Q_0 \in \ker \mathrm{ad}_A$. Letting $Q_j = (\delta k/\delta U)_j$ and $iA = U_0$ this is exactly the recursion relation obtained using the method of Poisson pairs. The recursion relation implies that $[U, Q_1] = -i[A, Q_2]$, so the three-wave equations are also written

$$\frac{dU}{dt} = i[A, Q_2] \ . \tag{68}$$

Carrying out the recursion (65)–(67) explicitly for the three-wave equations with $N = 1$ and $\mathrm{diag}(Q_j) = 0$ for $j > 0$, it is found that $Q_0 = \mathrm{diag}(\beta_1^0, \beta_2^0, \beta_3^0)$,

$$Q_1 = i \begin{pmatrix} 0 & \frac{\beta_2^0-\beta_1^0}{\beta_2-\beta_1} q_1 & \frac{\beta_3^0-\beta_1^0}{\beta_3-\beta_1} q_2 \\ \frac{\beta_2^0-\beta_1^0}{\beta_2-\beta_1} r_1 & 0 & \frac{\beta_3^0-\beta_2^0}{\beta_3-\beta_2} q_3 \\ \frac{\beta_3^0-\beta_1^0}{\beta_3-\beta_1} r_2 & \frac{\beta_3^0-\beta_2^0}{\beta_3-\beta_2} r_3 & 0 \end{pmatrix} , \tag{69}$$

and

$$Q_2 = \sigma \begin{pmatrix} 0 & q_2 r_3 & q_1 q_3 \\ r_2 q_3 & 0 & r_1 q_2 \\ r_1 r_3 & q_1 r_2 & 0 \end{pmatrix} , \tag{70}$$

where

$$\sigma = \frac{(\beta_2 - \beta_1)\beta_3^0 + (\beta_1 - \beta_3)\beta_2^0 + (\beta_3 - \beta_2)\beta_1^0}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)(\beta_2 - \beta_3)} \ . \tag{71}$$

Note that while Q_3 is cubic in the q_k, the recursion terminates with $Q_4 = 0$ because the diagonal terms in Q_j for $j > 0$ are set to zero.

With these definitions, $dU/dt = i[A, Q_2]$ yields

$$\frac{dq_1}{dt} = i(\beta_1 - \beta_2)\sigma q_2 r_3 \ , \tag{72}$$
$$\frac{dq_2}{dt} = i(\beta_3 - \beta_1)\sigma q_1 q_3 \ , \tag{73}$$
$$\frac{dq_3}{dt} = i(\beta_2 - \beta_3)\sigma q_2 r_1 \ , \tag{74}$$

(75)

which are the three-wave equations in (1) if $s_1\gamma_1 = m_1(\beta_1 - \beta_2)\sigma$, $s_2\gamma_2 = m_2(\beta_3 - \beta_1)\sigma$, $s_3\gamma_3 = m_3(\beta_2 - \beta_3)\sigma$ and $r_k = m_k\bar{q}_k$. Notice that the γ_k sum to zero as required. These definitions also show that for $dU/dt = [Q_1, U]$, $\alpha_1 = (\beta_2^0 - \beta_1^0)/(\beta_2 - \beta_1)$, $\alpha_2 = (\beta_3^0 - \beta_1^0)/(\beta_3 - \beta_1)$, $\alpha_3 = (\beta_3^0 - \beta_2^0)/(\beta_3 - \beta_2)$. Two higher-order equations are obtained using Q_3; they have cubic and quadratic coupling terms. For diag$(Q_j) \neq 0$ $j > 0$ integrable equations with terms of higher than quadratic order are obtained.

Conservation Laws and Hamiltonians. The Q_j are gradients of Hamiltonian functions, and $Q_j = -\delta H_j/\delta U$, where the Hamiltonians

$$H_{j+1} = -\mathrm{Tr}(UQ_j)/(j+1) \ .$$

Here, $(j+1)$ is the highest power of q_k in H_{j+1}. The cubic Hamiltonian, H_3, defined here is proportional to the one associated with the scaled canonical structure from above. The quadratic Hamiltonian, H_2, is associated with the Lie-Poisson structure.

These conserved quantities are found in a number of ways. The method of Poisson pairs produces invariants and their involutivity. The so called master conservation law is obtained by showing that the equation

$$\left\langle U, \frac{dD}{dt} \right\rangle = \left\langle U, [Q^{(1)}, D] \right\rangle \ , \tag{76}$$

reduces to

$$\frac{d}{dt}\langle D, U\rangle = \xi \langle D, [U, Q_0]\rangle \ . \tag{77}$$

Then using the recursion relation and in this case $D = Q^{(2)}$, one finds that $d\langle U, D\rangle/dt = 0$. In this way the Hamiltonians

$$H_2 = -\frac{1}{2}\langle Q_1, U\rangle \ , \quad H_3' = -\frac{1}{3}\langle Q_2, U\rangle \ , \tag{78}$$

are obtained, where $H_3' = -2\sigma H_3$.

7 Discussion

Equations (61) and (62) provide alternate methods for solving the three-wave equations. They are used to construct the Lax pair of (63), which are linear equations for the evolution of an associated eigenfunction. Recall that as D evolves, its determinant and the values of Trace(D^k) remain invariant. Since the coefficients of the spectral curve,

$$\Gamma = \mathrm{Det}(D - y\mathbb{I}) = 0 , \tag{79}$$

involve only these quantities Γ is also invariant. By constructing the Baker-Akheizer functions of the associated linear spectral problem or by constructing new coordinates using D, algebro-geometric methods are applied to integrate the system in terms of theta functions on Riemann surfaces.

Finally, recall that (63) is the Lax equation for P. If P and $Q = Q^{(1)}$ are linear in ξ then (63) contains the three-wave equations, as shown above; (63) is then the so called λ-representation for the three-wave equations (see [19,33]). The three-wave system exhibits a rich Hamiltonian structure that has only been partially discussed here. Note for instance that this system can be expressed in terms of the R-matrix representation. Also note that the λ-representation for the three-wave equations is a reduction of the loop algebra associated with $\mathfrak{su}(3)$ or $\mathfrak{su}(2,1)$. A more complete treatment of the general structure of integrable equations of this type is found for instance in Refs. [31,32,34].

The family of n-wave interactions is connected to the groups $SU(n)$ and $SU(p,q)$. The structures described above for the three-wave example also follow for these higher-dimensional groups. Here integrability of the n-wave interaction on $\mathbb{C}^n$ is connected with the fact that there are a series of $U(1)$ subgroups in $SU(n)$ and $SU(p,q)$ that reduce the equations on $\mathbb{C}^n$ to equations on surfaces in $\mathbb{R}^3$. In Kummer [20] the resonant Hamiltonian system with n-frequencies was analyzed using the reduction procedure discussed here for the three-wave system. Using $n-1$ independent S^1 symmetries the n-wave system is ultimately reduced to quadratures. Phase formulas for the reconstruction then follow. A simple generalization of our phase formulas for the three-wave system yields $\phi_k = [A(\Sigma)/2 + nHT/4]/K_k$.

Solutions of the three-wave system analyzed here are also traveling wave or stationary solutions of an integrable partial differential equation (for solution of the partial differential equation see Refs. [1,6–12]). In this sense the integrable structure outlined above generalizes to the structure of the partial differential equation. More generally, each integrable system of ordinary differential equations is associated with a hierarchy of evolution equations through (61)–(62) by letting $\xi \to \partial/\partial x$, $d/dt \to \partial/\partial t$ and associating D, P, and Q

with an appropriate group. For instance, the three-wave system is closely connected to the rigid body. The Euler equations are on the real subspace formed by taking $\mathfrak{su}^*(3) \to \mathfrak{so}^*(3)$. It follows that there is a related real partial differential equation for which the Euler equations are stationary or traveling wave solutions.

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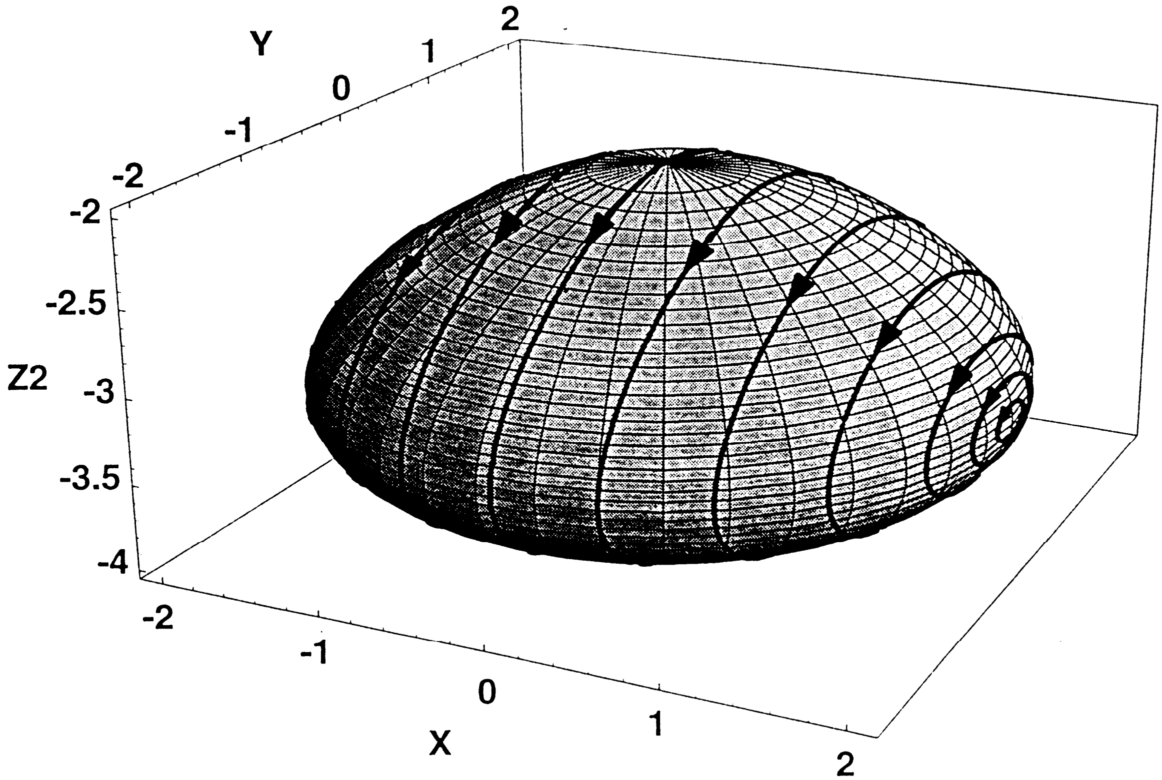
Fig. 1. A three-wave surface is drawn in (X, Y, Z_2) coordinates for the decay interaction. Trajectories are also drawn showing the phase space of the reduced three-wave equations on the three-wave surface when $(s_1, s_2, s_3) = (1, 1, 1)$, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, -2)$, and $(K_1, K_2) = (2, -1)$.

Fig. 2. The geometry used to reconstruct solutions in $\mathbb{C}^3$ from periodic orbits on the three-wave surfaces is shown.


Y
2
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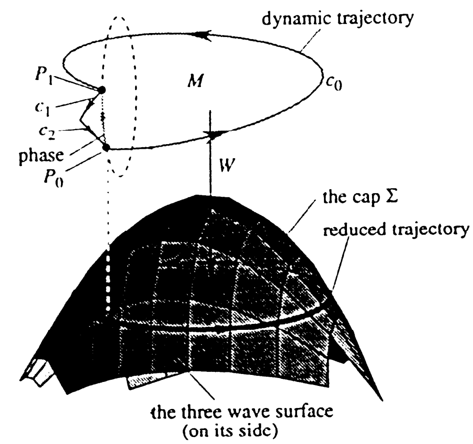
dynamic trajectory
P_1
M
c_0
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phase
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the cap Σ
reduced trajectory
the three wave surface
(on its side)