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Dynamics of Min-Max Functions

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Function $F : \mathbf{R}^n \to \mathbf{R}^n$ which are expansive in the $l\infty$ norm and homogeneous, $F_i(x_1 + h,\cdots,x_n + h) = F_i(x_1,\cdots,x_n) + h$, (so-called topical functions) have appeared recently in the work of several authors. They include (after suitable transformation) nonnegative matrices, Leontieff substitution systems, Bellman operators of games and of Markov decisions processes, examples arising from discrete event systems (digital circuits, computer networks, etc) and the min-max functions studied in this paper. Any topical function F can be approximated by min-max functions in a way which preserves some of the dynamics of F. We attempt, therefore, to clarify the dynamics of min-max functions, with a view to developing a generalised Perron-Frobenius theory for topical functions. Our main concern is with the existence of generalised fixed points, where $F(x_1,\cdots,x_n) = (x_1+h,\cdots,x_n+h)$, which correspond to nonlinear eigenvectors, and with the cycle time vector, $\chi(F) = \lim_{k\to\infty} F^k(x)/k \in \mathrm{R}^n$, which generalises the spectral radius. The Duality Conjecture for min-max functions asserts that this limit always exists, shows how it can be calculated and implies the strong result that F has a fixed point if, and only if, $\chi(F) = (h,\cdots,h)$. We use an analogue of Howard's policy improvement scheme from optimal control to prove a fixed point theorem of equivalent strength, which is independent of the Duality Conjecture, and we give a proof of the Conjecture itself for $n = 2$.

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1 Introduction

A min-max function $F : \mathbb{R}^n \to \mathbb{R}^n$ is built from terms of the form $x_i + a$, where $1 \leq i \leq n$ and $a \in \mathbb{R}$, by application of finitely many max and min operations in each component. For example,

$$\begin{aligned} F_1(x_1, x_2) &= \max(\min(\max(x_1+1, x_2-1.2), \max(x_1, x_2+2)), \min(x_1+0.5, x_2+1)) \\ F_2(x_1, x_2) &= \min(\max(x_1+7, x_2+4.3), \min(x_1-5, x_2-3)) \ . \end{aligned}$$

(A different notation is used in the body of the paper; see §1.1.) Such functions are homogeneous, $F_i(x_1+h, \cdots, x_n+h) = F_i(x_1, \cdots, x_n) + h$ for all $1 \leq i \leq n$, and nonexpansive in the ℓ_∞ norm, $\|F(\vec{x}) - F(\vec{y})\| \leq \|\vec{x} - \vec{y}\|$. Functions with these properties have emerged recently in the work of several authors, [3, 23, 29, 34, 46]. We shall follow Gunawardena and Keane and call them topical functions. They include (possibly after suitable transformation) nonnegative matrices, Leontieff substitution systems and Bellman operators of games and of Markov decisions processes, [24]. They also include examples less well-known to mathematicians, which arise from modelling discrete event systems, such as digital circuits, computer networks or automated manufacturing plants. This application is discussed in more detail in §1.2.

Any topical function T can be approximated by min-max functions in such a way that some of the dynamical behaviour of T is inherited by its approximations (see Lemma 1.1). In this paper we study the dynamics of min-max functions, motivated partly by the applications to discrete event systems and partly by the idea of developing a generalised Perron-Frobenius theory for topical functions. For any topical function, a (generalised) fixed point—or non-linear eigenvector—is a vector, $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, for which there exists $h \in \mathbb{R}$, such that $F(\vec{x}) = (x_1+h, \cdots, x_n+h)$. The cycle time vector, $\chi(F) = \lim_{k\to\infty} F^k(\vec{x})/k \in \mathbb{R}^n$, which arises as a performance measure for discrete event systems, provides the appropriate nonlinear generalisation of the Perron root, or spectral radius (see §1.3).

The cycle time vector is believed to exist for all min-max functions. This would be implied by the Duality Conjecture for min-max functions, first stated in [19], which asserts that χ, when considered as a functional from min-max functions to $\mathbb{R}^n$, is "almost" a homomorphism of lattices (see Theorem 2.2). The conjecture not only gives the existence of χ but also a method for calculating it in terms of the structure of F. One of the main results of the present paper is a proof of the conjecture in dimension 2 (see §4). For general topical functions, the situation is more complex. The cycle time does not always exist, [23, Theorem 3.1], and it is an important open problem to determine when it does, [24].

Unlike the conventional spectral radius, the cycle time is a vector quantity and immediately gives a necessary condition for the existence of a fixed point. If F has a fixed point, then it is easy to see that $\chi(F)$ exists and that $\chi(F) = (h, \cdots, h)$: the cycle time must have the same value in each component. For min-max functions, the Duality Conjecture implies that the converse is also true:

$$\exists \vec{x} \in \mathbb{R}^n, \text{ such that } F(\vec{x}) = (x_1+h, \cdots, x_n+h) \text{ if, and only if, } \chi(F) = (h, \cdots, h) \ . \quad (1)$$

The other main result of the present paper is a new fixed point theorem, equivalent in strength to (1) but independent of the Duality Conjecture and of the existence of χ (see §3). The proof is based on a min-max analogue of Howard's policy improvement scheme for stochastic decision processes (see §3). We recover, as a corollary, Olsder's fixed point theorem, [37], which applies to a restricted class of min-max functions.

If F has Lipschitz constant strictly less than 1, so that $\|F(\vec{x}) - F(\vec{y})\| \leq \lambda\|\vec{x} - \vec{y}\|$ where $0 < \lambda < 1$, then the Banach Contraction Theorem completely determines the dynamics: there is an unique fixed point to which every trajectory converges at an exponential rate, [16, Theorem 2.1]. When $\lambda = 1$, the situation is quite different. The existence of fixed points alone is a classical problem of nonlinear analysis; very little appears to be known about the general dynamics.

The metric approach to the fixed point problem for nonexpansive maps has focussed on finding properties of the ambient Banach space—usually convexity properties—which guarantee that every nonexpansive function has a fixed point, [16, Chapter 4]. Because these properties do not hold for the ℓ_∞ norm, these results are not very relevant here and the approach we take is different: we seek properties of the function rather than of the space. (In this respect it resembles the topological approach to fixed points arising from the theorems of Brouwer and Lefschetz.) What is interesting is that the properties turn out to be dynamical in nature: the cycle time vector is an asymptotic average over an orbit of the underlying dynamical system.

The methods of the present paper rely on a special class of min-max functions which can be studied by linear methods, albeit of an unusual nature. This is the class of matrices over the max-plus semiring, $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, where addition and multiplication are defined as max and $+$, respectively, the latter being distributive over the former (see §1.4). Matrices over $\mathbb{R}_{\max}$ (satisfying an appropriate nondegeneracy condition) correspond to min-max functions in which the min operation is never used. Matrix algebra over $\mathbb{R}_{\max}$ has been extensively studied, [2, 11, 12, 31, 36]. Min-max functions can be represented by finite collections of max-plus matrices and the dynamical properties of the latter, known from the linear theory over $\mathbb{R}_{\max}$, can be used to infer those of the former.

The attraction in studying this area stems from the emergence of a single vantage point from which a number of hitherto distinct problems can be viewed: modelling of discrete event systems, Perron-Frobenius theory, max-plus matrix algebra, fixed point theorems for nonexpansive functions, dynamics of nonlinear functions, etc. In view of its recent origins and the fact that it knits together so many different threads, the remainder of this Introduction is devoted to amplifying the outline above. In doing so the main concepts will be introduced and the ground prepared for the main results proved in subsequent sections. We hope this will give the reader a better sense of the scope of the present work, despite the resulting increase in its length.

Special cases of min-max functions were studied by Olsder in [37]. Min-max functions themselves were introduced in [21]. The present paper incorporates the results of [8, 19, 20], as well as new material. The authors gratefully acknowledge discussions with Michael Keane, Roger Nussbaum, Geert-Jan Olsder, Jean-Pierre Quadrat, Colin Sparrow and Sjoerd Verduyn Lunel. This work was partially supported by the European Community Framework IV programme through the research network ALAPEDES ("The Algebraic Approach to Performance Evaluation of Discrete Event Systems").

1.1 Min-max functions

We begin with some notation. Vectors in $\mathbb{R}^n$ will be denoted $\vec{x}, \vec{a}, etc.$ For vector valued quantities in general, such as functions $F : X \to \mathbb{R}^n$, the notation F_i will denote component i: $F(x) = (F_1(x), \cdots, F_n(x))$. However, to avoid clutter, we use x_i for the components of $\vec{x}$. The partial order on $\mathbb{R}$ will be denoted in the usual way by $a \leq b$ but it will be convenient to use infix forms for the lattice operations of least upper bound and greatest lower bound:

$$\begin{aligned} a \vee b &= \text{lub}(a, b) \\ a \wedge b &= \text{glb}(a, b) \,. \end{aligned}$$

(The word "lattice" is used in this paper to refer to a partial order in which any two elements have a least upper bound and a greatest lower bound, [27, §1.1]. We do not require, however, that a lattice has a greatest and a least element.) The same generic notation will be used for lattices derived from $\mathbb{R}$, such as the function space $X \to \mathbb{R}$. The partial order here is the pointwise ordering on functions: $f \leq g$ if, and only if, $f(x) \leq g(x)$ for all $x \in X$. If $\mathbb{R}^n$ is identified with $\{1, \cdots, n\} \to \mathbb{R}$, this specialises to the product ordering on vectors.

To reduce notational overhead we shall use the following vector-scalar convention: if, in a binary operation or relation, a vector and a scalar are mixed, the relevant operation is performed, or the relevant relation is required to hold, on each component of the vector. For instance, if $h \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, then $\vec{x} + h$ will denote the vector $(x_1 + h, \cdots, x_n + h)$, and $\vec{x} = h$ will imply $x_i = h$ for each $1 \leq i \leq n$. Throughout this paper, we shall use h to denote a real number without specifying so explicitly. Formulae such as $\vec{x} = h$ should therefore always be interpreted using the vector scalar convention.

The notation $\|\vec{x}\|$ will denote the ℓ_∞ norm on $\mathbb{R}^n$: $\|\vec{x}\| = |x_1| \vee \cdots \vee |x_n|$. If $F, G : X \to X$, then function composition will be denoted, as usual, by FG: $FG(x) = F(G(x))$.

Definition 1.1 *A min-max function of type* $(n, 1)$ *is any function* $f : \mathbb{R}^n \to \mathbb{R}^1$, *which can be written as a term in the following grammar:*

$$f := x_1, x_2, \cdots, x_n \mid f + a \mid f \wedge f \mid f \vee f \qquad (a \in \mathbb{R}) \,. \tag{2}$$

The notation used here is the Backus-Naur form familiar in computer science. The vertical bars separate the different ways in which terms can be recursively constructed. The simplest term is one of the n variables, x_i, thought of as the i-th component function. Given any term, a new one may be constructed by adding $a \in \mathbb{R}$; given two terms, a new one may be constructed by taking a greatest lower bound or a least upper bound. Only these rules may be used to build terms. Of the three terms

$$\begin{gathered} ((((x_1 + 2) \vee (x_2 - 0.2)) \wedge x_3) \vee (x_2 + 3.5)) - 1 \\ x_1 \vee 2 \\ (x_1 + x_2) \wedge (x_3 + 1) \end{gathered}$$

the first is a min-max function but neither the second nor the third can be generated by (2).

We shall assume that $+$ has higher precedence than $\vee$ or $\wedge$, allowing us to write the first example more simply:

$$(((x_1 + 2 \vee x_2 - 0.2) \wedge x_3) \vee x_2 + 3.5) - 1\,.$$

Although the grammar provides a convenient syntax for writing terms, we are interested in them only as functions, $\mathbb{R}^n \to \mathbb{R}$. Terms can therefore be rearranged using the associativity and distributivity of the lattice operations, as well as the fact that addition distributes over both $\wedge$ and $\vee$. The example above can hence be simplified further to

$$(x_1 + 1 \vee x_2 + 2.5) \wedge (x_3 - 1 \vee x_2 + 2.5)\,.$$

It is clear that any term can be reduced in a similar way to a minima of maxima, or, dually, a maxima of minima. We shall discuss the corresponding canonical forms in §2.

Definition 1.2 ([21, Definition 2.3]) *A min-max function of type (n, m) is any function $F : \mathbb{R}^n \to \mathbb{R}^m$, such that each component F_i is a min-max function of type $(n, 1)$.*

The set of min-max functions of type (n, m) will be denoted $\mathsf{MM}(n, m)$. We shall mostly be concerned with functions of type (n, n), which we refer to as functions of dimension n. It is convenient to single out some special cases. Let $f \in \mathsf{MM}(n, 1)$. If f can be represented by a term which uses $\vee$ but not $\wedge$, it is said to be max-only. If f requires $\wedge$ but not $\vee$, it is min-only. If f is both max-only and min-only, it is simple. The same terminology extends to functions $F \in \mathsf{MM}(n, m)$ by asking that each component F_i has the property in question. If $F \in \mathsf{MM}(n, m)$ and each F_i is either max-only or min-only, F is said to be separated. Of the following functions in $\mathsf{MM}(2, 2)$,

$$S = \begin{pmatrix} x_1 + 1 \\ x_2 - 1 \end{pmatrix} \qquad T = \begin{pmatrix} x_2 + 1 \\ x_2 - 1 \end{pmatrix} \qquad U = \begin{pmatrix} x_1 + 1 \vee x_2 + 1 \\ x_1 + 2 \end{pmatrix},$$

S and T are both simple, U is max-only, $S \wedge T$ is min-only and $(S \vee T) \wedge U$ is separated.

Proposition 1.1 *Let $F, G \in \mathsf{MM}(n, n)$ and $\vec{a}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $h \in \mathbb{R}$. The following hold.*

1. *$F + \vec{a}$, FG, $F \vee G$ and $F \wedge G$ all lie in $\mathsf{MM}(n, n)$.*
2. Homogeneity: *$F(\vec{x} + h) = F(\vec{x}) + h$.* H
3. Monotonicity: *if $\vec{x} \leq \vec{y}$ then $F(\vec{x}) \leq F(\vec{y})$.* M
4. Nonexpansiveness: *$\|F(\vec{x}) - F(\vec{y})\| \leq \|\vec{x} - \vec{y}\|$.* N

The first three parts follow easily from Definition 1.2, while the fourth is a consequence of the following observation of Crandall and Tartar, [10].

Proposition 1.2 *If $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies* H *then* M *is equivalent to* N.

(See [23, Proposition 1.1] for a proof adapted to the present context.) $\mathsf{MM}(n,n)$ forms a distributive lattice under $\vee$ and $\wedge$. The interplay between function composition, FG, and the lattice operations, $F \vee G$ and $F \wedge G$, forms the backdrop for much of what we study.

The nonexpansiveness property is very significant in constraining the dynamics of min-max functions. Let $\omega(\vec{x})$ denote the omega limit set of the orbit of $\vec{x}$: the set of limits of subsequences of $F^k(\vec{x})$ as $k \to \infty$.

Theorem 1.1 ([35, 42]) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy property* N *and choose $\vec{x} \in \mathbb{R}^n$. If $\omega(\vec{x})$ is compact then it is finite and there is a bound on its size which depends only on n.*

In particular, nonexpansive functions cannot have periodic orbits of arbitrarily large size. Determining the optimal upper bound remains an important open problem. Nussbaum's Conjecture asserts that it is 2^n, which has been verified only up to $n = 3$, [30]. (It follows from the Aronszajn-Panitchpakdi Theorem, [1], that there are periods of size 2^n in dimension n.) Blokhuis and Wilbrink have given an attractive short proof that $(2n)^n$ is an upper bound, [4]. Nussbaum's survey, [33], should be consulted for more details. For min-max functions, there are periodic orbits of size ${}^nC_{[\frac{n}{2}]}$ in dimension n and this is conjectured to be best possible, [24].

The monotonicity property is also known, at least in the context of flows, to have a constraining influence on dynamics, [25, 43]. The relationship between these various "constraints on dynamics" has yet to be properly explored.

The homogeneity property suggests a modification of the conventional notion of fixed, or periodic, point.

Definition 1.3 *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies property* H. *We say that $\vec{x} \in \mathbb{R}^n$ is a fixed point of F, if $F(\vec{x}) = \vec{x} + h$ for some $h \in \mathbb{R}$, and that $\vec{x}$ is a periodic point of F with period p, if $\vec{x}$ is a fixed point of F^p, but not of F^k for any $0 < k < p$.*

A fixed point of F in this sense is a fixed point of $F - h$ in the conventional sense. Unless otherwise stated, the phrases "fixed point" and "periodic point" will have the meaning given by Definition 1.3 throughout this paper.

Min-max functions first arose in applications and these applications continue to provide important insights. In the next two sub-sections we review this material.

1.2 Discrete event systems

A discrete event system is, roughly speaking, a system comprising a finite set of events which occur repeatedly. For instance, a digital circuit, in which an event might be a voltage change on a wire, from binary 1 to 0 or vice versa; or a distributed computer system, in which an event might be the arrival of a message packet at a computer; or an automated manufacturing plant, in which an event might be the completion of a job on a machine. Discrete event systems are ubiquitous in modern life and the focus of much current interest in engineering circles, [2, 9, 15, 26]. They are dynamical systems, in the sense that they evolve in time, but their analysis leads to quite different mathematics to that used to model dynamic behaviour in continuous and differentiable systems.

If n is the number of events in the system, let $\vec{x} \in \mathbb{R}^n$ be such that x_i is the time of first occurrence of event i, relative to some arbitrary origin of time when the system is started. Suppose that the system can be modelled in such a way that, for some function $F : \mathbb{R}^n \to \mathbb{R}^n$, $F_i(\vec{x})$ gives the time of next occurrence of event i. In this case the dynamic behaviour of the system can be modelled by the discrete dynamic system F.

It might be thought that such a model is so simplified as to not occur in practice. It turns out, however, that the problem of clock schedule verification in digital circuits leads directly to a model of this kind. To each such circuit may be associated an element $F \in \mathsf{MM}(n,n)$, where n is one more than the number of storage latches. (Circuits may have as many as 10^4 latches.) The clock schedule verification problem may be solved by finding a fixed point of F. We shall not discuss this application further here; the reader should consult [18, 40, 45] for more details. It does suggest, however, the importance of understanding the fixed points of min-max functions. This is one of the main concerns of the present paper.

Notwithstanding this application, it is clear that the model outlined above is severely restricted. It can be broadened considerably in several ways. First, by using the semigroup generated by a set of functions, $\{F(\alpha) \mid \alpha \in A\}$, [13, 41]. This allows for the possibility of nondeterminism: if the system is in state $\vec{x}$, it may evolve to any of the states $F(\alpha)(\vec{x})$. For instance, demanding £20 from an automatic cash machine may sometimes result in two ten pound notes and sometimes in one ten and two fives. A second extension comes by taking $F(\alpha)$ to be a random variable from some suitable measure space into the space of allowed functions. This permits stochastic behaviour to be modelled. In a digital circuit it is conventional to consider only the maximum or minimum delays through a component (the manufacturer provides a data book which lists these values) but in a distributed computer system the time taken by a message packet will vary widely and a probabilistic approach is more appropriate, [2, Chapter 7], [39]. Finally, one can choose a sufficiently broad class of allowed functions. Proposition 1.1 suggests a class which appears very suitable.

Definition 1.4 ([23, Definition 1.1]) *A topical function is any function $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying properties* H *and* M.

Property H may be interpreted as saying that the origin of time is irrelevant, property M as saying that if times of occurrence are delayed, then the times of next occurrence cannot be sooner. These are intuitively very reasonable and are observed in some form in most discrete event systems. Recent work has suggested that "semigroups of random topical functions" are both mathematically tractable and capable of modelling a wide variety of discrete event systems, [3, 46].

What role do min-max functions play within the class of topical functions? In turns out to be an unexpectedly central one, as shown by the following observation of Gunawardena, Keane and Sparrow.

Lemma 1.1 ([24]) *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function and let $S \subseteq \mathbb{R}^n$ be any finite set of points. There exists $H \in \mathsf{MM}(n,n)$ such that $T \leq H$ and $T(\vec{u}) = H(\vec{u})$ for all $\vec{u} \in S$.*

It follows that min-max functions approximate topical functions, not only in the topological sense that $\mathsf{MM}(n,n)$ is dense in the set of topical functions (in the compact-open topology), but also in a lattice theoretic sense: any topical function is the lower envelope of a family of min-max functions. More importantly, this approximation preserves some aspects of the dynamics. Using the notation of Lemma 1.1, it follows from property M that $T^k \leq H^k$. In particular, the cycle time vector of T will be bounded by that of H (provided both exist). It also follows from Lemma 1.1 that every periodic orbit of a topical function is the orbit of some min-max function. Lemma 1.1 and its consequences provide one of the principal motivations for the present paper: to study min-max functions as a foundation for analysing topical functions.

We have presented topical functions as arising naturally from attempts to find a mathematical model for discrete event systems. However, they also have intrinsic mathematical interest because they include a number of classical examples which have been extensively studied in quite different contexts.

1.3 Topical functions and cycle times

Let $\mathbb{R}^+$ denote the positive reals: $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. The whole space, $\mathbb{R}^n$, can be put into bijective correspondence with the positive cone, $(\mathbb{R}^+)^n$, via the mutually inverse functions $\exp : \mathbb{R}^n \to (\mathbb{R}^+)^n$ and $\log : (\mathbb{R}^+)^n \to \mathbb{R}^n$, which do exp and log on each component: $\exp(\vec{x})_i = \exp(x_i)$, for $\vec{x} \in \mathbb{R}^n$, and $\log(\vec{x})_i = \log(x_i)$, for $\vec{x} \in (\mathbb{R}^+)^n$. Let A be any $n \times n$ matrix all of whose entries are nonnegative. Elements of $\mathbb{R}^n$ can be thought of as column vectors and A acts on them on the left as $A\vec{x}$. We further suppose the nondegeneracy condition that no row of A is zero:

$$\forall 1 \leq i \leq n, \ \exists 1 \leq j \leq n, \ \text{such that} \ A_{ij} \neq 0\,. \qquad (3)$$

In this case, A maps the positive cone onto itself, $A : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$. Let $\mathcal{E}(A) : \mathbb{R}^n \to \mathbb{R}^n$ denote the conjugate $\mathcal{E}(A)(\vec{x}) = \log(A(\exp(\vec{x})))$. Clearly, $\mathcal{E}(AB) = \mathcal{E}(A)\mathcal{E}(B)$, so that the dynamics of A and $\mathcal{E}(A)$ are entirely equivalent.

The point of this is that $\mathcal{E}(A)$ is always a topical function: property H is the additive equivalent of the fact that A commutes with scalar multiplication, while property M follows from the nonnegativity of A. We see that the dynamics of topical functions includes as a special case that of nonnegative matrices; in other words, Perron-Frobenius theory. It can be shown that a number of classical examples in optimal control, game theory and mathematical economics also give rise to topical functions. The geography of the space of topical functions is discussed in more detail in [24].

If $\vec{x} \in \mathbb{R}^n$ is a fixed point of $\mathcal{E}(A)$, so that $\mathcal{E}(A)(\vec{x}) = \vec{x} + h$, then $\exp(\vec{x})$ is an eigenvector of A with eigenvalue $\exp(h)$. Fixed points of $\mathcal{E}(A)$ therefore correspond bijectively to positive eigenvectors of A. That is, to eigenvectors lying in the positive cone. What about the eigenvalue? Can this also be generalised to the nonlinear context? The clue to doing this came from the applications.

A frequent demand from system designers is to estimate performance, [5, 18]. If the system can be modelled by a single function, $F : \mathbb{R}^n \to \mathbb{R}^n$, as described above, an estimate can be made on the basis of the time elapsed between successive occurrences: $F(\vec{x}) - \vec{x}$. Better still is an average over several occurrences:

$$(F^k(\vec{x}) - F^{k-1}(\vec{x}) + \cdots + F(\vec{x}) - \vec{x})/k\,.$$

Letting $k \to \infty$, we get $\lim_{k\to\infty} F^k(\vec{x})/k$. This is a vector quantity, which measures the asymptotic average slowness in each component. Does this limit exist?

Lemma 1.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy property* N. *If $\lim_{k\to\infty} F^k(\vec{x})/k$ exists somewhere, then it exists everywhere and has the same value.*

Proof Suppose that $\lim_{k\to\infty} F^k(\vec{x})/k = \vec{a}$ and let $\vec{y}$ be another point of $\mathbb{R}^n$. Choose $\epsilon > 0$. By property N, for all sufficiently large k,

$$\|\vec{a} - F^k(\vec{y})/k\| \leq \|\vec{a} - F^k(\vec{x})/k\| + \|F^k(\vec{x})/k - F^k(\vec{y})/k\| \leq \epsilon + \|\vec{x} - \vec{y}\|/k\,.$$

From which the result follows immediately.

□

Definition 1.5 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy property* N. *The cycle time vector of F, denoted $\chi(F) \in \mathbb{R}^n$, is defined to be*

$$\lim_{k\to\infty} F^k(\vec{x})/k \tag{4}$$

when this limit exists for some $\vec{x} \in \mathbb{R}^n$, and to be undefined otherwise.

Suppose that $F = \mathcal{E}(A)$ and that A has a positive eigenvector with eigenvalue λ. (Perron-Frobenius tells us that λ is real and equals the spectral radius of A.) It then follows, as above, that $\chi(\mathcal{E}(A)) = \log(\lambda)$. We see from this that χ is a nonlinear vector generalisation of the spectral radius. It can be shown that if A is any nonnegative matrix satisfying (3) then χ exists and can be determined in terms of the spectral radii of the irreducible components of A, [24]. Indeed, by using $\mathcal{E}^{-1}$ on (4), we see that the cycle time vector corresponds to the usual spectral radius formula, albeit disintegrated into individual components.

The cycle time vector immediately yields a necessary condition for a fixed point. Suppose that F is a topical function with a fixed point, so that $F(\vec{x}) = \vec{x}+h$. By repeated application of H, we see that $F^k(\vec{x}) = \vec{x} + k.h$. Hence, χ does exist and $\chi(F) = h$. We recall by the vector-scalar convention that this means each component of $\chi(F)$ has the same value h. By Lemma 1.2, h is independent of the choice of fixed point and is characteristic of the function; we shall calculate its value in Proposition 2.2. There is extensive evidence that, when F is a min-max function, the converse result, (1), also holds. As discussed earlier, this would follow from the Duality Conjecture which is stated in §2.

It can be shown that the cycle time vector exists for all topical functions in dimension 2, [23], in particular, for min-max functions. Sparrow has shown further that any min-max function in dimension 3 has a cycle time vector, [44]. However, these results do not give the Duality Conjecture and do not yield methods for calculating χ.

It is not the case that all topical functions have cycle times, [23, Theorem 3.1], and it is a major open problem to identify those that do, [24]. Furthermore, one cannot expect such a strong fixed point theorem as (1) even for those topical functions that do have cycle times. If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then $\chi(\mathcal{E}(A)) = (0,0)$ but A does not have a positive eigenvector. This example raises a number of issues which take us beyond the scope of the present paper and we defer further discussion to [24].

1.4 Max-only functions and max-plus matrices

It should be clear from the remarks before Definition 1.2 that if $f \in \mathsf{MM}(n, 1)$ is max-only, it can be reduced to the form

$$f = x_1 + a_1 \vee \cdots \vee x_n + a_n$$

where the absence of a term $x_j + a_j$ is indicated by setting $a_j = -\infty$. This can be thought of as an element adjoined to $\mathbb{R}$ which is less than any real number and acts as an absorbing element for addition: $a + (-\infty) = -\infty$. Hence, if $F \in \mathsf{MM}(n, n)$ is max-only, it can be represented by a matrix A with entries in $\mathbb{R} \cup \{-\infty\}$:

$$\begin{array}{rcl} F_1 & = & x_1 + A_{11} \vee \cdots \vee x_n + A_{1n} \\ \vdots & \vdots & \vdots \\ F_n & = & x_1 + A_{n1} \vee \cdots \vee x_n + A_{nn}\,. \end{array} \tag{5}$$

Since each component of F must have some value, A satisfies a nondegeneracy property formally similar to (3):

$$\forall 1 \leq i \leq n,\ \ \exists 1 \leq j \leq n,\ \text{ such that }\ A_{ij} \neq -\infty\,. \tag{6}$$

Suppose that the algebraic operations on $\mathbb{R} \cup \{-\infty\}$ are now redefined so that the sum operation becomes maximum and the multiplication operation becomes addition. The element $-\infty$ then becomes a zero for sum, while 0 becomes a unit for multiplication. Since addition distributes over maximum, this forms a semiring, called the max-plus semiring, and denoted $\mathbb{R}_{\max}$. If vectors in $\mathbb{R}^n$ are thought of as column vectors, (5) can be rewritten as a matrix equation

$$F(\vec{x}) = A\vec{x} \tag{7}$$

in which the matrix operations are interpreted in $\mathbb{R}_{\max}$. It follows that $F^k(\vec{x}) = A^k\vec{x}$ and the dynamics of F reduce to matrix algebra, albeit of an unusual sort. We have, in effect, linearised an apparently nonlinear problem.

Cuninghame-Green was perhaps the first to realise the implications of matrix algebra over max-plus, [11]. Since that time the idea has been rediscovered and redeveloped several times and there are now several standard texts on the subject, [2, 6, 12, 31, 48]. For a recent overview, see [14].

In this paper we shall not adopt max-plus notation. That is, $+$ and $\times$ will always have their customary meanings. We shall use $\vee$ and $+$ for the corresponding max-plus operations. Similarly, 0 will always have its customary meaning and we shall use $-\infty$ for the zero in $\mathbb{R}_{\max}$. If A and B are, respectively, $n \times p$ and $p \times m$ matrices over $\mathbb{R}_{\max}$, then AB will always mean the matrix product over $\mathbb{R}_{\max}$:

$$(AB)_{ij} = \bigvee_{1 \leq k \leq p} A_{ik} + A_{kj}\ .$$

(Recall that $+$ has higher precedence than $\vee$.) The customary ordering on $\mathbb{R}$ extends to $\mathbb{R}_{\max}$ in the obvious way, so that $-\infty \leq x$ for all $x \in \mathbb{R}_{\max}$. The same symbol is used for the product ordering on vectors: if $\vec{x}, \vec{y} \in (\mathbb{R}_{\max})^n$ then $\vec{x} \leq \vec{y}$ if, and only if, $x_i \leq y_i$ for all i. An $n \times n$ matrix over $\mathbb{R}_{\max}$, A, acts on the whole space $(\mathbb{R}_{\max})^n$ and it is easy to see that it is monotonic with respect to the product ordering: if $\vec{x} \leq \vec{y}$ then $A\vec{x} \leq A\vec{y}$. We recall that $\vec{x} \in (\mathbb{R}_{\max})^n$ is an eigenvector of A for the eigenvalue $h \in \mathbb{R}_{\max}$, if $A\vec{x} = \vec{x} + h$. If A satisfies the nondegeneracy condition (6), so that A can also be considered as a min-max function, then fixed points of A correspond bijectively to eigenvectors of A lying in $\mathbb{R}^n$. (This restriction is formally similar to that needed for nonnegative matrices and their eigenvectors in §1.3.) In this paper, the word "eigenvector" will indicate an element of $(\mathbb{R}_{\max})^n$ while the phrase "fixed point" will imply that the element in question lies in $\mathbb{R}^n$.

We need to recall various standard results in max-plus theory. The reader seeking more background should consult [2, Chapter 3].

Let A be an $n \times n$ matrix over $\mathbb{R}_{\max}$. The precedence graph of A, denoted $\mathcal{G}(A)$, is the directed graph with labelled edges which has nodes $\{1, \cdots, n\}$ and an edge from j to i if, and only if, $A_{ij} \neq -\infty$. The label on this edge is then the real number A_{ij}. (Some authors use the opposite convention for the direction of edges.) We shall denote an edge from j to i by $i \leftarrow j$. A path in this graph has the usual meaning of a chain of directed edges: a path from i_m to i_1 is a sequence of nodes $i_1, \cdots, i_m$ such that $1 < m$ and $i_j \leftarrow i_{j+1}$ for $1 \leq j < m$. A circuit is a path which starts and ends at the same node: $i_1 = i_m$. A circuit is elementary if the nodes $i_1, \cdots, i_{m-1}$ are all distinct. A node j is upstream from i, denoted $i \Leftarrow j$, if either $i = j$ or there is a path in $\mathcal{G}(A)$ from j to i. (A node is always upstream from itself.) A circuit g is upstream from node i, denoted $i \Leftarrow g$, if some node on the circuit is upstream from i. The weight of a path p, $|p|_{\mathsf{w}}$, is the sum of the labels on the edges in the path:

$$|p|_{\mathsf{w}} = \sum_{j=1}^{m-1} A_{i_j i_{j+1}}.$$

It follows from this that matrix multiplication has a nice interpretation in terms of path weights: A^s_{ij} is the maximum weight among all paths of length s from j to i. The length of a path, $|p|_{\ell}$, is the number of edges in the path: $|p|_{\ell} = m - 1$. If g is a circuit, its cycle mean, denoted $\mathsf{m}(g)$ is defined by $\mathsf{m}(g) = |g|_{\mathsf{w}}/|g|_{\ell}$. If A is an $n \times n$ matrix over $\mathbb{R}_{\max}$, let $\mu(A) \in (\mathbb{R}_{\max})^n$ be defined by

$$\mu_i(A) = \max\{\mathsf{m}(g) \mid i \Leftarrow g\}. \tag{8}$$

This is well defined: although there may be infinitely many circuits in $\mathcal{G}(A)$, only the elementary ones are needed to determine $\mu(A)$. By convention, the maximum of an empty set is taken to be $-\infty$. Hence, if there are no circuits upstream from node i, $\mu_i(A) = -\infty$. If A satisfies the nondegeneracy condition (6) then every node has an upstream circuit and so $\mu(A) \in \mathbb{R}^n$.

It is convenient at this point to single out the functions $\mathsf{t}, \mathsf{b} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\begin{aligned} \mathsf{t}(\vec{x}) &= x_1 \vee \cdots \vee x_n \\ \mathsf{b}(\vec{x}) &= x_1 \wedge \cdots \wedge x_n . \end{aligned}$$

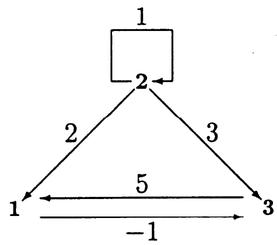
If c is any vector valued quantity, we shall often simplify this notation by writing $\mathsf{t}c$ and $\mathsf{b}c$ in place of $\mathsf{t}(c)$ and $\mathsf{b}(c)$, respectively. It follows from (8) that $\mathsf{t}\mu(A)$ is the maximum cycle mean over all circuits. A critical circuit is an elementary circuit with cycle mean $\mathsf{t}\mu(A)$.

Before proceeding further, it may be helpful to see an example. The max-only function

$$\begin{aligned} F_1(x_1,x_2,x_3) &= x_2+2 \vee x_3+5 \\ F_2(x_1,x_2,x_3) &= x_2+1 \\ F_3(x_1,x_2,x_3) &= x_1-1 \vee x_2+3 \end{aligned} \tag{9}$$

has associated max-plus matrix, precedence graph and μ vector shown below

$$\begin{pmatrix} -\infty & 2 & 5 \\ -\infty & 1 & -\infty \\ -1 & 3 & -\infty \end{pmatrix}$$



$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}. \tag{10}$$

The maximum cycle mean is 2 and $1 \leftarrow 3 \leftarrow 1$ is the unique critical circuit.

Proposition 1.3 *If $F \in \mathsf{MM}(n,n)$ is max-only and A is the associated matrix over $\mathbb{R}_{\max}$, then χ exists and $\chi(F) = \mu(A)$.*

Proof Let $\mu_1(A) = h$. Suppose initially that $h = 0$ and consider the sequence of numbers $\alpha(s) = (A^s\vec{0})_1$. It follows from one of the remarks above that we can interpret $\alpha(s)$ as the maximum weight among paths in $\mathcal{G}(A)$ of length s which terminate at node 1. If we consider any path terminating at node 1 then the only positive contribution to the weight of the path can come from those edges which are not repeated on the path: a repeated edge would be contained in a circuit, whose contribution to the path weight is at most 0. Since there are only finitely many edges, the weight of any path must be bounded above by $\sum_{A_{ij}>0} A_{ij}$. Hence $\alpha(s)$ is bounded above. Since $h = 0$, we know that there is some circuit upstream from node 1 whose weight is 0. Call this circuit g. For s sufficiently large, we can construct a path, $p(s)$, terminating at node 1 whose starting point cycles round the circuit g. The weight of this path can only assume a finite set of values because $|g|_{\mathsf{w}} = 0$. Since $\alpha(s)$ is the path of maximum weight of length s, it follows that $\alpha(s) \geq |p(s)|_{\mathsf{w}}$ and so $\alpha(s)$ is also bounded below. We have shown that there exist $m, M \in \mathbb{R}$ such that, for all $s \geq 0$, $m \leq \alpha(s) \leq M$. It follows immediately that $\lim_{s\to\infty} \alpha(s)/s = 0$. Hence,

$$\lim_{s\to\infty} (F^s(\vec{0}))_1/s = \mu_1(A).$$

If $h \neq 0$ then replace F by $G = F - h$. G is also a max-only function and if B is its associated matrix, then $B_{ij} = A_{ij} - h$. Hence $\mu_1(B) = 0$ and we can apply the argument above to show that $\lim_{s\to\infty}(G^s(\vec{0}))_1/s = 0$. But since $F = G+h$, it follows from property H that $\lim_{s\to\infty}(F^s(\vec{0}))_1/s = h = \mu_1(A)$. The same argument can be applied to any component of F and the result follows.

□

If F, has a fixed point, so that $F(\vec{x}) = \vec{x} + h$, then $h = \mu(A)$. In particular, $h = \mathsf{t}\mu(A)$, the maximum cycle mean over all circuits in $\mathcal{G}(A)$. This is the eigenvalue associated to any eigenvector of A lying in $\mathbb{R}^n$. It is the analogue for max-plus matrices of the Perron root, or spectral radius, for nonnegative matrices, [2, Theorem 3.23].

Suppose that A is an $n \times n$ matrix over $\mathbb{R}_{\max}$. Suppose further that $\mathrm{t}\mu(A) = 0$, so that all circuits of $\mathcal{G}(A)$ have nonpositive weight. Since any path p in $\mathcal{G}(A)$, with $|p|_\ell \geq n$, must contain a circuit, it is not difficult to see that

$$(A^s)_{ij} \leq A_{ij} \vee \cdots \vee (A^n)_{ij} \tag{11}$$

for all $s \geq n$. Let $(A^+)_{ij} = \sup\{(A^s)_{ij} \mid 1 \leq s\}$, which is well defined as an element of $\mathbb{R}_{\max}$ by the previous observation. (It is well-known in max-plus theory that, $A^+ = A \vee \cdots \vee A^n$, [2, Theorem 3.20], but we shall not need this here.) Note that it is still possible for $(A^+)_{ij} = -\infty$, since it may be the case that there are no paths from j to i. Let $\mathcal{C}(A) \subseteq \{1, \cdots, n\}$ be the set of those nodes of $\mathcal{G}(A)$ which lie on some critical circuit. Let P_A be the $\mathbb{R}_{\max}$ matrix defined as follows:

$$(\mathsf{P}_A)_{ij} = \bigvee_{u \in \mathcal{C}(A)} (A^+)_{iu} + (A^+)_{uj} \ . \tag{12}$$

P_A is sometimes called a spectral projector, [2, §3.7.3]. Part 3 of Lemma 1.4 will show that it encapsulates information about the eigenvectors of A.

The next lemma is a standard result in max-plus matrix theory.

Lemma 1.3 ([2, Theorem 3.105]) *Suppose that A is an $n \times n$ matrix over $\mathbb{R}_{\max}$ such that $\mathrm{t}\mu(A) = 0$. Then $(\mathsf{P}_A)A = A(\mathsf{P}_A) = \mathsf{P}_A$ and $(\mathsf{P}_A)^2 = \mathsf{P}_A$.*

The next lemma collects together a number of useful observations. Some of them are well-known in max-plus theory, [2, Chapter 3], but none of them appear in a convenient form in the literature.

Lemma 1.4 *Suppose that A is an $n \times n$ matrix over $\mathbb{R}_{\max}$ such that $\mathrm{t}\mu(A) = 0$. Suppose further that $\vec{x}, \vec{y} \in (\mathbb{R}_{\max})^n$. The following statements hold.*

1. *If $A\vec{x} \leq \vec{x}$ then $\mathsf{P}_A\vec{x} \leq \vec{x}$.*
2. *$A\vec{x} = \vec{x}$ if, and only if, $\mathsf{P}_A\vec{x} = \vec{x}$.*
3. *The image of $\mathsf{P}_A : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$ is the eigenspace of A for the eigenvalue 0.*
4. *If $A\vec{x} = \vec{x}$, $A\vec{y} = \vec{y}$ and $x_i = y_i$ for all $i \in \mathcal{C}(A)$, then $\vec{x} = \vec{y}$.*
5. *If $i \in \mathcal{C}(A)$ then $(\mathsf{P}_A)_{ii} = 0$.*
6. *If $A\vec{x} \leq \vec{x}$ then $(A\vec{x})_i = x_i$ for all $i \in \mathcal{C}(A)$.*
7. *If $\mu(A) = 0$ then $\mathsf{P}_A : \mathbb{R}^n \to \mathbb{R}^n$.*

Proof 1. Since $A\vec{x} \leq \vec{x}$, it follows that $A^s\vec{x} \leq \vec{x}$ and so $A^+\vec{x} \leq \vec{x}$. Hence $(A^+)_{uj} + x_j \leq (A^+\vec{x})_u \leq x_u$. Choose $1 \leq i \leq n$. Then, by (12), $(\mathsf{P}_A\vec{x})_i \leq \bigvee_{u\in\mathcal{C}(A)}(A^+)_{iu} + x_u \leq (A^+\vec{x})_i \leq x_i$. Hence $\mathsf{P}_A\vec{x} \leq \vec{x}$ as required.

2. If $\mathsf{P}_A\vec{x} = \vec{x}$ then it follows immediately from Lemma 1.3 that $A\vec{x} = \vec{x}$. So suppose that $A\vec{x} = \vec{x}$. By part 1, $\mathsf{P}_A\vec{x} \leq \vec{x}$. Choose $1 \leq i \leq n$. If $x_i = -\infty$, then certainly $(\mathsf{P}_A\vec{x})_i = x_i$, so we may assume that $x_i > -\infty$. For each s, $\vec{x} = A^s\vec{x}$. Hence there exists $1 \leq j \leq n$ such that $x_i = (A^s)_{ij} + x_j$. $(A^s)_{ij}$ is the weight of some path of length s from node j to node i. If we choose $s = n$, then this path must contain a circuit g. It is not difficult to see that, because $x_i > -\infty$, we must have $\mathsf{m}(g) = 0$. It follows that there exists $v \in \mathcal{C}(A)$ such that, for some s, $x_i = (A^s)_{iv} + x_v$. But then,

$$x_i \leq (A^+)_{iv} + x_v \leq \bigvee_{u\in\mathcal{C}(A)} (A^+)_{iu} + x_u \leq (A^+\vec{x})_i = x_i \ ,$$

the last equality holding because $\vec{x}$ is evidently an eigenvector of A^+. It follows that $x_i = \bigvee_{u\in\mathcal{C}(A)}(A^+)_{iu} + x_u$. Hence, by (12), $(\mathsf{P}_A\vec{x})_i = \bigvee_{u\in\mathcal{C}(A)}(A^+)_{iu} + (A^+\vec{x})_u = x_i$. Hence $\mathsf{P}_A\vec{x} = \vec{x}$.

3. According to Lemma 1.3, $(\mathsf{P}_A)^2 = \mathsf{P}_A$. Hence the image of P_A coincides with the set of eigenvectors of P_A. By part 2, the eigenvectors of P_A are exactly the eigenvectors of A.

4. Choose $1 \leq i \leq n$. Since $\vec{x}$ and $\vec{y}$ are eigenvectors of A, they are also eigenvectors of A^+. Hence, since $x_u = y_u$ for all $u \in \mathcal{C}(A)$,

$$(\mathsf{P}_A\vec{x})_i = \bigvee_{u\in\mathcal{C}(A)} (A^+)_{iu} + x_u = \bigvee_{u\in\mathcal{C}(A)} (A^+)_{iu} + y_u = (\mathsf{P}_A\vec{y})_i \ .$$

By part 2, $\vec{x} = \vec{y}$.

5. If $i \in \mathcal{C}(A)$ then there exists some k such that $(A^k)_{ii} = 0$. Hence $(A^+)_{ii} = 0$. It then follows from (12) that $(\mathsf{P}_A)_{ii} = 0$.

6. Since $A\vec{x} \leq \vec{x}$ it follows that $A^k\vec{x} \leq A^{k-1}\vec{x}$. Hence, by (11), $A^+\vec{x} \leq A\vec{x} \leq \vec{x}$. If $i \in \mathcal{C}(A)$ then as in the previous part, $(A^+)_{ii} = 0$. Hence, $x_i = (A^+)_{ii} + x_i \leq (A^+\vec{x})_i \leq x_i$. It follows that $(A\vec{x})_i = x_i$ as required.

7. If $\mu(A) = 0$ then there must be a critical circuit upstream from every node of $\mathcal{G}(A)$. Hence, for any $1 \leq i \leq n$, there exists some $k \in \mathcal{C}(A)$, such that $A_{ik} > -\infty$. It follows from Lemma 1.3 and part 5 that $(\mathsf{P}_A)_{ik} \geq A_{ik} + (\mathsf{P}_A)_{kk} > -\infty$. Hence $\mathsf{P}_A : \mathbb{R}^n \to \mathbb{R}^n$, as required.

□

Proposition 1.4 *Let $F \in \mathsf{MM}(n,n)$ be max-only. The fixed point result* (1) *holds for F.*

Proof If F has a fixed point then clearly $\chi(F) = h$. So suppose $\chi(F) = h$. Assume first that $h = 0$. Let A be the max-plus matrix associated to F. By Proposition 1.3 we see that $\mu(A) = 0$ and, in particular, $\mathsf{t}\mu(A) = 0$. Let $\vec{c} = \mathsf{P}_A(1, \cdots, 1)$. By Lemma 1.3, $\vec{c}$ is an eigenvector of A with eigenvalue 0. Furthermore, since $\mu(A) = 0$, part 7 of Lemma 1.4 shows that $\vec{c} \in \mathbb{R}^n$. Hence F has a fixed point. If $h \neq 0$ then the same reasoning shows that $(F - h)$ has a fixed point: $(F - h)(\vec{c}) = \vec{c}$. Hence, $F(\vec{c}) = \vec{c} + h$, as required.

□

Dual results to those above hold for min-only functions. To each such function is associated a matrix over the min-plus semiring: $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ with minimum as sum and addition as multiplication. $\mathbb{R}_{\min}$ is isomorphic, as a semiring, to $\mathbb{R}_{\max}$ and any result holding over one of them has a dual over the other in which the roles of max and min are interchanged. We leave it to the reader to formulate these and any associated definitions; we shall not state them separately. It will be helpful, however, to use a different notation for the dual of the μ-vector. If B is an $n \times n$ matrix over $\mathbb{R}_{\min}$, which satisfies the (dual) nondegeneracy condition (6), $\eta(B) \in \mathbb{R}^n$ will denote the vector of minimun upstream cycle means in $\mathcal{G}(B)$:

$$\eta_i(B) = \min\{\mathsf{m}(g) \mid i \Leftarrow g\}\,.$$

If F is the corresponding min-only function, then by Proposition 1.3, $\chi(F) = \eta(B)$.

With this extended preparation we are finally in a position to study the main concerns of the present paper: the existence and calculation of the cycle time for min-max functions and its relationship to fixed points.

2 The Duality Conjecture

Definition 2.1 *Let $F \in \mathsf{MM}(n,m)$. A subset $S \subseteq \mathsf{MM}(n,m)$ is said to be a max-representation of F if S is a finite set of max-only functions such that $F = \bigwedge_{H \in S} H$.*

It should be clear from the remarks before Definition 1.2 that every min-max function has a max-representation and a (dual) min-representation. Since we know the cycle time vectors of max-only functions, we can approximate that of F. Suppose that $\chi(F)$ exists. For any $H \in S$, $F \leq H$. By property M, $\chi(F) \leq \chi(H)$. Hence,

$$\chi(F) \leq \bigwedge_{H \in S} \chi(H)\,. \tag{13}$$

The difficulty with this is that there may be different max-representations of F. The min-max function

$$\begin{array}{rcl} F_1(x_1,x_2,x_3) & = & (x_2+2 \vee x_3+5) \wedge x_1 \\ F_2(x_1,x_2,x_3) & = & x_2+1 \wedge x_3+2 \\ F_3(x_1,x_2,x_3) & = & x_1-1 \vee x_2+3 \end{array} \tag{14}$$

has both the max-representation

$$\left\{ \begin{pmatrix} x_2+2 \vee x_3+5 \\ x_2+1 \\ x_1-1 \vee x_2+3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_3+2 \\ x_1-1 \vee x_2+3 \end{pmatrix} \right\}$$

and the max-representation

$$\left\{ \begin{pmatrix} x_2+2 \vee x_3+5 \\ x_3+2 \\ x_1-1 \vee x_2+3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2+1 \\ x_1-1 \vee x_2+3 \end{pmatrix} \right\}.$$

The cycle time vectors of the constituent max-only functions can be calculated by the methods of the previous section. We leave it to the reader to show that they are, in the order in which they appear above,

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.5 \\ 2.5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

It follows that the estimate (13) gives, for the first max-representation, $(0,1,2)$, while for the second, $(0,1,1)$.

To get the best estimate, the information in all the max-representations of F must be used. Observe that the set of min-max functions $\mathsf{MM}(n,m)$ has a natural representation as an m-fold Cartesian product: $\mathsf{MM}(n,m) = \mathsf{MM}(n,1) \times \cdots \times \mathsf{MM}(n,1)$. If $S \subseteq A_1 \times \cdots \times A_m$ is a subset of such a Cartesian product, let $\pi_i(S) \subseteq A_i$ denote its projection on the i-th factor.

Definition 2.2 *The rectangularisation of S, denoted $\mathsf{Rec}(S)$, is defined by*

$$\mathsf{Rec}(S) = \pi_1(S) \times \cdots \times \pi_m(S).$$

S is said to be rectangular if $S = \mathsf{Rec}(S)$.

It is, of course, always the case that $S \subseteq \mathsf{Rec}(S)$. It is also clear that $\pi_i(S) = \pi_i(\mathsf{Rec}(S))$. It follows that, if $S \subseteq \mathsf{MM}(n,m)$ is finite, then

$$\bigwedge_{H \in S} H = \bigwedge_{H \in \mathsf{Rec}(S)} H \quad \text{and} \quad \bigvee_{H \in S} H = \bigvee_{H \in \mathsf{Rec}(S)} H, \tag{15}$$

since the partial order on $\mathsf{MM}(n,m)$ is defined componentwise. Furthermore, if S contains only max-only functions, then so does $\mathsf{Rec}(S)$. It is worth observing that neither of the max-representations used above were rectangular.

Suppose that $S \subseteq P$, where $(P, \leq)$ is a partially ordered set. Denote by $\mathsf{Min}(S)$ the subset of least elements of S,

$$\mathsf{Min}(S) = \{x \in S \mid y \in S,\ y \leq x \implies y = x\},$$

and by $\mathsf{Max}(S)$ the corresponding set of greatest elements. If $x \in S$, then there exists $u \in \mathsf{Min}(S)$ and $v \in \mathsf{Max}(S)$ such that $u \leq x \leq v$.

Now suppose that P is a product partial order: $P = A_1 \times \cdots \times A_m$, with the partial order on P defined componentwise from those on the A_i.

Lemma 2.1 *Let $S_i \subseteq A_i$ be finite subsets for $1 \leq i \leq m$. Then*

$$\mathsf{Min}(S_1 \times \cdots \times S_m) = \mathsf{Min}(S_1) \times \cdots \times \mathsf{Min}(S_m).$$

Proof It is clear that both $L = \mathsf{Min}(S_1 \times \cdots \times S_m)$ and $R = \mathsf{Min}(S_1) \times \cdots \times \mathsf{Min}(S_m)$ are irredundant: no two elements are related by the partial order. If $x \in S_1 \times \cdots \times S_m$ then, by definition of the least element subset, we can find $u \in L$ such that $u \leq x$. By a similar argument on each component, we can find $v \in R$ such that $v \leq x$. It follows easily that $L = R$.

□

Theorem 2.1 *Let $F \in \mathsf{MM}(n,m)$ and suppose that $S, T \subseteq \mathsf{MM}(n,m)$ are rectangular max-representations of F. Then $\mathsf{Min}(S) = \mathsf{Min}(T)$.*

Proof For $m = 1$ this is a restatement of one of the main results of an earlier paper, which asserts the existence of a canonical form for min-max functions, [21, Theorem 2.1]. Now suppose that $m > 1$. Since $\pi_i(S)$ and $\pi_i(T)$ are evidently max-representations of F_i, it follows from the first case that $\mathsf{Min}(\pi_i(S)) = \mathsf{Min}(\pi_i(T))$. But then, since S and T are rectangular, it follows from Lemma 2.1 that

$$\mathsf{Min}(S) = \mathsf{Min}(\pi_1(S)) \times \cdots \times \mathsf{Min}(\pi_m(S)) = \mathsf{Min}(\pi_1(T)) \times \cdots \times \mathsf{Min}(\pi_m(T)) = \mathsf{Min}(T)\,,$$

as required.

□

Corollary 2.1 *Let $F \in \mathsf{MM}(n,n)$ and suppose that $S, T \subseteq \mathsf{MM}(n,n)$ are rectangular max-representations of F. Then*

$$\bigwedge_{H \in S} \chi(H) = \bigwedge_{G \in T} \chi(G)\,.$$

Proof Since χ is monotonic, it must be the case that $\bigwedge_{H \in \mathsf{Min}(S)} \chi(H) = \bigwedge_{H \in S} \chi(H)$. The result follows immediately from Theorem 2.1.

□

The max-representations used for example (14) have identical rectangularisations, obtained by taking the union of the two representations. The best estimate for the cycle time, on the basis of Corollary 2.1, is therefore $\chi(F) \leq (0,1,1)$.

Such calculations suggest an approach to proving the existence of χ. Suppose that $F \in \mathsf{MM}(n,n)$ and that $S, T \subseteq \mathsf{MM}(n,n)$ are, respectively, a max-representation and a min-representation of F. Choose $\vec{x} \in \mathbb{R}^n$ and $\epsilon > 0$. If $G \in T$ and $H \in S$, then $G^s \leq F^s \leq H^s$. Hence, for all sufficiently large s, $\chi(G) - \epsilon \leq F^s(\vec{x})/s \leq \chi(H) + \epsilon$. It follows that, for all sufficiently large s,

$$\left(\bigvee_{G \in T} \chi(G)\right) - \epsilon \;\leq\; \frac{F^s(\vec{x})}{s} \;\leq\; \left(\bigwedge_{H \in S} \chi(H)\right) + \epsilon\,. \tag{16}$$

It is natural, in the light of this, to make the following guess, which is supported by extensive calculations.

Conjecture 2.1 (The Duality Conjecture) *Let $F \in \mathsf{MM}(n,n)$. Suppose that $S,T \subseteq \mathsf{MM}(n,n)$ are rectangular max and min-representations, respectively, of F. Then,*

$$\bigvee_{G\in T} \chi(G) = \bigwedge_{H\in S} \chi(H)\,. \tag{17}$$

The implication of this should be clear from (16). We record it below for future reference.

Lemma 2.2 *If the Duality Conjecture holds for F then $\chi(F)$ exists and has the value*

$$\bigvee_{G\in T} \chi(G) = \chi(F) = \bigwedge_{H\in S} \chi(H)\,. \tag{18}$$

Consider once again example (14), for which we have already shown that the right hand side of (17) is $(0,1,1)$. A min-representation can be constructed from (14) by using the distributivity of $\wedge$ over $\vee$ to interchange the two operations in F_1:

$$F_1(x_1,x_2,x_3) = (x_1 \wedge x_2 + 2) \vee (x_1 \wedge x_3 + 5)\,.$$

A rectangular min-representation of F is then given by

$$\left\{ \begin{pmatrix} x_1 \wedge x_2 + 2 \\ x_2 + 1 \wedge x_3 + 2 \\ x_1 - 1 \end{pmatrix}, \begin{pmatrix} x_1 \wedge x_3 + 5 \\ x_2 + 1 \wedge x_3 + 2 \\ x_2 + 3 \end{pmatrix}, \begin{pmatrix} x_1 \wedge x_2 + 2 \\ x_2 + 1 \wedge x_3 + 2 \\ x_2 + 3 \end{pmatrix}, \begin{pmatrix} x_1 \wedge x_3 + 5 \\ x_2 + 1 \wedge x_3 + 2 \\ x_1 - 1 \end{pmatrix} \right\}$$

and we leave it to the reader to show that the corresponding cycle time vectors are

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that the left hand side of (17) is also $(0,1,1)$, confirming the Duality Conjecture. The calculation gives very little hint as to why the same numbers come out at the end but we have at least shown that χ exists for example (14) and equals $(0,1,1)$. (In fact, we already knew from Sparrow's result, [44], that $\chi(F)$ exists but we did not know how to calculate it. The existence of the cycle time vector does not imply the Duality Conjecture.)

It is easy to see that at least half of (17) must always be true. Under the same assumptions as Conjecture 2.1, choose $G \in T$ and $H \in S$. Since $G \leq H$, $\chi(G) \leq \chi(H)$. Hence,

$$\bigvee_{G\in T} \chi(G) \leq \bigwedge_{H\in S} \chi(H) \tag{19}$$

In two dimensions, it can be shown by a direct calculation that the Duality Conjecture is true. We shall give the proof in §4. Before doing so, we shall first reformulate the conjecture in a more attractive way and then discuss two special cases where the conjecture can be easily proved.

Theorem 2.2 *The following statements are equivalent.*

- *The Duality Conjecture holds for all functions in* $\mathsf{MM}(n,n)$.

- *The cycle time vector defines a functional,* $\chi : \mathsf{MM}(n,n) \to \mathbb{R}^n$*, which is a homomorphism of lattices on rectangular subsets of the domain. That is, if* $S \subseteq \mathsf{MM}(n,n)$ *is nonempty, finite and rectangular, then*

$$\begin{array}{rcl} \chi\left(\bigwedge_{F\in S} F\right) & = & \bigwedge_{F\in S} \chi(F) \\ \chi\left(\bigvee_{F\in S} F\right) & = & \bigvee_{F\in S} \chi(F) \ . \end{array} \tag{20}$$

Proof Suppose first that χ is a functional with the stated properties. Choose $F \in \mathsf{MM}(n,n)$. Let S be a rectangular max-representation of F. Then, by the homomorphic property of χ,

$$\chi(F) = \chi\left(\bigwedge_{H\in S} H\right) = \bigwedge_{H\in S} \chi(H) \ .$$

A similar argument for a min-representation establishes the Duality Conjecture.

Now suppose that the Duality Conjecture holds for all functions in $\mathsf{MM}(n,n)$. Evidently, χ defines a functional, $\chi : \mathsf{MM}(n,n) \to \mathbb{R}^n$. We have to show that the formulae in (20) hold. Choose $S \subseteq \mathsf{MM}(n,n)$ to be nonempty, finite and rectangular. Each $f \in \pi_i(S)$ is a min-max function of type $(n,1)$. Choose a max-representation $\lambda_i(f) \subseteq \mathsf{MM}(n,1)$ for each such f. If $F \in S$, let $\lambda(F) \subseteq \mathsf{MM}(n,n)$ be defined by $\lambda(F) = \lambda_1(F_1) \times \cdots \times \lambda_n(F_n)$. Evidently, $\lambda(F)$ is a rectangular max-representation of F. Now define $\Lambda \subseteq \mathsf{MM}(n,n)$ to be the union of all the $\lambda(F)$: $\Lambda = \bigcup_{F\in S} \lambda(F)$. Λ consists entirely of max-only functions and

$$\bigwedge_{H\in\Lambda} H = \bigwedge_{F\in S} \bigwedge_{H\in\lambda(F)} H = \bigwedge_{F\in S} F \ .$$

We claim that Λ is rectangular. To see this, note that, by elementary set theory,

$$\pi_i(\Lambda) = \pi_i\left(\bigcup_{F\in S} \lambda(F)\right) = \bigcup_{F\in S} \pi_i(\lambda(F)) = \bigcup_{F\in S} \lambda_i(F_i) = \bigcup_{f\in\pi_i(S)} \lambda_i(f) \quad .$$

Furthermore,

$$\left(\bigcup_{f_1\in\pi_1(S)} \lambda_1(f_1)\right) \times\cdots\times \left(\bigcup_{f_n\in\pi_n(S)} \lambda_n(f_n)\right) = \bigcup_{(f_1,\cdots,f_n)\in\pi_1(S)\times\cdots\times\pi_n(S)} \lambda_1(f_1)\times\cdots\times\lambda_n(f_n) \ .$$

Since S is rectangular, the right hand side of this can be rewritten as

$$\bigcup_{F\in S} \lambda_1(F_1) \times \cdots \times \lambda_n(F_n) = \bigcup_{F\in S} \lambda(F) = \Lambda \ .$$

It follows that

$$\pi_1(\Lambda) \times \cdots \times \pi_n(\Lambda) = \Lambda \ ,$$

which establishes the claim. We have shown that Λ is a rectangular max-representation of $\bigwedge_{F\in S} F$. By the Duality Conjecture for this min-max function, making use of the rectangular max-representation Λ, and the Duality Conjecture for $F \in S$, making use of the rectangular max-representation $\lambda(F)$, we see that

$$\chi\left(\bigwedge_{F\in S} F\right) = \bigwedge_{H\in\Lambda} \chi(H) = \bigwedge_{F\in S} \bigwedge_{H\in\lambda(F)} \chi(H) = \bigwedge_{F\in S} \chi(F) \ .$$

This establishes the first formula of (20). The second follows by a dual argument. This completes the proof.

□

This formulation contains all the information necessary to calculate the cycle time. A min-max function F is usually specified in such a way that a max-representation, $S \subseteq \mathsf{MM}(n,n)$, can be easily found, perhaps after some algebraic simplification. The Duality Conjecture then reduces the calculation of χ to that of the max-only functions appearing in $\mathsf{Rec}(S)$:

$$\chi(F) = \chi\left(\bigwedge_{H\in S} H\right) = \chi\left(\bigwedge_{H\in\mathsf{Rec}(S)} H\right) = \bigwedge_{H\in\mathsf{Rec}(S)} \chi(H) \ ,$$

where we have used (15). For max-only functions, the Duality Conjecture suggests a further reduction. Any max-only function has a min-representation in which the min-only functions are simple functions. For instance, example (9) has the min-representation

$$\left\{ \begin{pmatrix} x_2+2 \\ x_2+1 \\ x_1-1 \end{pmatrix}, \begin{pmatrix} x_3+5 \\ x_2+1 \\ x_2+3 \end{pmatrix} \right\} .$$

The Duality Conjecture suggests that the cycle time of (9) can be calculated by rectangularising this set. This turns out to correspond exactly to the prescription given by Proposition 1.3.

Proposition 2.1 *If F is max-only or min-only then the Duality Conjecture holds for F.*

Proof Suppose that F is max-only and let A be the associated max-plus matrix. Corollary 2.1 shows that it does not matter what rectangular max-representation is chosen so choose the one consisting only of F. The right hand side of (17) is then $\mu(A)$. A min-representation, T, can be constructed as follows, as in the example above,

$$T = \prod_{1\leq i\leq n} \{x_k + A_{ik} \mid A_{ik} \neq -\infty\} .$$

This is patently rectangular and each element of T is not just min-only, but simple. We have to show that $\bigvee_{G\in T}\chi(G) = \mu(A)$. Choose $1 \leq i \leq n$ and $G \in T$. We can clearly regard the precedence graph of G as contained in $\mathcal{G}(A)$; it simply picks out the edges used by G. Because G is simple, there is an unique circuit, g, upstream from node i in $\mathcal{G}(G)$. It follows that $\chi_i(G) = \mathsf{m}(g)$. Furthermore, any circuit upstream from i in $\mathcal{G}(A)$ must appear in this way through some element of T. Hence, $\bigvee_{G\in T}\chi_i(G)$ is the maximum cycle mean among all circuits upstream from node i. But, according to (8), this is exacly $\mu_i(A)$. Since i was chosen arbitrarily, $\bigvee_{G\in T}\chi(G) = \mu(A)$, as required.

□

We see from this proof that rectangularisation is precisely what is needed to ensure that each circuit in $\mathcal{G}(A)$ makes a contribution to $\chi(F)$.

There is one other case where the Duality Conjecture can be easily shown to hold. Suppose that $F \in \mathsf{MM}(n,n)$ and that χ is known to exist for F. Then,

$$\bigvee_{G\in T} \mathsf{b}\chi(G) \leq \bigvee_{G\in T} \chi(G) \leq \chi(F) \leq \bigwedge_{H\in S} \chi(H) \leq \bigwedge_{H\in S} \mathsf{t}\chi(H)\,. \tag{21}$$

The outermost inequalities are obvious while the innermost follow from (13).

Proposition 2.2 *With the same assumptions as in Conjecture 2.1, suppose in addition that F has a fixed point, where $F(\vec{x}) = \vec{x} + h$. Then (21) collapses to an equality. In particular, the Duality Conjecture holds for F.*

Proof Suppose that $F(\vec{x}) = \vec{x} + h$. We know in this case that χ does exist and that $\chi(F) = h$. Since S is rectangular, there must be some $H \in S$ such that $H(\vec{x}) = \vec{x} + h$. But then $h = \chi(H) = \mathsf{t}\chi(H)$. It follows that $h = \bigwedge_{H\in S} \mathsf{t}\chi(H)$. The dual argument, using the rectangularity of T, shows that $h = \bigvee_{G\in T} \mathsf{b}\chi(G)$. The result follows.

□

It follows that $h = \bigwedge_{H\in S} \mathsf{t}\chi(H)$, which expresses h in terms of the maximum cycle means of the max-plus matrices associated to elements of S. Similarly, $h = \bigvee_{G\in T} \mathsf{b}\chi(G)$. Corollary 3.3 below shows that both these properties together imply that F has a fixed point.

3 Fixed points of min-max functions

The main goal of this section is to derive a fixed point theorem which is independent of the Duality Conjecture but which, nevertheless, is equally powerful to (1). The proof of this occupies the first sub-section. We then discuss algorithmic issues connected with finding fixed points. In the final sub-section we show that our general fixed point theorem leads to a straightforward proof of an earlier fixed point result due to Olsder, [37].

3.1 The fixed point theorem

It will be convenient, from this point onwards, to drop the distinction between max-only or min-only functions and their associated matrices. If A is a max-only or min-only function, we shall use the same symbol to stand for its associated $\mathbb{R}_{\max}$ or $\mathbb{R}_{\min}$ matrix, respectively. Furthermore, taking advantage of Proposition 1.3, we shall use $\chi(A)$ interchangeably with $\mu(A)$ and $\eta(A)$.

If U and V are sets, let $U\backslash V$ denote the complement of V in U: $U\backslash V = \{i \in U \mid i \notin V\}$. The next result is the key technical lemma of this section.

Lemma 3.1 *Suppose that $F \in \mathsf{MM}(n,n)$ and that $S \subseteq \mathsf{MM}(n,n)$ is a rectangular max-representation of F. Choose any family of n functions in S: $A_1, \cdots, A_n \in S$. There exists a function $K \in S$ such that $\mathsf{t}\chi(K) = \bigvee_{1\leq i\leq n} \chi_i(A_i)$.*

Proof Let $h_i = \mu_i(A_i)$ and $h = \bigvee_{1 \le i \le n} h_i$. We have to find $K \in S$ such that $\mathsf{t}\chi(K) = h$. We can assume, without loss of generality, that $h_1 = h$. Let $U_i \subseteq \{1, \cdots, n\}$ be the subset of nodes upstream from i in $\mathcal{G}(A_i)$:

$$U_i = \{k \in \{1, \cdots, n\} \mid i \Leftarrow k \text{ in } \mathcal{G}(A_i)\}\,.$$

By convention, a node is always upstream from itself, so that $i \in U_i$. It follows that the sets $\{U_i\}$ provide a cover of $\{1, \cdots, n\}$: $U_1 \cup \cdots \cup U_n = \{1, \cdots, n\}$. Let $V_r = U_1 \cup \cdots \cup U_r$ for $1 \le r \le n$. The sets $\{V_r\}$ provide a filtration of $\{1, \cdots, n\}$: $U_1 = V_1 \subseteq \cdots \subseteq V_n = \{1, \cdots, n\}$. Define a function $\ell : \{1, \cdots, n\} \to \{1, \cdots, n\}$ by the filtration level at which a number first appears:

$$\ell(i) = \begin{cases} 1 & \text{if } i \in U_1 \\ r & \text{if } i \in U_r \backslash U_{r-1} \text{ for } r > 1 \end{cases}$$

Now define a new matrix K according to the following rule: $K_{ij} = (A_{\ell(i)})_{ij}$. Since S is rectangular, $K \in S$. It remains to show that K has the required property.

Let i, j be nodes of $\mathcal{G}(K)$ such that $i \leftarrow j$. Let $\ell(i) = r$, so that $i \in U_r$. By construction of U_r, $r \Leftarrow i$ in $\mathcal{G}(A_r)$. Since $i \leftarrow j$ in $\mathcal{G}(K)$, it must be the case that $K_{ij} \ne -\infty$ and so $(A_r)_{ij} \ne -\infty$. Hence, $i \leftarrow j$ in $\mathcal{G}(A_r)$ and therefore also $r \Leftarrow j$. It follows that $j \in U_r$. But then $\ell(j) \le r$. We have shown that if $i \leftarrow j$ in $\mathcal{G}(K)$, then $\ell(i) \ge \ell(j)$.

Suppose that $g = i_1 \leftarrow \cdots \leftarrow i_m$ is a circuit in $\mathcal{G}(K)$, where $m > 1$ and $i_1 = i_m$. Let $\ell(i_1) = r$. By the previous paragraph, it must be the case that $\ell(i_j) = r$ for $1 \le r \le m$. Hence g is also a circuit in $\mathcal{G}(A_r)$ and furthermore $r \Leftarrow g$. But then, by virtue of (8), $\mathsf{m}(g) \le \mu_r(A_r) = h_r$. In particular, $\mathsf{m}(g) \le h$. Since g was chosen arbitrarily, it follows that $\mathsf{t}\mu(K) \le h$.

Finally, let g be a critical circuit of A_1 upstream from node 1. Since we assumed that $h = h_1$, it follows that $\mathsf{m}(g) = h$. Every node on g and every node on the path from g to 1 are in U_1. By construction of K, g is also upstream from node 1 in $\mathcal{G}(K)$. Hence, $\mathsf{t}\mu(K) \ge h$. It follows that $\mathsf{t}\mu(K) = h$, as claimed. This completes the proof.

□

Lemma 3.1 has a number of useful consequences which we collect in the following Lemmas.

Lemma 3.2 *Under the same conditions as Lemma 3.1, the function* $\{1, \cdots, n\} \times S \to \mathbb{R}$: $(i, H) \to \chi_i(H)$ *has a saddle point:*

$$\mathsf{t}\left(\bigwedge_{H \in S} \chi(H)\right) = \bigwedge_{H \in S} \mathsf{t}\chi(H)\,.$$

Proof It is well known that half the conclusion always holds; we briefly recall the argument. Choose $1 \le j \le n$. For any $H \in S$, $\chi_j(H) \le \mathsf{t}\chi(H)$. Hence, $\bigwedge_{H \in S} \chi_j(H) \le \bigwedge_{H \in S} \mathsf{t}\chi(H)$. Since j was chosen arbitrarily, it follows that $\mathsf{t}\left(\bigwedge_{H \in S} \chi(H)\right) \le \bigwedge_{H \in S} \mathsf{t}\chi(H)$.

Let $H_i \in S$ be a max-only function for which $\chi_i(H_i) = \bigwedge_{H \in S} \chi_i(H)$. Let $h = \bigvee_{1 \le i \le n} \chi_i(H_i)$. It follows from Lemma 3.1 that there exists $K \in S$ such that $\mathsf{t}\chi(K) = h$. Hence,

$$\bigwedge_{H \in S} \mathsf{t}\chi(H) \;\le\; h \;=\; \mathsf{t}\left(\bigwedge_{H \in S} \chi(H)\right)\,.$$

The result follows.

□

Lemma 3.3 *Under the same conditions as Lemma 3.1, if* $\bigwedge_{H\in S}\chi(H) = h$*, there exists* $K \in S$ *such that* $\chi(K) = h$*.*

Proof It follows from Lemma 3.2 that $h = \mathrm{t}\,(\bigwedge_{H\in S}\chi(H)) = \bigwedge_{H\in S}\mathrm{t}\chi(H)$. Let $K \in S$ be such that $\mathrm{t}\chi(K) = h$. Then,

$$h = \bigwedge_{H\in S}\chi(H) \leq \chi(K) \leq h\,. \tag{22}$$

It follows that $\chi(K) = h$, as required.

□

The next result is the main theorem of this section. It follows in detail an argument given by Cochet-Terrasson and Gaubert in [8]. The additional ingredient which appears here is Lemma 3.1, in the guise of Lemma 3.3, which allows a stronger result to be derived than that in [8].

The proof is based on a min-max analogue of Howard's policy improvement algorithm for stochastic control problems with average or ergodic cost (see, for example, [47, Ch. 31–33],[38]). Typically, Howard's algorithm finds a fixed point of $F(\vec{x}) = \bigwedge_{u\in U}\vec{c}_u + P_u\vec{x}$ where U is a finite set and, for all $u \in U$, $\vec{c}_u \in \mathbb{R}^n$ is a cost vector and P_u is a row-stochastic matrix. (In this paragraph matrix operations are to be interpreted in the usual algebra.) It is easy to see using Proposition 1.1 that functions of this form are in fact topical. At each step, Howard's algorithm selects a function $A(\vec{x}) = \vec{c} + P\vec{x}$ in $S = \mathsf{Rec}\{\vec{c}_u + P_u\vec{x} \mid u \in U\}$ and finds a fixed point of it. It is necessary to assume that such a fixed point can be found, which is the case, for instance, if each matrix P_u is positive. If this point is not also a fixed point of F, then the function A is replaced by $A' \in S$ which satisfies $F(\vec{x}) = A'(\vec{x})$ and the process is repeated. Under appropriate conditions it can be shown that this leads, after finitely many steps, to a fixed point of F.

The convergence proof for the traditional Howard algorithm relies on a form of maximum principle: algebraically, the fact that the inverse of $I - P$ is monotone, for a nonnegative matrix P whose spectral radius is strictly less than one. The analogue of this in the proof below is the monotonicity property of the spectral projector which appears as part 1 of Lemma 1.4.

Theorem 3.1 *Let* $F \in \mathsf{MM}(n,n)$ *and suppose that* $S, T \in \mathsf{MM}(n,n)$ *are rectangular and, respectively, a max-representation and a min-representation of* F*. The following conditions are equivalent.*

1. *F has a fixed point with* $F(\vec{x}) = \vec{x} + h$*.*
2. $\bigwedge_{H\in S}\chi(H) = h$*.*
3. $\bigvee_{G\in T}\chi(G) = h$*.*

Proof It follows from Proposition 2.2 that *1* implies both *2* and *3*. Assume that *2* holds. We shall deduce *1*. The fact that *3* also implies *1* follows by a dual argument.

We may assume, as usual, that $h = 0$. It follows from Lemma 3.3 that there is $A_1 \in S$ such that $\chi(A_1) = 0$. By Proposition 1.4, A_1 has a fixed point: $A_1(\vec{a}_1) = \vec{a}_1$. Hence, $F(\vec{a}_1) \le \vec{a}_1$. Since S is rectangular, we can find $A_2 \in S$ such that $A_2(\vec{a}_1) = F(\vec{a}_1)$. We can ensure, furthermore, that if $F_i(\vec{a}_1) = (\vec{a}_1)_i$, then $(A_2)_i = (A_1)_i$. Since $A_2(\vec{a}_1) \le \vec{a}_1$, it follows by property M that $\mu(A_2) \le 0$ and so, by a similar argument to (22) that $\mu(A_2) = 0$. As a consequence, it follows from part 6 of Lemma 1.4, that $(A_2(\vec{a}_1))_i = (\vec{a}_1)_i$ for all $i \in \mathcal{C}(A_2)$. Hence, $F_i(\vec{a}_1) = (A_2(\vec{a}_1))_i = (\vec{a}_1)_i$, and so, by construction, $(A_2)_i = (A_1)_i$ for all $i \in \mathcal{C}(A_2)$. It is then not difficult to see that $\mathcal{C}(A_2) \subseteq \mathcal{C}(A_1)$.

Since $\mu(A_2) = 0$, it also follows that A_2 has a fixed point. By part 7 of Lemma 1.4, $\mathsf{P}_{A_2} : \mathbb{R}^n \to \mathbb{R}^n$. Hence we may choose the fixed point of A_2 to be $\vec{a}_2 = \mathsf{P}_{A_2}(\vec{a}_1)$. Since $A_2(\vec{a}_1) \le \vec{a}_1$, it follows from part 1 of Lemma 1.4 that $\vec{a}_2 \le \vec{a}_1$. At the same time, if $i \in \mathcal{C}(A_2)$, then by part 5 of Lemma 1.4, $(\mathsf{P}_{A_2})_{ii} = 0$ and so $(\vec{a}_2)_i \ge (\vec{a}_1)_i$. Hence, $(\vec{a}_2)_i = (\vec{a}_1)_i$ for all $i \in \mathcal{C}(A_2)$.

We can now carry on and generate a sequence $(A_s, \vec{a}_s)$ for $s = 1, 2, \cdots$, such that the following properties hold:

$$\begin{array}{lrcl}
1) & A_s \in S & \text{and} & a_s \in \mathbb{R}^n \\
2) & A_s(\vec{a}_{s-1}) & = & F(\vec{a}_{s-1}) \\
3) & A_s(\vec{a}_s) & = & \vec{a}_s \\
4) & \vec{a}_s & \le & \vec{a}_{s-1} \\
5) & (\vec{a}_s)_i & = & (\vec{a}_{s-1})_i \text{ for all } i \in \mathcal{C}(A_s) \\
6) & \mathcal{C}(A_s) & \subseteq & \mathcal{C}(A_{s-1})\,.
\end{array}$$

Evidently, since S is finite, we must have $A_k = A_l$ for some $k < l$. By property 6, $\mathcal{C}(A_k) = \cdots = \mathcal{C}(A_l)$. Hence, by property 5, $(\vec{a}_k)_i = \cdots = (\vec{a}_l)_i$ for all $i \in \mathcal{C}(A_l)$. It follows from property 3 that $\vec{a}_k$ and $\vec{a}_l$ are fixed points of A_l which agree on $\mathcal{C}(A_l)$. Hence, by part 4 of Lemma 1.4, $\vec{a}_k = \vec{a}_l$. By property 4, $\vec{a}_k = \cdots = \vec{a}_l$. In particular, $\vec{a}_l = \vec{a}_{l-1}$. By property 2, $\vec{a}_l = A_l(\vec{a}_l) = A_l(\vec{a}_{l-1}) = F(\vec{a}_{l-1}) = F(\vec{a}_l)$. It follows that $\vec{a}_l$ is a fixed point of F. This completes the proof.

□

Corollary 3.1 ([19, §3]) *If $F \in \mathsf{MM}(n,n)$ satisfies the Duality Conjecture, then the fixed point result (1) holds.*

Proof Evident.

□

Corollary 3.2 ([8]) *Suppose that $F \in \mathsf{MM}(n,n)$ has a max-representation S such that each $H \in S$ has a fixed point. Then F has a fixed point.*

Proof Since $\chi(H) = \mathrm{t}\chi(H)$ for each $H \in S$, the result follows immediately from the Theorem.

□

Corollary 3.3 ([21, Theorem 5.1]) *Let $F \in \mathsf{MM}(n,n)$. F has a fixed point, with $F(\vec{x}) = \vec{x} + h$, if, and only if,*

$$\bigvee_{G \in T} \mathsf{b}\chi(G) = h = \bigwedge_{H \in S} \mathsf{t}\chi(H)\,.$$

Proof If F has a fixed point, this is just Proposition 2.2. If the formula holds, then it follows from (21) that condition 2 and condition 3 of Theorem 3.1 hold and hence that F has a fixed point.

□

3.2 Algorithmic issues

Finding fixed points of min-max functions is an important problem in applications. For instance, the clock schedule verification problem mentioned in §1.2 is equivalent to finding a fixed point of a min-max function associated to a digital circuit. The particular form of the min-max functions which arise in this application leads to efficient algorithms for finding fixed points. For general min-max functions the situation is less clear. Although the methods of the previous section are constructive in nature, they do not give rise to an efficient general algorithm.

The problem stems from the fact that a min-max function is typically presented in the form $F = \bigwedge_{H \in S} H$ where S is a subset of max-only functions which is not necessarily rectangular. In order to make use of the method in Theorem 3.1, it is necessary to find $A \in \mathsf{Rec}(S)$, such that A has a fixed point and $\chi(A)$ is minimal; this is the starting point for the iteration. Searching all of $\mathsf{Rec}(S)$ to find such a function is prohibitively expensive. However, it is sometimes the case that all functions $H \in \mathsf{Rec}(S)$ have fixed points. This occurs, for instance, when S consists of functions for which the corresponding max-only matrices have no $-\infty$ entries. In this case it is easy to see, using Proposition 1.3, that each function $H \in \mathsf{Rec}(S)$ satisfies $\chi(H) = h$, for some $h \in \mathbb{R}$. Hence, by Proposition 1.4, each H has a fixed point. This situation does arise in applications. We can adapt the method of Theorem 3.1 to give a tractable algorithm in this case.

It will be convenient to extend the spectral projector notation P_A (see (12)) to general matrices: if $\mathsf{t}\mu(A) \neq 0$, let $\tilde{A} = -\mathsf{t}\mu(A) + A$, so that $\mathsf{t}\mu(\tilde{A}) = 0$, and define $\mathsf{P}_A = \mathsf{P}_{\tilde{A}}$.

Suppose that a a min-max function F is given in the form:

$$F_i(\vec{x}) = \bigwedge_{u \in U(i)} A_{iu}\vec{x}\ , \qquad (23)$$

where $U(1), \cdots, U(n)$ are finite sets and A_{iu} are row vectors with entries in $\mathbb{R}_{\max}$. Borrowing the vocabulary of optimal control, we say that a policy is a map $\pi : \{1, \cdots, n\} \rightarrow \bigcup_{1 \leq i \leq n} U(i)$, such that $\pi(i) \in U(i)$, for all $1 \leq i \leq n$. The corresponding policy matrix $A[\pi]$ is defined by $A[\pi]_i = A_{i\pi(i)}$. By construction, the set of policy matrices $A[\pi]$ is rectangular.

The fixed point algorithm takes as input a min-max function of the form (23) each of whose policy matrices have a fixed point. Equivalently, by Proposition 1.4, for each policy matrix, π, there exists $h_\pi \in \mathbb{R}$ such that $\chi(A[\pi]) = h_\pi$. The algorithm produces as output $\vec{x} \in \mathbb{R}^n$ and $h \in \mathbb{R}$ such that $F(\vec{x}) = \vec{x} + h$. The steps are as follows.

1. *Initialisation.* Select an arbitrary policy π_1. Set $s = 1$ and let $A_1 = A[\pi_1]$. Find $\vec{x}_1 \in \mathbb{R}^n$ and $h_1 \in \mathbb{R}$, such that $A_1\vec{x}_1 = \vec{x}_1 + h_1$.

2. If $F(\vec{x}_s) = \vec{x}_s + h_s$, then stop.

3. *Policy improvement.* Define π_{s+1} by

$$\forall 1 \leq i \leq n, \qquad \bigwedge_{u \in U(i)} A_{iu}\vec{x}_s = A_{i\pi_{s+1}(i)}\vec{x}_s \ .$$

   The choice should be conservative, in the sense that $\pi_{s+1}(i) = \pi_s(i)$ whenever possible. Let $A_{s+1} = A[\pi_{s+1}]$.

4. *Value determination.*
   (a) If $\mu(A_{s+1}) < h_s$, then, take any fixed point $\vec{x}_{s+1}$ of A_{s+1}.
   (b) If $\mu(A_{s+1}) = h_s$, then, take the particular fixed point $\vec{x}_{s+1} = P_{A_{s+1}}\vec{x}_s$.

5. Increment s by one and go to step 2.

The cycle time vector $\mu(A)$ of a max-plus matrix A can be computed by Karp's algorithm, [2, 28] while Gondran and Minoux give algorithms in [17, Chapter 3, §4] which can be adapted for computing the spectral projector P_A.

The proof that the algorithm terminates is a straightforward generalisation of the method of Theorem 3.1 and is left as an exercise to the reader. The following example illustrates how the algorithm works in practice.

Consider the Min-Max function:

$$\begin{aligned}
F_1(x_1, x_2, x_3) &= (x_1 \vee x_2 + 1 \vee x_3 + 1) \wedge (x_1 + 1 \vee x_2 + 1 \vee x_3 + 1) \\
F_2(x_1, x_2, x_3) &= (x_1 + 1 \vee x_2 + 2 \vee x_3 + 1) \wedge (x_1 \vee x_2 + 1 \vee x_3) \\
F_3(x_1, x_2, x_3) &= (x_1 + 1 \vee x_2 \vee x_3 + 2) \wedge (x_1 \vee x_2 + 1 \vee x_3 + 2)
\end{aligned}$$

Alternatively, F can be written in the form (23), with

$$U(1) = U(2) = U(3) = \{1, 2\}, \qquad \left\{\begin{matrix} A_{11} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \\ A_{21} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \\ A_{31} = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \end{matrix}\right\} \quad \left\{\begin{matrix} A_{12} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\ A_{22} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ A_{32} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \end{matrix}\right\}$$

Note that each policy matrix of F has all its entries finite and so, as discussed above, F satisfies the conditions required by the algorithm.

Initialisation. Select $\pi_1(1) = 1, \pi_1(2) = 1, \pi_1(3) = 1$ so that

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} .$$

Since $\mu(A_1) = 2$, we set $h_1 = 2$ and we choose some fixed point of A_1, for instance,

$$\vec{x}_1 = \begin{pmatrix} -2 & -1 & -1 \end{pmatrix}^T .$$

Policy improvement. We have $F(\vec{x}_1) = A_2\vec{x}_1$ where

$$A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} .$$

Here, $\pi_2(1) = 1$, $\pi_2(2) = 2$ and $\pi_2(3) = 1$.

Value determination. We have $h_2 = h_1 = 2$. Accordingly, we select the particular fixed point

$$\vec{x}_2 = \begin{pmatrix} -2 & -3 & -1 \end{pmatrix}^T = \mathsf{P}_{A_2}\vec{x}_1 ,$$

where

$$\mathsf{P}_{A_2} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -4 & -2 \\ -1 & -2 & 0 \end{pmatrix} .$$

Since $F(\vec{x}_2) = \vec{x}_2 + h_2$, the algorithm terminates.

The complexity of one iteration of the algorithm is $O(n^3 + (\sum_{1\leq i\leq n} |U(i)|)n)$, where n is the dimension of the ambient space, and $|X|$ denotes the cardinality of the set X. Indeed, Karp's algorithm for computing the cycle time vector $\mu(A)$ of a matrix A has time complexity $O(n^3)$ as do the algorithms of [17]. It follows that one value determination step costs $O(n^3)$ time. Clearly, one policy improvement step requires $(\sum_{1\leq i\leq n} |U(i)|)$ scalar products, which can be done in time $O((\sum_{1\leq i\leq n} |U(i)|)n)$.

It seems difficult to bound accurately the number of iterations of the algorithm. Experiments suggest that its average value is well below n, at least when $|U(i)|$ is $O(n)$, for all i. The situation seems very similar to that of conventional policy improvement algorithms, which are known to be excellent in practice although no polynomial bound is known in general for their execution time.

3.3 Derivation of Olsder's Theorem

Olsder has proved a fixed point theorem for certain separated min-max functions, [37, Theorem 2.1]. Although this applies only to a restricted class, it was the first result to be proved on min-max functions beyond the $\mathbb{R}_{\max}$ linear setting. We now show that it follows from Theorem 3.1.

Let $F \in \mathsf{MM}(n, n)$ be a separated function. We can assume, without loss of generality, that F has the following form

$$F_i = \begin{cases} x_1 + K_{i1} \vee \cdots \vee x_n + K_{in} & \text{if } 1 \leq i \leq s \\ x_1 + K_{i1} \wedge \cdots \wedge x_n + K_{in} & \text{if } s+1 \leq i \leq n \end{cases}$$

where $1 \leq s < n$ and K_{ij} is an $n \times n$ matrix of elements satisfying $K_{ij} \in \mathbb{R}_{\min}$ for $1 \leq i \leq s$ and $K_{ij} \in \mathbb{R}_{\max}$ for $s+1 \leq i \leq n$. Let $t = n - s$. Recall the notation introduced at the end of §1.4: if B is a matrix over $\mathbb{R}_{\min}$, then $\eta(B)$ denotes the vector of minimum upstream cycle means and $\chi(B) = \eta(B)$.

K is neither a matrix over $\mathbb{R}_{\max}$ nor over $\mathbb{R}_{\min}$ but it is convenient to break it into blocks which are. Let A, C be matrices over $\mathbb{R}_{\max}$ of size $s \times s$ and $s \times t$, respectively, corresponding to the top left and top right blocks of K and let D, B be matrices over $\mathbb{R}_{\min}$ of size $t \times s$ and $t \times t$, respectively, corresponding to the bottom left and bottom right blocks of K:

$$\begin{array}{rcllll} A_{ij} & = & K_{ij} & i \in \{1, \cdots, s\} & j \in \{1, \cdots, s\} \\ C_{i(j-s)} & = & K_{ij} & i \in \{1, \cdots, s\} & j \in \{s+1, \cdots, s+t\} \\ D_{(i-s)j} & = & K_{ij} & i \in \{s+1, \cdots, s+t\} & j \in \{1, \cdots, s\} \\ B_{(i-s)(j-s)} & = & K_{ij} & i \in \{s+1, \cdots, s+t\} & j \in \{s+1, \cdots, s+t\} . \end{array}$$

Suppose that F has a fixed point: $F(\vec{x}) = \vec{x} + h$. Let $\vec{y} \in \mathbb{R}^s$ be the vector obtained from $\vec{x}$ by truncating the last t components: $y_i = x_i$ for $1 \leq i \leq s$. Evidently, $A(\vec{y}) \leq \vec{y} + h$, so that $\mu(A) \leq h$. Equivalently, in scalar terms, $\mathsf{t}\mu(A) \leq h$. A dual argument shows that $h \leq \mathsf{b}\eta(B)$. Hence, a necessary condition for F to have a fixed point is that $\mathsf{t}\mu(A) \leq \mathsf{b}\eta(B)$.

Olsder's result is by way of a converse to this but requires more assumptions on the structure of A, B, C and D. To discuss it, we need to review some further material from matrix theory. For more details, see [2].

Let A be an $n \times n$ matrix over $\mathbb{R}_{\max}$. A is said to be irreducible if there does not exist any permutation matrix P such that $P^{\mathsf{t}}AP$ is in upper triangular block form. (This is identical to the notion of irreducibility for nonnegative matrices, [32, §1.2].) An equivalent condition is that $\mathcal{G}(A)$ is strongly connected. That is, if i and j are any two nodes in $\mathcal{G}(A)$, then they are upstream from each other: $i \Leftarrow j$ and $j \Leftarrow i$. If $i \leftarrow j$ in $\mathcal{G}(A)$ then it is easy to see that $\mu_i(A) \geq \mu_j(A)$. It follows that if A is irreducible then $\mu(A) = \mathsf{t}\mu(A)$. (By Proposition 1.4 we see that A has an eigenvector lying in $\mathbb{R}^n$. This is a max-plus version of the Perron-Frobenius Theorem, [2, Theorem 3.23].) If $U \subseteq \{1, \cdots, n\}$ is a subset of nodes, all of which are upstream from each other—$i \Leftarrow j$ for all $i, j \in \{1, \cdots, n\}$—then we shall say that U is upstream (respectively, downstream) from some node $k \in \{1, \cdots, n\}$, if there is some $i \in U$ such that $k \Leftarrow i$ (respectively, $i \Leftarrow k$).

Theorem 3.2 *Suppose that $F \in \mathsf{MM}(n,n)$ is separated. Using the notation above, suppose further that A and B are irreducible and that both C and D have at least one real entry. Then the Duality Conjecture holds for F. Furthermore,* ([37, Theorem 2.1]), *F has a fixed point if, and only if,* $\mathsf{t}\mu(A) \leq \mathsf{b}\eta(B)$.

Proof Define rectangular max and min-representations of F, $S, T \subseteq \mathsf{MM}(n,n)$, similar to those used in Proposition 2.1:

$$\begin{array}{rcl} S & = & \{F_1\} \times \cdots \times \{F_s\} \times \prod_{s+1 \leq i \leq s+t} \{x_j + K_{ij} \mid K_{ij} \neq +\infty\} \\ T & = & \prod_{1 \leq i \leq s} \{x_j + K_{ij} \mid K_{ij} \neq -\infty\} \times \{F_{s+1}\} \times \cdots \times \{F_{s+t}\} . \end{array}$$

Let $\vec{\mu} = \bigwedge_{H \in S} \mu(H)$ and $\vec{\eta} = \bigvee_{G \in T} \eta(G)$. We know that $\vec{\eta} \leq \vec{\mu}$. To establish the Duality Conjecture, we have to show that $\vec{\eta} = \vec{\mu}$.

For any $H \in S$, $H_i = F_i$ for $i \in \{1, \cdots, s\}$. Hence the top left block of H is equal to A: $H_{ij} = A_{ij}$ for $i, j \in \{1, \cdots, s\}$. Since A is irreducible by hypothesis, it follows that, for any $i, j \in \{1, \cdots, s\}$, $\mu_i(H) = \mu_j(H)$. Furthermore, $\mathsf{t}\mu(A) \leq \mu_i(H)$ for all $H \in S$. Hence,

$$\mu_i = \bigwedge_{H \in S} \mu_i(H) = \bigwedge_{H \in S} \mu_j(H) = \mu_j .$$

It follows that $\mu_1 = \cdots = \mu_s$. Let us call the common value μ. Evidently, $\mathsf{t}\mu(A) \leq \mu$. Dually, $\eta_{s+1} = \cdots = \eta_{s+t} = \eta$ and $\mathsf{b}\eta(B) \geq \eta$.

Consider the min-plus matrix B. It has a rectangular max-representation, $R \subseteq \mathsf{MM}(n,n)$, of the same form (but dual) to the one used in the proof of Proposition 2.1. Each element of R is a simple function. Since B is irreducible, $\eta(B) = \mathsf{b}\eta(B)$. By Lemma 3.3 there exists a simple function $U \in R$ such that $\eta(U) = \mathsf{b}\eta(B)$. Now construct an element $K \in S$ by choosing $K_j = U_{j-s}$ for $j \in \{s+1, \cdots, s+t\}$. It is clear that $\mu_j(K) = \mathsf{b}\eta(B)$ for $j \in \{s+1, \cdots, s+t\}$. By hypothesis, $C_{ik} \neq -\infty$ for some $i \in \{1, \cdots, s\}$ and $k \in \{s+1, \cdots, s+t\}$. It follows that $i \leftarrow k$ in $\mathcal{G}(K)$. It is not difficult to see that $\mu_i(K) = \mathsf{t}\mu(A) \vee \mathsf{b}\eta(B)$. Hence,

$$\mu_i(K) = \begin{cases} \mathsf{t}\mu(A) \vee \mathsf{b}\eta(B) & \text{if } i \in \{1, \cdots, s\} \\ \mathsf{b}\eta(B) & \text{if } i \in \{s+1, \cdots, s+t\}\,. \end{cases}$$

It follows that $\mu \leq \mathsf{t}\mu(A) \vee \mathsf{b}\eta(B)$ and $\mu_i \leq \mathsf{b}\eta(B)$ for $i \in \{s+1, \cdots, s+t\}$. By a dual construction we can show that $\eta \geq \mathsf{b}\eta(B) \wedge \mathsf{t}\mu(A)$ and $\eta_i \geq \mathsf{t}\mu(A)$ for $i \in \{1, \cdots, s\}$. We can summarise what we have shown in the table below.

$$\begin{array}{rcccccl} \mathsf{t}\mu(A) & \leq & \eta_1 & \leq & \mu & \leq & \mathsf{t}\mu(A) \vee \mathsf{b}\eta(B) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathsf{t}\mu(A) & \leq & \eta_s & \leq & \mu & \leq & \mathsf{t}\mu(A) \vee \mathsf{b}\eta(B) \\ \mathsf{b}\eta(B) \wedge \mathsf{t}\mu(A) & \leq & \eta & \leq & \mu_{s+1} & \leq & \mathsf{b}\eta(B) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathsf{b}\eta(B) \wedge \mathsf{t}\mu(A) & \leq & \eta & \leq & \mu_{s+t} & \leq & \mathsf{b}\eta(B) \end{array}$$

Suppose that $\mathsf{b}\eta(B) \leq \mathsf{t}\mu(A)$. Then it is easy to see from the table above that $\vec{\eta} = \vec{\mu}$. Hence the Duality Conjecture holds.

Now suppose that $\mathsf{t}\mu(A) \leq \mathsf{b}\eta(B)$. As we saw above, this is a necessary condition for F to have a fixed point. We shall now show that it is also sufficient.

Choose $j \in \{s+1, \cdots, s+t\}$. We claim that $\mu_j \leq \mu$. To see this, choose $H \in S$ such that $\mu_1(H) = \cdots = \mu_s(H) = \mu$. By construction of S, H is simple in the components $s+1, \cdots, s+t$. Hence the node j must have an unique edge leading to it in $\mathcal{G}(H)$: say, $j \leftarrow k$. If $k \in \{s+1, \cdots, s+t\}$ then it has a similar property and we can proceed in this way until one of two mutually exclusive possibilities occur. Either the path remains entirely among the nodes in the range $\{s+1, \cdots, s+t\}$ or it contains a node $i \in \{1, \cdots, s\}$. In the latter case, $\mu_j(H) = \mu_i(H) = \mu$ since the path out of j is unique until it reaches a node in $\{1, \cdots, s\}$. Hence, $\mu_j \leq \mu$.

In the former case, j is not downstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H)$. Suppose, to begin with, that there is no node in the range $\{s+1, \cdots, s+t\}$ which is downstream from $\{1, \cdots, s\}$. Because C has at least one real entry, some node in this range is upstream from $\{1, \cdots, s\}$. Since every circuit of $\mathcal{G}(H)$ in the range $\{s+1, \cdots, s+t\}$ must also be a circuit in $\mathcal{G}(B)$, it follows that $\mu \geq \mathsf{b}\eta(B)$. From the table, we see that $\mu_j \leq \mathsf{b}\eta(B)$ and so $\mu_j \leq \mu$.

We may now assume that there exists $u \in \{s+1, \cdots, s+t\}$ downstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H)$. Since B is irreducible, there exists a path in $\mathcal{G}(B)$ from u to j:

$$j = u_1 \leftarrow u_2 \leftarrow \cdots \leftarrow u_m = u\,, \tag{24}$$

where $1 < m$, $\{u_1, \cdots, u_m\} \subseteq \{s+1, \cdots, s+t\}$. We may assume furthermore, without loss of generality, that $u_1, \cdots, u_{m-1}$ are not downstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H)$. It follows that $\mu_{u_i}(H) \geq \mathsf{b}\eta(B)$ for $1 \leq i < m$. Define $H' \in S$ by altering H as follows:

$$\begin{array}{rcll} (H')_i & = & H_i & \text{if } i \notin \{u_1, \cdots, u_{m-1}\} \\ (H')_{u_i} & = & B_{(u_i - s)(u_{i+1} - s)} & \text{if } 1 \leq i \leq m-1 \end{array}$$

By construction, (24) is also a path in $\mathcal{G}(H')$ and j is downstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H')$. It follows that $\mu_j(H') = \mu_1(H')$. The only difference between $\mathcal{G}(H')$ and $\mathcal{G}(H)$ is at the nodes $u_1, \cdots, u_{m-1}$ which may have different edges leading to them.

Suppose that $\mu_1(H') > \mu$. This can only happen if one of the edges on (24) has created a new circuit upstream from 1 in $\mathcal{G}(H')$. Let u_r be the first node on (24) which is upstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H')$. We may assume that $1 \leq r < m$, for if $r = m$, then, contrary to what was just said, no edge of (24) can have caused the change. It must now be the case that u_r was also upstream from $\{1, \cdots, s\}$ in $\mathcal{G}(H)$. Hence, $\mu = \mu_1(H) \geq \mu_{u_r}(H)$. But, as we saw above, $\mu_{u_r}(H) \geq \mathsf{b}\eta(B)$. It follows once again that $\mu_j \leq \mu$. Hence, we may assume that $\mu_1(H') = \mu$. But then $\mu_j \leq \mu_j(H') = \mu_1(H') = \mu$. In either case, $\mu_j \leq \mu$, which establishes the claim.

Now suppose that for some $j \in \{s+1, \cdots, s+t\}$, it is the case that $\mu_j < \mu$. Choose $H \in S$ such that $\mu_j(H) = \mu_j$. If p is the path in $\mathcal{G}(H)$ leading to node j then p cannot start from any node $i \in \{1, \cdots, s\}$. For if it did, $\mu \leq \mu_i(H) = \mu_j(H) < \mu$, which is nonsense. Hence p must terminate in a circuit g, which must also be a circuit in $\mathcal{G}(B)$. Hence $\mathsf{m}(g) \geq \mathsf{b}\eta(B)$. But evidently, $\mu_j(H) = \mathsf{m}(g)$, since g is the only circuit upstream from j. Hence $\mu_j(H) \geq \mathsf{b}\eta(B)$, from which it follows that $\mathsf{b}\eta(B) \geq \mu > \mu_j(H) \geq \mathsf{b}\eta(B)$, which is also nonsense. It follows that for all $j \in \{s+1, \cdots, s+t\}$, $\mu_j = \mu$. We have shown that $\bigwedge_{H \in S} \chi(H) = \mu$. By Theorem 3.1, F has a fixed point. By Proposition 2.2, the Duality Conjecture holds for F. This completes the proof.

□

The above proof is straightforward in comparison with Olsder's original argument and fits within the general framework established by the Duality Conjecture.

The case $\mathsf{b}\eta(B) < \mathsf{t}\mu(A)$ is the only situation in which the Duality Conjecture is known to hold for fixed point-free functions of high dimension outside the linear setting.

4 The Duality Conjecture in dimension 2

Choose $F \in \mathsf{MM}(2,2)$ and let $S, T \subseteq \mathsf{MM}(2,2)$ be rectangular max and min-representations, respectively, of F. Let $L = \bigvee_{B \in T} \eta_1(B)$ and $R = \bigwedge_{A \in S} \mu_1(A)$. We know that $L \leq R$.

Theorem 4.1 *Under the above assumptions, $R \leq L$. In particular, the Duality Conjecture holds for F.*

The proof occupies the whole of this section. We begin with some extensive book-keeping.

It will be convenient to regard the expressions for cycle times which appear in formulae such as (28) and (29) below as taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$. For instance, we will encounter expressions of the form $\bigwedge_{x \in S} x$ where $S \subseteq \mathbb{R}_{\max}$ and it may be the case that S is empty. By convention the minimum over an empty set is taken to be $+\infty$ (and, dually, the maximum over an empty set is taken to be $-\infty$). We regard $\mathbb{R} \cup \{-\infty, +\infty\}$ as equipped with the obvious extension of the partial order on $\mathbb{R}$ so that $-\infty < +\infty$. This will allow us, where necessary, to compare $-\infty$ with $+\infty$.

If $U \subseteq V$, let $\overline{U}$ denote the complement of U in V, $\overline{U} = V \backslash U$, whenever the superset V is clear from the context. If $\mathcal{L}$ is any set, let $\mathsf{P}^n(\mathcal{L})$ denote the set of partitions of $\mathcal{L}$ into n disjoint pieces:

$$\mathsf{P}^n(\mathcal{L}) = \{\{U_1, \cdots, U_n\} \mid U_i \subseteq \mathcal{L},\ \mathcal{L} = U_1 \cup \cdots \cup U_n \text{ and } U_i \cap U_j = \emptyset \text{ for } i \neq j\}.$$

It will be convenient to use the variables x and y in place of x_1 and x_2 in what follows.

The first step is to obtain expressions for L and R. We may write F in the form

$$\begin{array}{rcl} F_1(x, y) & = & \bigwedge_{1 \leq i \leq n_1}(a_i + x \vee b_i + y) \\ F_2(x, y) & = & \bigwedge_{1 \leq j \leq n_2}(c_j + x \vee d_j + y), \end{array} \tag{25}$$

where $a_i, b_i, c_j, d_j \in \mathbb{R}_{\max}$ and $1 \leq n_1, n_2$. We may assume that not both a_i and b_i are $-\infty$ and that the same applies to c_j and d_j. It will be convenient to identify the sets $I_1, I_2 \subseteq \{1, \cdots, n_1\}$ and $J_1, J_2 \subseteq \{1, \cdots, n_2\}$ where

$$\begin{array}{rclrcl} I_1 & = & \{i \mid b_i = -\infty\} & I_2 & = & \{i \mid a_i = -\infty\} \\ J_1 & = & \{j \mid d_j = -\infty\} & J_2 & = & \{j \mid c_j = -\infty\}. \end{array}$$

In view of the assumptions above, these sets satisfy the restrictions

$$I_1 \cap I_2 = \emptyset \text{ and } J_1 \cap J_2 = \emptyset\,. \tag{26}$$

We can evidently construct a rectangular max-representation of F, $S \subseteq \mathsf{MM}(2,2)$, in which each $A \in S$ is indexed by an element of $\{1, \cdots, n_1\} \times \{1, \cdots, n_2\}$. If (i, j) lies in this set, then the corresponding element of S is represented by the max-plus matrix

$$A(i,j) = \begin{pmatrix} a_i & b_i \\ c_j & d_j \end{pmatrix}.$$

We may write

$$R = \bigwedge_{(i,j) \in \{1,\cdots,n_1\} \times \{1,\cdots,n_2\}} \mu_1(A(i,j))\ .$$

By Proposition 1.3, $\mu_1(A(i,j))$ is given by

$$\mu_1(A(i,j)) = \begin{cases} a_i \vee (b_i + c_j)/2 \vee d_j & \text{if } b_i \neq -\infty \\ a_i & \text{otherwise} \end{cases}. \tag{27}$$

This equation is valid even when a_i, b_i, c_j and d_j take on the value $-\infty$; the point of the second formula being that, if $b_i = -\infty$, then node 2 is no longer upstream of node 1. In all other cases, the first formula gives the correct answer.

In order to compare R with L we shall need to write one of the expressions "the other way round". Let us do this with R and use the distributive law to interchange $\wedge$ and $\vee$. This amounts to choosing, for each $(i,j) \in \{1,\cdots,n_1\} \times \{1,\cdots,n_2\}$, one of the three possible terms which appear in (27),

$$a_i \ , \qquad (b_i + c_j)/2 \ , \qquad d_j \ ,$$

and doing so in all possible ways consistent with the restrictions in (27). If $\{S,T,U\} \in \mathsf{P}^3(\{1,\cdots,n_1\} \times \{1,\cdots,n_2\})$, let $\rho(S,T,U) \in \mathbb{R} \cup \{-\infty,+\infty\}$ be defined by

$$\rho(S,T,U) = \bigwedge_{p\in\pi_1 S} a_p \wedge \bigwedge_{(q,r)\in T} \frac{b_q + c_r}{2} \wedge \bigwedge_{u\in\pi_2 U} d_u \ . \tag{28}$$

This formula must be evaluated in $\mathbb{R} \cup \{-\infty,+\infty\}$ because it may be the case that both $-\infty$ and $+\infty$ appear when evaluating it. For instance, this happens if $S \cap I_2 \neq \emptyset$ and $U = \emptyset$. In this case, of course, $\rho(S,T,U) = -\infty$. We may now write

$$R = \bigvee_{\{S,T,U\}\in\mathsf{P}^3(\{1,\cdots,n_1\} \times \{1,\cdots,n_2\})} \rho(S,T,U) \ , \tag{29}$$

where we must assume that $\pi_1 T \cap I_1 = \pi_1 U \cap I_1 = \emptyset$ because of (27). We may as well get rid of all the partitions which make a contribution of $-\infty$ to (29). We may hence assume that the partitions in (29) satisfy the restrictions

$$\begin{array}{rcccl} \pi_1 S \cap I_2 & = & \emptyset & & \\ \pi_1 T \cap I_1 & = & \pi_2 T \cap J_2 & = & \emptyset \\ \pi_2 U \cap J_1 & = & \pi_1 U \cap I_1 & = & \emptyset \ . \end{array} \tag{30}$$

It is easy to deduce from these equations and the fact that $\{S,T,U\}$ is a partition, that $I_1 \times \{1,\cdots,n_2\} \subseteq S$. In particular,

$$I_1 \subseteq \pi_1 S. \tag{31}$$

We now turn to R. By using distributivity again, (25) can be rewritten as

$$\begin{array}{rcl} F_1(x,y) & = & \bigvee_{X\subseteq\{1,\cdots,n_1\}} \left([(\bigwedge_{p\in X} a_p) + x] \wedge [(\bigwedge_{q\in\overline{X}} b_q) + y] \right) \\ F_2(x,y) & = & \bigvee_{Y\subseteq\{1,\cdots,n_2\}} \left([(\bigwedge_{r\in Y} c_r) + x] \wedge [(\bigwedge_{u\in\overline{Y}} d_u) + y] \right) \ , \end{array} \tag{32}$$

where X and Y are subsets satisfying the restrictions

$$\begin{array}{rcccl} I_1 & \subseteq & X & \subseteq & \{1,\cdots,n_1\} \backslash I_2 \\ J_1 & \subseteq & Y & \subseteq & \{1,\cdots,n_2\} \backslash J_2 \ . \end{array} \tag{33}$$

It follows that we can construct a rectangular min-representation of F, $T \subseteq \mathsf{MM}(2,2)$, in which each $B \in T$ is indexed by pairs (X,Y) where $X \subseteq \{1,\cdots,n_1\}$ and $Y \subseteq \{1,\cdots,n_2\}$ satisfy the restrictions in (33). The matrix corresponding to (X,Y) is then

$$B(X,Y) = \begin{pmatrix} \bigwedge_{p\in X} a_p & \bigwedge_{q\in\overline{X}} b_q \\ \bigwedge_{r\in Y} c_r & \bigwedge_{u\in\overline{Y}} d_u \end{pmatrix} \ . \tag{34}$$

Our conventions ensure that $B(X,Y)$ is an $\mathbb{R}_{\mathsf{min}}$ matrix, provided the restrictions in (33) are satisified.

This completes the book-keeping. We can now embark on the proof proper.

Proof (of Theorem 4.1) We begin with an observation which sets out the direction which the proof will take.

Lemma 4.1 *With the details above, suppose that for each partition $\{S, T, U\}$ satisfying the restrictions in (30), it is possible to find X, Y satisfying the restrictions in (33) such that*

$$\rho(S,T,U) \leq \eta_1 B(X,Y).$$

Then $R \leq L$.

Proof: With the restrictions in (30) and (33), we have

$$R = \bigvee_{S,T,U} \rho(S,T,U) \leq \bigvee_{X,Y} \eta_1 B(X,Y) = L.$$

□

Choose a partition $\{S, T, U\}$ satisfying (30) and assume to begin with that $T = \emptyset$. It follows from (28) that

$$\rho(S, \emptyset, U) = \bigwedge_{p \in \pi_1 S} a_p \wedge \bigwedge_{u \in \pi_2 U} d_u.$$

Now suppose further that $\pi_2 U = \{1, \cdots, n_2\}$. It follows from (30) that $J_1 = \emptyset$. Let $X = \pi_1(S)$ and $Y = \emptyset$. It follows from (30) and (31) that X, Y satisfy (33). Since $Y = \emptyset$, the corresponding matrix $B(X,Y)$ in (34) has $+\infty$ in the bottom left corner. If $\overline{X} \neq \emptyset$, it follows from (27) that,

$$\eta_1 B(X,Y) = \bigwedge_{p \in X} a_p \wedge \bigwedge_{u \in \overline{Y}} d_u$$

and it is clear that $\rho(S, \emptyset, U) = \eta_1 B(X,Y)$. If $\overline{X} = \emptyset$ then certainly $\rho(S, \emptyset, U) \leq \eta_1 B(X,Y)$ since the latter omits the contribution from U. In either case we are done. Now suppose that $\pi_1 S \neq \{1, \cdots, n_1\}$. It then follows from (30) that $\pi_2 U = \{1, \cdots, n_2\}$ (a picture is quite useful at this point) and we have already done this case. So we may assume that $\pi_1 S = \{1, \cdots, n_1\}$ and hence that $I_2 = \emptyset$. Let $X = \{1, \cdots, n_1\}$, which certainly satisfies (33), and let Y be any subset of $\{1, \cdots, n_2\}$ which satisfies (33). We can always choose such a subset in view of (26). Because $\overline{X} = \emptyset$, it follows from (27) that $\eta_1 B(X,Y) = \bigwedge_{p \in X} a_p$. Hence $\rho(S, \emptyset, U) \leq \eta_1 B(X,Y)$ and once again we are done. This deals with all the possibilities when $T = \emptyset$.

We may now assume that $T \neq \emptyset$. The crux of the proof hinges on the shape of T. The form of $\eta_1 B(X,Y)$, as calculated from (the dual of) (27), shows that T corresponds to the set $\overline{X} \times Y$. This differs from T in being a rectangular subset of $\{1, \cdots, n_1\} \times \{1, \cdots, n_2\}$. This clue gives rise to the argument which follows. The idea is to replace the partition (S, T, U) by a new partition (S', T', U') which still satisfies (30) but for which $\rho(S,T,U) \leq \rho(S', T', U')$. The new partition will have T' rectangular. Such partitions can be dealt with relatively simply. Rectangularisation is thus the crucial step.

Suppose that we can find $u \in \pi_1 T$ and $v \in \pi_2 T$ such that $(u, v) \notin T$. Then either $(u, v) \in S$ or $(u, v) \in U$. If $(u, v) \in S$ then let $D \subseteq \{1, \cdots, n_1\} \times \{1, \cdots, n_2\}$ be the set $D = \{x \in T \mid \pi_1 x = u\}$. Evidently, $D \neq \emptyset$. Construct a new partition $\{S', T', U'\}$ such that $S' = S \cup D$, $T' = T \setminus D$ and $U' = U$. It is clear that this is still a partition of $\{1, \cdots, n_1\} \times \{1, \cdots, n_2\}$. We need to check that it satisfies (30). Since T has got smaller, it follows that T' cannot violate (30) and, of course, U has not changed. As for S, it is easy to see that $\pi_1 S' = \pi_1 S$, so that S also satisfies (30). We thus have a good partition. Furthermore, since T' has got smaller while $\pi_1 S' = \pi_1 S$ and $\pi_2 U' = \pi_2 U$, it is easy to see that $\rho(S,T,U) \leq \rho(S',T',U')$. If $(u,v) \in U$ then we move elements from T to U and a similar argument works. We can now carry on constructing new partitions in this way. Since T is finite and strictly decreases each time, the process can only stop in two ways. Either we end up with $T = \emptyset$, which we have already dealt with, or we find that we can no longer choose (u,v) satisfying the requirements above. But it must then be the case that $T = \pi_1 T \times \pi_2 T$. Hence we may assume that T is non-empty and rectangular. The importance of this stems from the following elementary fact.

Lemma 4.2 *With the above details, if T is rectangular, then*

$$\bigwedge_{(q,r)\in T} (b_q + c_r)/2 = ((\bigwedge_{q\in\pi_1 T} b_q) + (\bigwedge_{r\in\pi_2 T} c_r))/2 \ .$$

Proof: The rectangularity of T implies that

$$\bigwedge_{(q,r)\in T} (b_q + c_r)/2 \quad = \quad \bigwedge_{q\in\pi_1 T} \bigwedge_{r\in\pi_2 T} (b_q + c_r)/2 \ .$$

We can now use the distributivity of $+$ over $\wedge$, twice, to rewrite this as follows:

$$\begin{aligned} &= \bigwedge_{q\in\pi_1 T} (b_q/2 + (\bigwedge_{r\in\pi_1 T} c_r/2)) \\ &= ((\bigwedge_{q\in\pi_1 T} b_q) + (\bigwedge_{r\in\pi_2 T} c_r))/2 \ . \end{aligned}$$

□

The remainder of the argument resembles the case when $T = \emptyset$. Suppose first that $\pi_2 U = \{1, \cdots, n_2\}$ so that $J_1 = \emptyset$. Let $X = \pi_1 S$ and $Y = \emptyset$. As before, these satisfy (33). The corresponding $B(X,Y)$ has $+\infty$ in the bottom left corner. It follows from (27) that $\rho(S,T,U) \leq \eta_1 B(X,Y)$ since the latter simply omits the contribution coming from T, if $\overline{X} \neq \emptyset$, and from both T and U, if $\overline{X} = \emptyset$. Now suppose that $\pi_1 S = \{1, \cdots, n_1\}$ so that $I_2 = \emptyset$. Let $X = \{1, \cdots, n_1\}$, which certainly satisfies (33), and choose any Y which also satisfies (33), which we may always do by (26). The corresponding $B(X,Y)$ has $+\infty$ in the top right corner. It follows from (27) that $\rho(S,T,U) \leq \eta_1 B(X,Y)$ since the latter omits the contributions from both T and U. Now let $\overline{X} = \pi_1 T$. If $X \not\subseteq \pi_1 S$ then it follows from (30) that $\pi_2 U = \{1, \cdots, n_2\}$, which we have already considered. So we may assume that $X \subseteq \pi_1 S$ and so $\bigwedge_{p\in\pi_1 S} a_p \leq \bigwedge_{p\in X} a_p$. Furthermore, it is easy to see that X satisfies (33). Let $Y = \pi_2 T$ and suppose that $\overline{Y} \not\subseteq \pi_2 U$. Then, in a similar way, it must be the case that $\pi_1 S = \{1, \cdots, n_1\}$, which we have also considered. Hence, we may also assume that $\overline{Y} \subseteq \pi_2 U$ and so $\bigwedge_{u\in\pi_2 U} d_u \leq \bigwedge_{u\in\overline{Y}} d_u$. Furthermore, Y also satisfies (33). But now, $\rho(S,T,U) \leq \eta_1 B(X,Y)$ because in the latter the contribution from $\overline{X} \times Y$ is equal to that from T by Lemma 4.2, while the other contributions have got larger. It follows that $R \leq L$.

This completes the proof of Theorem 4.1.

□

A similar argument can be attempted in higher dimensions, albeit at the cost of increased book-keeping. It is not the book-keeping that defeats this, however, so much as the fact that Lemma 4.1 is no longer of any use. There is an example in dimension 3 such that, for a given partition of the form $\{S, T, \cdots, U\}$ (but now requiring 8 entries) there is no single $\mathbb{R}_{\min}$ matrix $B(X, Y, Z)$ for which the hypothesis of Lemma 4.1 holds. Different matrices are required for different values of the parameters, a_i, b_i, etc. It is a convenient accident that this does not happen in dimension 2.

5 Conclusion

The Duality Conjecture remains the main stumbling block to further progress in this area. The methods of this paper suggest two distinct lines of approach to it.

A close reading of the proof of Theorem 4.1 suggests that the interplay between distributivity and rectangularity lies at the heart of the conjecture. We can consider this in a more abstract setting as follows. If P is a partially ordered set, satisfying suitable conditions, we can generate from it a free distributive lattice D. (In our context, $P = \{1, \cdots, n\} \times \mathbb{R}$, which represents terms of the form $x_i + a$. The fact that D corresponds to $\mathsf{MM}(n, 1)$ is a restatement of [21, Theorem 2.1].) The free construction satisfies the property that if $f : P \to \mathbb{R}$ is a montone function on P then f extends uniquely to a morphism of distributive lattices $D \to \mathbb{R}$. Now consider $P \times P$. (In our context we would need to consider $P \times \cdots \times P$, with n factors, but the essential idea is clear with two factors.) It is still the case that $D \times D$ is generated as a distributive lattice by $P \times P$ but it is no longer freely generated. For instance, there are such relations as:

$$(a, b) \vee (c, d) = (a, d) \vee (c, b) \tag{35}$$

where $a, b, c, d \in P$. Given a monotone function $f : P \times P \to \mathbb{R}$, we can ask if f can be extended to a rectangular morphism of distributive lattices $D \times D \to \mathbb{R}$. It can be shown, using essentially the same ideas as in §2, that f can be extended via rectangularity in two possible ways to $D \times D$. One extension, call it f^+, satisfies the first equation of (20) while the other, f^-, satisfies the second. For instance, f^+ would be defined on the element (35) by

$$f^+((a, b) \vee (c, d)) = f(a, b) \vee f(c, d) \vee f(a, d) \vee f(c, b) \ .$$

A necessary and sufficient condition for there to be an extension of f which is a rectangular morphism of distrbutive lattices is that $f^+ = f^-$. Under what circumstances on P and f does this occur?

Although this question appears quite natural in the context of lattice theory, there appears to have been no work done on it. If a general answer were available, we could apply it immediately to test whether or not the Duality Conjecture holds. Although this seems a harder question than the conjecture itself, the abstract setting reveals the fundamental combinatorial problem with greater clarity.

An alternative approach to the conjecture is suggested by the policy improvement methods of §3.1. It can be shown that if H is a max-only function then there exists $\vec{x} \in \mathbb{R}^n$ such that $H^k(\vec{x}) = \vec{x} + k\chi(H)$. We can think of this as a generalised fixed point appropriate for the situation in which $\chi(H) \neq h$. If a similar generalised fixed point existed for an arbitrary min-max function, the policy improvement methods of §3.1 might be adapted to find it, thereby establishing the Duality Conjecture.

We hope to report on progress in these directions in future work.

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