Periodic Orbit Maslov Indices for Systems with Time-Reversal Symmetry

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We show that in time-reversal invariant systems, a pair of periodic orbits related by time-reversal have the same Maslov index. Previously this result had been implicitly assumed in the semiclassical derivation of the Gaussian Orthogonal Ensemble (GOE) spectral form factor.

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That the energy levels of a (spinless) classically chaotic quantum system, invariant under time reversal symmetry, have the same statistics as the eigenvalues of the Gausssian Orthogonal Ensemble (GOE) of random matrices is one of the central results of quantum chaology (Bohigas et al 1984, Berry 1987, Bohigas 1991, Mehta 1991). The purpose of this note is first to point out that the semiclassical derivation of GOE statistics for such systems depends on pairs of time-reversal-related periodic orbits having the same Maslov index (previously this passed as an unnoticed assumption in the argument), and then to give a demonstration of this result.

Throughout we consider a system of N degrees of freedom, defined on cartesian configuration space R^N, whose classical dynamics is chaotic and time-reversal invariant. We make no additional assumptions about the form of the Hamiltonian. Let us consider the spectral form factor

$$K(T) = \frac{1}{\bar{d}} \int_{-\infty}^{\infty} \exp(\mathrm{i}xT/\hbar) \langle d(E+x/2) d(E-x/2) \rangle \mathrm{d}x - 2\pi\hbar\bar{d}\delta(T), \quad (1)$$

the Fourier transform of the two-point correlation function of the density of states $d(E) = \sum_n \delta(E - E_n)$. In (1), $\langle \cdots \rangle$ denotes an energy average, $\bar{d}(E) = \langle d(E) \rangle$ is the mean density of states, and the normalization is chosen so that $K(T) \to 1$ as $T \to \infty$. Following Berry (1985), the semiclassical evaluation of $K(T)$ proceeds by substituting for $d(E)$ the Gutzwiller trace formula (Gutzwiller 1990),

$$\mathrm{d}(E) \approx \bar{d}(E) + \frac{1}{\pi\hbar} \mathrm{Re} \sum_j A_j \exp(\mathrm{i}S_j/\hbar - \mathrm{i}\mu_j\pi/2), \quad (2)$$

where the sum is taken over periodic orbits with energy E (assumed to be isolated and unstable). $S_j(E)$ is the periodic orbit action, $A_j = T_j/|\det(\mathbf{M}_j - \mathbf{I})|^{1/2}$, T_j is the period, $\mathbf{M}_j(E)$ is the Poincaré map linearized about the orbit and μ_j is the Maslov index. From (1) and (2),

$$K(T) = \frac{1}{T_H} \sum_{j,k} \left\langle A_j A_k \exp(\mathrm{i}\{S_j - S_k\}/\hbar - \mathrm{i}\{\mu_j - \mu_k\}\pi/2) \delta\left(T - \frac{T_j + T_k}{2}\right) \right\rangle_E, \quad (3)$$

where $T_H = 2\pi\hbar\bar{d}$ is the Heisenberg time.

Next, we restrict the sum in (3) to the diagonal ($S_j = S_k$) terms. These terms dominate the expression for $K(T)$ for $T \ll T_H$, and as shown by Bogolmolny & Keating (1996), the contributions from the off-diagonal terms, which become important for larger values of T, can be evaluated to leading order in terms of the diagonal ones. For systems with time-reversal symmetry, there are generically two kinds of diagonal terms. In the first, the labels j and k refer to the same orbit, and so $A_j = A_k$, $T_j = T_k$ and $\mu_j = \mu_k$. In the second, the orbit labelled k is the time-reverse of the orbit labelled j; we denote this by writing $k = \bar{j}$. Then $A_j = A_{\bar{j}}$ and $T_j = T_{\bar{j}}$. The diagonal contribution may thus be written as

$$K^{(diag)}(T) = \frac{1}{T_H} \sum_j \left\langle A_j^2 (1 + \exp(-\mathrm{i}(\mu_j - \mu_{\bar{j}})\pi/2)) \delta(T - T_j) \right\rangle_E. \quad (4)$$

If we take $\mu_{\bar{j}} = \mu_j$, the GOE form factor $K^{GOE}(T) \approx 2T/T_H$ for $T << T_H$ is then recovered from the Hannay-Ozorio de Almeida (1984) sum rule,

$$\sum_j \left\langle A_j^2 \delta(T - T_j) \right\rangle_E \approx T. \tag{5}$$

In previous discussions, the fact that a periodic orbit and its time-reverse have the same Maslov index appears to have been implicitly assumed. Here we give an explicit demonstration. This will be based on the following topological characterization of the trace formula Maslov index (Creagh *et al* 1990, Robbins 1991).

Associated to the unstable periodic orbit $\mathbf{Z}_j(t)$ are its N-dimensional stable and unstable manifolds $W_j^{s,u}$, consisting of points which approach the orbit asymptotically in forward and backward time respectively. The N-dimensional planes tangent to the stable and unstable manifolds at $\mathbf{Z}_j(t)$, which we denote by $\lambda_j^s(t)$ and $\lambda_j^u(t)$, are Lagrangian planes; that is, if $\delta\mathbf{z}_1 = (\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)$ and $\delta\mathbf{z}_2 = (\boldsymbol{\xi}_2, \boldsymbol{\eta}_2)$ are two vectors in $\lambda_j^s(t)$ for example (here the $\boldsymbol{\xi}$'s and $\boldsymbol{\eta}$'s denote the q and p components of the $\delta\mathbf{z}$'s), then $\boldsymbol{\xi}_1 \cdot \boldsymbol{\eta}_2 = \boldsymbol{\xi}_2 \cdot \boldsymbol{\eta}_1$. Therefore, over a period T, $\lambda_j^s(t)$ and $\lambda_j^u(t)$ describe closed curves in the space of N-dimensional Lagrangian planes. The space of Lagrangian planes $\Lambda(N)$ has nontrivial topology; in particular, closed curves in $\Lambda(N)$ can be classified by an integer winding number. A calculation shows that the winding numbers $\mathrm{wn}\,\lambda_j^s(t)$ and $\mathrm{wn}\,\lambda_j^u(t)$ (which turn out to be the same) are just the Maslov indices μ_j which appear in the trace formula (2), so that

$$\mu_j = \mathrm{wn}\,\lambda_j^s(t) = \mathrm{wn}\,\lambda_j^u(t). \tag{6}$$

We shall make use of the following explicit formula for the winding number (Robbins 1992). Let the vectors $(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)(t), \ldots (\boldsymbol{\xi}_N, \boldsymbol{\eta}_N)(t)$ comprise a basis for $\lambda_j^s(t)$. For present purposes it is convenient to stipulate that these vectors be periodic themselves (although in general they need not be). Consider the complex N-dimensional matrix

$$L_{\alpha\beta}(t) = \xi_{\alpha\beta}(t) + \mathrm{i}\eta_{\alpha\beta}(t) \tag{7}$$

(ie, the matrix whose rows are the vectors $\boldsymbol{\xi}_\alpha(t) + \mathrm{i}\boldsymbol{\eta}_\alpha(t)$). The phase of the determinant $\det \mathbf{L}(t)$ is periodic, and $\mathrm{wn}\,\lambda_j^s(t)$ is just the number of times $\det \mathbf{L}(t)$ encircles the origin of the complex plane in the counterclockwise sense. Explicitly,

$$\mathrm{wn}\,\lambda_j^s(t) = \frac{1}{2\pi}\mathrm{Im} \int_0^T \frac{d}{dt} \ln \det \mathbf{L}(t)\, dt. \tag{8}$$

Now consider a Hamiltonian invariant under time reversal $\mathbf{z} = (\mathbf{q}, \mathbf{p}) \rightarrow \overline{\mathbf{z}} = (\mathbf{q}, -\mathbf{p})$. This implies the Hamiltonian flow $\Phi_t(\mathbf{z})$ satisfies

$$\overline{\Phi}_t(\mathbf{z}) = \Phi_{-t}(\overline{\mathbf{z}}), \tag{9}$$

so that if the t-origins of the unstable periodic orbits $\mathbf{Z}_j(t)$ and $\mathbf{Z}_{\bar{j}}(t)$ are appropriately chosen, then

$$\mathbf{Z}_{\bar{j}}(t) = \overline{\mathbf{Z}}_j(-t). \tag{10}$$

From the definition of the stable and unstable manifolds, it is obvious that (9) also implies that $W_{\bar{j}}^{s,u} = \overline{W_j^{u,s}}$. Therefore

$$\lambda_{\bar{j}}^{s,u}(t) = \overline{\lambda_j^{u,s}}(-t), \tag{11}$$

where, in general, the time-reverse $\overline{\lambda}$ of a Lagrangian plane λ is obtained by changing the sign of the momentum components of the vectors in λ. From (6), (7), (8) and (11) we have that

$$\begin{aligned}
\mu_{\bar{j}} = \mathrm{wn}\,\lambda_{\bar{j}}^{u}(t) &= \mathrm{wn}\,\overline{\lambda_j^{s}}(-t) \\
&= \frac{1}{2\pi}\mathrm{Im}\int_0^T \frac{d}{dt}\ln\det\mathbf{L}^*(-t)\,dt \\
&= -\frac{1}{2\pi}\mathrm{Im}\int_0^T \frac{d}{dt'}\ln\det\mathbf{L}^*(t')\,dt' \\
&= \frac{1}{2\pi}\mathrm{Im}\int_0^T \frac{d}{dt'}\ln\det\mathbf{L}(t')\,dt' = \mathrm{wn}\,\lambda_j^{s}(t) = \mu_j
\end{aligned} \tag{12}$$

(in the substitution $t' = -t$ we have used the periodicity of $\mathbf{L}(t)$), which gives the desired result.

The equality of the Maslov indices μ_j and $\mu_{\bar{j}}$ can be understood to follow from the cancellation of two sign factors. The first is due to the change in the sense of traversal of the time-reversed orbit (cf (10) and (11)). The second is due to the effect of time-reversal on the space of Lagrangian planes. The transformation $\lambda \mapsto \overline{\lambda}$ defines a continuous map with continuous inverse (ie, a homeomorphism) on $\Lambda(N)$, and therefore induces an automorphism on its fundamental group, namely the integer winding numbers. There are just two automorphisms of the integers, namely $n \mapsto n$ and $n \mapsto -n$, so on general grounds we can expect that under $\lambda(t) \mapsto \overline{\lambda}(t)$, either all the winding numbers remain the same or they all change sign. The above calculation shows it is the second alternative which holds.

The result generalizes to anticanonical symmetries (Robnik and Berry 1986). These are the classical analogues of antiunitary symmetries — an anticanonical transformation Ψ is obtained by composing time-reversal with a canonical transformation Φ, so that $\Psi(\mathbf{z}) = \Phi(\overline{\mathbf{z}})$. If the Hamiltonian is invariant under Ψ, and $\mathbf{Z}_j(t)$ and $\mathbf{Z}_{\bar{j}}(t)$ are two unstable periodic orbits related by $\mathbf{Z}_{\bar{j}}(t) = \Psi(\mathbf{Z}_j(-t))$, then their Maslov indices μ_j and $\mu_{\bar{j}}$ are equal. This follows by noting that $\lambda_{\bar{j}}^{u}(t) = D\Phi(\mathbf{Z}_j(t))\cdot\overline{\lambda_j^{s}}(-t)$, where $D\Phi$ is the tangent map of Φ. The calculation (12) shows that $\mathrm{wn}\,\lambda_j^{s}(t) = \mathrm{wn}\,\overline{\lambda_j^{s}}(-t)$, while $\mathrm{wn}\,\overline{\lambda_j^{s}}(-t) = \mathrm{wn}\,\{D\Phi(\mathbf{Z}_j(t))\cdot\overline{\lambda_j^{s}}(-t)\}$ follows from the invariance of the winding number under canonical transformations (see, eg, Littlejohn and Robbins (1987)).

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