Indistinguishability for Quantum Particles: Spin, Statistics and the Geometric Phase

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The quantum mechanics of two identical particles with spin S in three dimensions is reformulated by employing not the usual fixed spin basis but a transported spin basis that exchanges the spins along with the positions. Such a basis, required to be smooth and parallel-transported, can be generated by an 'exchange rotation' operator resembling angular momentum. This is constructed from the four harmonic oscillators from which the two spins are made according to Schwinger's scheme. It emerges automatically that the phase factor accompanying spin exchange with the transported basis is just the Pauli sign, that is $(-1)^{2s}$

Singlevaluedness of the total wavefunction, involving the transported basis, then implies the correct relation between spin and statistics. The Pauli sign is a geometric phase factor of topological origin, associated with noncontractible circuits in the doubly-connected (and nonorientable) configuration space of relative positions with identified antipodes. The theory extends to more than two particles.

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1. Introduction

The status of the relation between the spin and the statistics of identical particles in nonrelativistic quantum mechanics has been unsatisfactory, for three reasons. First, "It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation... This probably means that we do not have a complete understanding of the fundamental principle involved" (Feynman et al., 1965, p4-3). Second, it represents a departure from the principle in the simplest form of quantum mechanics that indistinguishable situations (e.g. positions with angle coordinates ϕ and $\phi+2\pi$) are described by the same wavefunction: for identical fermions, the exchanged and unexchanged states differ by a sign. (Of course it is always possible to use multivalued wavefunctions, but at the price of introducing gauge potentials into the Hamiltonian.) Third, the existing proofs incorporate ideas beyond elementary quantum mechanics: relativistic quantum field theory (Pauli, 1940, Streater and Wightman, 1964), topological solitons (Finkelstein and Rubinstein, 1968, Mickelson, 1984), or the existence of antiparticles (Tscheuschner, 1989, Balachandran et al., 1993).

Here we will argue that "the fundamental principle" that Feynman sought is the correct incorporation of identity into an augmented quantum kinematics in which the space of wavefunctions has, built into it, the indistinguishability of states related by exchange of positions and spins. When this is done, the physics of exchange emerges naturally from the nonrelativistic Schrödinger equation with singlevalued wavefunctions (§3). The construction of a configuration space in which exchanged configurations are identified (e.g. the points $(\mathbf{r}_1, \mathbf{r}_2)$ and $(\mathbf{r}_2, \mathbf{r}_1)$) is not a new idea (Leinaas and Myrheim, 1977, Laidlaw and DeWitt, 1971), but we complete it by incorporating spin (§§2,4). In order to accomplish this, the spin must be embedded into a larger Hilbert space. The unexpected result is that for particles with spin quantum number S the exchange ('Pauli') sign $(-1)^{2S}$ emerges automatically, as an unfamiliar type of geometric phase (§5).

Our essentially three-dimensional argument was inspired by, and can be regarded as a mathematization of, a well known geometrical trick with a belt (Hartung, 1979, Feynman, 1987, Gould, 1995, Guerra and

Marra, 1984), suggestive of the Pauli principle. Consider first a single object, tied to one end of a belt, with the other end held fixed. If the object is turned by 4π (about any axis) this introduces a double twist in the belt, which can however be eliminated by translating the object with its orientation fixed. Such 'tethered' rotations, where an even number of turns is equivalent to no turns but an odd number is not, are regarded as analogous to fermions, where the wavefunction changes sign after one turn. (Ordinary untethered objects, for which any number of turns is equivalent to no turns, are analogous to bosons.) Now consider two objects, tethered to each other by a belt. Exchanging them introduces a twist into the belt, which can be eliminated by turning one of them once. This suggests that exchange of identical fermions is equivalent to a single turn of one of them, that is to a sign change.

We consider the essence of the connection between spin and statistics to lie in the exchange of two particles, and our argument will be presented in detail for this case. However the central ideas generalize to permutations of N particles, as will be explained in §6.

2. Transported spin basis

The wavefunction for two identical particles with spin S depends on their positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Exchange involves the vector $\mathbf{r} \equiv \mathbf{r}_2-\mathbf{r}_1$ of relative position - it is unnecessary to consider the centre of mass vector. Under exchange of positions, $\mathbf{r}$ becomes $-\mathbf{r}$. The spins must be exchanged as well. We will describe spin states of the two particles with the quantum numbers m_1 and m_2 ($|m_1, m_2| \leq S$) representing their spin components in the z direction, and employ the convenient notations

$$M \equiv \{m_1, m_2\}, \quad \overline{M} \equiv \{m_2, m_1\} \tag{1}$$

To incorporate spin exchange, we employ an $\mathbf{r}$-dependent ('transported') basis $|M(\mathbf{r})\rangle$ rotated (in a sense to be described in §4) from the commonly-employed fixed basis $|M\rangle$. Thus

$$|M(\mathbf{r})\rangle = \mathsf{U}(\mathbf{r})|M\rangle \tag{2}$$

where $\mathsf{U}(\mathbf{r})$ is a unitary operator. The notations $|M(\mathbf{r})\rangle$ and $\mathsf{U}(\mathbf{r})$ imply that the basis is uniquely determined by **r**; we are therefore excluding the inconvenience of multivalued bases.

We require the basis (2) to possess the following properties.

a. Smoothness: the basis must be a smooth and nonsingular function for all $\mathbf{r}\neq 0$, e.g. there must be no Dirac strings.

b. Exchange:

$$|M(-\mathbf{r})\rangle = (-1)^K |\overline{M}(\mathbf{r})\rangle \tag{3}$$

where K is an integer. It is possible to envisage more general phase factors, but the restriction to a sign is forced by the fact that the basis is uniquely determined by **r**, applied after a double exchange. The sign cannot depend on **r**, because this would imply discontinuities (where the sign switches), and it cannot depend on M because (as an easy argument shows) this would imply a preferred quantization axis: for example, eigenstates of the x component of spin would not be exchanged.

c. Parallel transport:

$$\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = 0 \tag{4}$$

for arbitrary values of the quantum numbers M, M'. This is the simplest rule for the transport of spins. With parallel transport, and the fact that the basis is a function of **r**, there are no local geometric phases (abelian or nonabelian) associated with contractible circuits of **r**.

It is important to emphasize at this point that parallel transport rules out the possibility of constructing operators $\mathsf{U}(\mathbf{r})$ that generate exchange according to (2) and (3) by using just the usual spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$. The reason is that then the states $|M(\mathbf{r})\rangle$ in (2) would span the whole Hilbert space of spins, and (4) would imply that $\mathsf{U}(\mathbf{r})$ is constant. Therefore we will need to work with a representation of spin that incorporates the usual one but is larger, in that it allows additional operations that can generate exchange while preserving parallel transport ('flat exchange').

In §4 we shall construct a transported basis that satisfies these requirements. It is far from obvious *a priori* that this can be done - for example, it is impossible to satisfy the analogue of a for the eigenstates $|m(\mathbf{r})\rangle$ of $\mathbf{r}\,.\mathbf{S}$ representing the ordinary rotation of a single spin. Having constructed the basis we will derive (in §4) the centrally important relation

$$K = 2S\,, \quad \text{i.e. } |\overline{M}(-\mathbf{r})\rangle = (-1)^{2S}|M(\mathbf{r})\rangle \tag{5}$$

In a sense this encapsulates the belt trick, because it asserts that $\mathbf{r} \to -\mathbf{r}$ and $M \to \overline{M}$, that is exchange of positions and spins, is equivalent to the sign change $(-1)^{2S}$ from the rotation of one spin.

3. Identifications

With the spins thus attached to the positions, we can represent any spin state $|\Psi(\mathbf{r})\rangle$ of the two particles by the $(2S+1)^2$-dimensional vector $\psi_M(\mathbf{r})$, where

$$|\Psi(\mathbf{r})\rangle = \sum_{m_1, m_2} \psi_M(\mathbf{r})|M(\mathbf{r})\rangle \tag{6}$$

Now we must identify the points $\mathbf{r}$ and $-\mathbf{r}$, since these correspond to complete interchange of the particles (positions and spins) and so are indistinguishable. Singlevaluedness of the wavefunction - applied here as elsewhere in quantum mechanics - requires

$$|\Psi(\mathbf{r})\rangle = |\Psi(-\mathbf{r})\rangle \tag{7}$$

Now, from (5),

$$\begin{aligned} |\Psi(-\mathbf{r})\rangle &= \sum_M \psi_M(-\mathbf{r})(-1)^{2S}|\overline{M}(\mathbf{r})\rangle \\ &= \sum_M \psi_{\overline{M}}(-\mathbf{r})(-1)^{2S}|M(\mathbf{r})\rangle \end{aligned} \tag{8}$$

so that singlevaluedness implies

$$\psi_{\overline{M}}(-\mathbf{r}) = (-1)^{2S}\psi_M(\mathbf{r}) \tag{9}$$

This resembles the usual spin-statistics relation. However, before we can assert that it is the same as the usual relation we must show that the coefficients

$$\psi_M(\mathbf{r}) = \langle M(\mathbf{r})|\Psi(\mathbf{r})\rangle \tag{10}$$

in (6) satisfy the same Schrödinger equation as the coefficients in the fixed basis, where the wavefunctions are

$$|\Psi(\mathbf{r})\rangle_{\text{fixed}} = \sum_M \psi_{M,\text{fixed}}(\mathbf{r})|M\rangle \tag{11}$$

Thus

$$\psi_{M,\text{fixed}}(\mathbf{r}) = \langle M|\Psi(\mathbf{r})\rangle_{\text{fixed}} \tag{12}$$

In order to show that the quantities defined by (10) and (12) are the same, we must first define dynamical variables (e.g. momentum and spin) in the transported basis. Of course, these must satisfy the same commutation relations as in the fixed basis. This can be accomplished by generating the transported dynamical variables from their counterparts in the fixed basis by the same unitary transformation $\mathsf{U}(\mathbf{r})$ that generates the basis itself. In particular, the momentum operator, which in the fixed basis has the usual form

$$\mathbf{P}_{\text{fixed}} = -i\hbar\nabla \tag{13}$$

becomes, in the transported basis

$$\mathbf{P}(\mathbf{r}) = \mathsf{U}(\mathbf{r})\mathbf{P}_{\text{fixed}}\mathsf{U}^\dagger(\mathbf{r}) \tag{14}$$

Similarly, the transported spin operators $\mathbf{S}(\mathbf{r})$ (=$\{\mathbf{S}_1, \mathbf{S}_2\}$) will be defined in terms of the usual fixed spins $\mathbf{S}_{\text{fixed}}$ by

$$\mathbf{S}(\mathbf{r}) = \mathsf{U}(\mathbf{r})\mathbf{S}_{\text{fixed}}\mathsf{U}^\dagger(\mathbf{r}) \tag{15}$$

$\mathbf{P}(\mathbf{r})$ and $\mathbf{S}(\mathbf{r})$ must be employed instead of $\mathbf{P}_{\text{fixed}}$ and $\mathbf{S}_{\text{fixed}}$ when constructing the Hamiltonian to express the Schrödinger equation in the transported basis. For the momenta, an easy calculation leads to

$$\langle M(\mathbf{r})|\mathbf{P}(\mathbf{r})|\Psi(\mathbf{r})\rangle = -i\hbar\nabla\psi_M(\mathbf{r}) \tag{16}$$

while of course

$$\langle M|\mathbf{P}_{\text{fixed}}|\Psi(\mathbf{r})\rangle_{\text{fixed}} = -i\hbar\nabla\psi_{M,\text{fixed}}(\mathbf{r}) \tag{17}$$

and similarly for spins. Therefore the 'transported' and 'fixed' quantities defined by (10) and (12) do satisfy the same Schrödinger equation (including boundary conditions) and so are the same function. It follows that (9) is indeed the usual spin-statistics relation, here derived by requiring the wavefunction to be singlevalued. In effect, we have shown that although $|\Psi(\mathbf{r})\rangle_{\text{fixed}}$ need not be singlevalued under exchange, $|\Psi(\mathbf{r})\rangle = \mathrm{U}(\mathbf{r})|\Psi(\mathbf{r})\rangle_{\text{fixed}}$ must be.

Mathematically, what we are doing is setting up quantum mechanics on a 'two-spin bundle', whose six-dimensional base is the configuration space $\mathbf{r}_1$, $\mathbf{r}_2$ with exchanged configurations identified and coincidences $\mathbf{r}_1=\mathbf{r}_2$ excluded (to make the base a manifold). The fibres are the two-spin Hilbert spaces spanned by the transported basis $|M(\mathbf{r})\rangle$. The full Hilbert space consists of global sections of the bundle, i.e. singlevalued wavefunctions. The base manifold has nontrivial topology. It can be regarded as the product of the centre of mass with the space of relative coordinates $\mathbf{r}$, with the latter parameterized by the separation distance $r\equiv|\mathbf{r}|>0$ and a point on the projective plane (two-sphere with identified antipodes) of relative directions. Exchanges of positions correspond to noncontractible closed loops in this (nonorientable) configuration space.

4. Exchange rotation

As we saw in §2, we need to construct an enlarged representation of spin that incorporates the usual one but allows additional operations that can generate flat exchange. To the extent that these additional operations are unphysical (e.g. by allowing the spins of the two particles to differ) they must be unobservable: their only role is to accomplish the exchange.

This can be achieved, and the exchange sign calculated, by adapting Schwinger's representation of spin in terms of harmonic oscillators (Schwinger, 1965, Sakurai, 1994). For a single spin, two independent (that is, commuting) oscillators are required - the *a* oscillator, with annihilation and creation operators a and $a^\dagger$, and the b oscillator, with b and $b^\dagger$. From these can be constructed $\mathbf{S}=(S_x, S_y, S_z)$:

$$\mathbf{S} = \tfrac{1}{2}\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} a \\ b \end{pmatrix},$$

$$\text{i.e. } S_z = \tfrac{1}{2}\left(a^\dagger a - b^\dagger b\right), \tag{18}$$

$$S_+ \equiv S_x + iS_y = a^\dagger b, \quad S_- \equiv S_x - iS_y = b^\dagger a$$

where $\boldsymbol{\sigma}$ denotes the vector of Pauli matrices. The components of $\mathbf{S}$ satisfy the commutation rules for angular momentum:

$$\mathbf{S} \times \mathbf{S} = i\mathbf{S} \tag{19}$$

(Here and hereafter we omit $\hbar$ in expressions involving spins, so that, for example, the eigenvalues of S_z are spin quantum numbers - integer or half-integer - rather than dynamical spins.) In this representation, the eigenstates of $\mathbf{S}^2$ and S_z, with quantum numbers S and m, are number states of the oscillators: if there are n_a quanta in the a oscillator and n_b quanta in the b oscillator, then it follows from (18) that

$$S = \tfrac{1}{2}\left(n_a + n_b\right), \quad m = \tfrac{1}{2}\left(n_a - n_b\right) \tag{20}$$

For two spins, we require four oscillators: a_1, b_1, a_2, b_2. The individual spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$ are constructed by analogy with (18). To create exchange, we mix the 1 and 2 oscillators rather than the a and b oscillators. The rationale behind this is that since an ordinary spin rotation from z to $-z$, generated by $\mathbf{S}$, changes the sign of the m quantum number and so, by (20), interchanges the quanta in the a and b oscillators, so rotations generated by operators where a and b are replaced by 1 and 2 will interchange the spins. There are two ways of mixing 1 and 2, involving the a and b operators separately, yielding the operator $\mathbf{E}_a$, given by

$$\mathbf{E}_a = \tfrac{1}{2}\begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

$$\text{i.e. } E_{az} = \tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \tag{21}$$

$$E_{a+} \equiv E_{ax} + iE_{ay} = a_1^\dagger a_2, \quad E_{a-} \equiv E_{ax} - iE_{ay} = a_2^\dagger a_1$$

and similarly the operator $\mathbf{E}_b$. Obviously $[\mathbf{E}_a, \mathbf{E}_b]=0$. The components of $\mathbf{E}_a$ satisfy angular momentum commutation relations, as do those of $\mathbf{E}_b$. The linear combination

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b \tag{22}$$

uniquely shares this property, namely

$$\mathbf{E} \times \mathbf{E} = \mathrm{i}\mathbf{E} \tag{23}$$

Moreover, by elementary calculations, it can be shown that

$$[\mathrm{E}_z, \mathbf{S}_1] = 0, \quad [\mathrm{E}_z, \mathbf{S}_2] = 0, \quad [\mathbf{E}, \mathbf{S}_{\mathrm{tot}}] = 0 \tag{24}$$

where $\mathbf{S}_{\mathrm{tot}}=\mathbf{S}_1+\mathbf{S}_2$. However, $\mathbf{S}_1$ and $\mathbf{S}_2$ do not commute with $\mathrm{E}_\pm=\mathrm{E}_x\pm\mathrm{i}\mathrm{E}_y$, and nor do S_1^2 and S_2^2. In addition, one can show that

$$\mathbf{E}^2 = \mathbf{S}_{\mathrm{tot}}^2 \tag{25}$$

We will call $\mathbf{E}$ *exchange angular momentum*, because the group of rotations it generates - *exchange rotations* - can be chosen to satisfy the requirement (3). We begin with a simple, geometrically motivated, construction: as the line joining the particles is turned from z to $\mathbf{r}$, the two-spin state is turned by the corresponding exchange rotation. A symmetrical choice for the turn is about an axis $\mathbf{n}(\mathbf{r})$ perpendicular to both $\mathbf{e}_z$ and $\mathbf{r}$. If (θ, ϕ) are the polar angles of the direction of $\mathbf{r}$, then

$$\mathbf{n}(\mathbf{r}) = -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi \tag{26}$$

With this choice, we let the transported basis be generated by

$$\mathrm{U}(\mathbf{r}) = \exp\{-\mathrm{i}\theta\mathbf{n}(\mathbf{r})\cdot\mathbf{E}\} \tag{27}$$

Now consider the state of the two spins corresponding to n_{1a}, n_{1b},n_{2a},n_{2b}, namely

$$|n_{1a}, n_{2a}, n_{1b}, n_{2b}\rangle = C\left(\mathrm{a}_1^\dagger\right)^{n_{1a}}\left(\mathrm{a}_2^\dagger\right)^{n_{2a}}\left(\mathrm{b}_1^\dagger\right)^{n_{1b}}\left(\mathrm{b}_2^\dagger\right)^{n_{2b}}|0,0,0,0\rangle \tag{28}$$

where C is a normalization constant. By an obvious extension of (20), this can be written in terms of the more familiar quantum numbers for the individual spins:

$$\begin{aligned} |n_{1a}, n_{2a}, n_{1b}, n_{2b}\rangle &= |S_1 + m_1, S_2 + m_2, S_1 - m_1, S_2 - m_2\rangle \\ &\equiv |S_1, S_2; M\rangle \end{aligned} \tag{29}$$

It then follows from (21) and (22) that

$$\mathsf{E}_z |S_1, S_2; M\rangle = (S_1 - S_2)|S_1, S_2; M\rangle \tag{30}$$

For identical spins, $S_1=S_2=S$, and when S need not be written explicitly we will revert to our previous notation, namely

$$|S, S; M\rangle \equiv |M\rangle \tag{31}$$

From (30), we find, for identical spins, the important result

$$\mathsf{E}_z |M\rangle = 0 \tag{32}$$

This ensures gauge invariance of U: any ineffective rotations about the z axis, applied before U is used to generate the transported basis from the fixed basis, will not introduce phase factors, since

$$\exp(-\mathrm{i}\alpha(\mathbf{r})\mathsf{E}_z)|M\rangle = |M\rangle \tag{33}$$

Now we have to show that U does indeed generate spin exchange according to (3). To evaluate the effect of the exchange rotation operator (27), we first note that

$$\mathsf{U}(\mathbf{r}) = \exp\{-\mathrm{i}\theta\mathbf{n}(\mathbf{r})\cdot\mathbf{E}_a\}\exp\{-\mathrm{i}\theta\mathbf{n}(\mathbf{r})\cdot\mathbf{E}_b\} \equiv \mathsf{U}_a(\mathbf{r})\mathsf{U}_b(\mathbf{r}) \tag{34}$$

and U_a and U_b commute, so we can consider their actions separately. For this we use the fact that in the Schwinger representation the actions of U_a and U_b on states of arbitrary spin can be evaluated in terms of 2x2 matrices multiplying the vectors of creation operators. Thus $\mathsf{U}_a(\mathbf{r})$ induces the transformation

$$\begin{aligned} \begin{pmatrix} \mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger \end{pmatrix} \to \begin{pmatrix} \mathsf{a}'^\dagger_1 & \mathsf{a}'^\dagger_2 \end{pmatrix} &= \mathsf{U}_a(\mathbf{r})\begin{pmatrix} \mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger \end{pmatrix}\mathsf{U}_a^\dagger(\mathbf{r}) \\ &= \begin{pmatrix} \mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger \end{pmatrix}\exp(-\tfrac{1}{2}\mathrm{i}\theta\mathbf{n}(\mathbf{r})\cdot\boldsymbol{\sigma}) \\ &= \begin{pmatrix} \mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger \end{pmatrix}\begin{pmatrix} \cos\tfrac{1}{2}\theta & -\exp(\text{-}\mathrm{i}\phi)\sin\tfrac{1}{2}\theta \\ \exp(\mathrm{i}\phi)\sin\tfrac{1}{2}\theta & \cos\tfrac{1}{2}\theta \end{pmatrix} \end{aligned} \tag{35}$$

and similarly for U_b.

It is instructive first to allow $\mathsf{U}(\mathbf{r})$ to act on the general number state (28), where the spins need not be the same. From (34) and (35),

$$\begin{aligned}
&\left(\mathsf{a}_1^\dagger\right)^{n_{1a}}\left(\mathsf{a}_2^\dagger\right)^{n_{2a}}\left(\mathsf{b}_1^\dagger\right)^{n_{1b}}\left(\mathsf{b}_2^\dagger\right)^{n_{2b}} \to \\
&\left(\cos\tfrac{1}{2}\theta\,\mathsf{a}_1^\dagger + \exp(\mathrm{i}\phi)\sin\tfrac{1}{2}\theta\,\mathsf{a}_2^\dagger\right)^{n_{1a}} \times \\
&\left(-\exp(\text{-}\mathrm{i}\phi)\sin\tfrac{1}{2}\theta\,\mathsf{a}_1^\dagger + \cos\tfrac{1}{2}\theta\,\mathsf{a}_2^\dagger\right)^{n_{2a}} \times \\
&\left(\cos\tfrac{1}{2}\theta\,\mathsf{b}_1^\dagger + \exp(\mathrm{i}\phi)\sin\tfrac{1}{2}\theta\,\mathsf{b}_2^\dagger\right)^{n_{1b}} \times \\
&\left(-\exp(\text{-}\mathrm{i}\phi)\sin\tfrac{1}{2}\theta\,\mathsf{b}_1^\dagger + \cos\tfrac{1}{2}\theta\,\mathsf{b}_2^\dagger\right)^{n_{2b}}
\end{aligned} \tag{36}$$

Similarly, under the action of $\mathsf{U}(\text{-}\mathbf{r})$, where θ is replaced by $\pi\text{-}\theta$ and ϕ by $\phi+\pi$,

$$\begin{aligned}
&\left(\mathsf{a}_1^\dagger\right)^{n_{1a}}\left(\mathsf{a}_2^\dagger\right)^{n_{2a}}\left(\mathsf{b}_1^\dagger\right)^{n_{1b}}\left(\mathsf{b}_2^\dagger\right)^{n_{2b}} \to \\
&\left(\sin\tfrac{1}{2}\theta\,\mathsf{a}_1^\dagger - \exp(\mathrm{i}\phi)\cos\tfrac{1}{2}\theta\,\mathsf{a}_2^\dagger\right)^{n_{1a}} \times \\
&\left(\exp(\text{-}\mathrm{i}\phi)\cos\tfrac{1}{2}\theta\,\mathsf{a}_1^\dagger + \sin\tfrac{1}{2}\theta\,\mathsf{a}_2^\dagger\right)^{n_{2a}} \times \\
&\left(\sin\tfrac{1}{2}\theta\,\mathsf{b}_1^\dagger - \exp(\mathrm{i}\phi)\cos\tfrac{1}{2}\theta\,\mathsf{b}_2^\dagger\right)^{n_{1b}} \times \\
&\left(\exp(\text{-}\mathrm{i}\phi)\cos\tfrac{1}{2}\theta\,\mathsf{b}_1^\dagger + \sin\tfrac{1}{2}\theta\,\mathsf{b}_2^\dagger\right)^{n_{2b}}
\end{aligned} \tag{37}$$

Common factors in (36) and (37) can be identified by pulling out phase factors, and there follows, in an obvious extension of our previous notation for the transported basis,

$$\begin{aligned}
&\left|n_{1a},n_{2a},n_{1b},n_{2b}(\mathbf{r})\right\rangle \equiv \exp\{-\mathrm{i}\theta\,\mathbf{n}(\mathbf{r})\cdot\mathbf{E}\}\left|n_{1a},n_{2a},n_{1b},n_{2b}\right\rangle \\
&= (-1)^{n_{2a}+n_{2b}}\left(\exp(\mathrm{i}\phi)\right)^{\left(-n_{2a}-n_{2b}+n_{1a}+n_{1b}\right)}\left|n_{2a},n_{1a},n_{2b},n_{1b}(-\mathbf{r})\right\rangle
\end{aligned} \tag{38}$$

showing that the operator $\mathbf{E}$ does indeed generate exchange of spins.

An alternative way of writing (38) is (cf. 29)

$$\left|S_1,S_2;M(\mathbf{r})\right\rangle = (-1)^{2S_2}\exp\{2\mathrm{i}(S_1-S_2)\phi\}\left|S_2,S_1;\overline{M}(-\mathbf{r})\right\rangle \tag{39}$$

from which the desired exchange relation (3) follows at once on setting S_1=S_2=S, as does the Pauli sign (5). This sign can be regarded as arising from the rotation of the second spin by 2π (if ϕ=0), or equivalently from the rotation of the first spin by 2π (if $\phi=\pi$), or equivalently from the rotation of both spins by π (as in the simplest version of the argument relating exchange to spin rotation (Feynman, 1987)).

Similar techniques establish that the transported basis $|M(\mathbf{r})\rangle$ is a smooth function of $\mathbf{r}$, notwithstanding the Dirac string singularity in $\mathsf{U}(\mathbf{r})$ at the south pole, arising from the half-angles in (36) (details are given in appendix A), and that it is parallel-transported according to (4) (details are given in appendix B). Therefore all the requirements laid down in §2 are satisfied by the transported basis generated by the exchange angular momentum $\mathbf{E}$ defined by (21) and (22).

In the foregoing analysis, the Pauli sign was derived using the particular choice (27) of $\mathsf{U}(\mathbf{r})$. In fact, this sign is a consequence of any exchange rotation satisfying the conditions a and b in §2 (the parallel-transport condition is automatically satisfied); the argument is given in appendix C. Thus the implications of the conditions are essentially topological. (An example of a more general exchange rotation is (27), multiplied on the left by a smooth exchange rotation even in $\mathbf{r}$.)

As we have seen, parallel transport implies that the spin space must be enlarged. If such enlargement is abandoned (and with it flat exchange), a 'counterconstruction' can be devised, satisfying the conditions a and b in §2 but leading to bizarre relations between spin and statistics, that depend on the value of spin in the four classes S(mod 2). The counterconstruction is described in appendix D.

Now we elucidate the meaning of the relation (32) by writing it in the transported basis. Since

$$\mathbf{E}\cdot\mathbf{r}/r = \mathsf{U}(\mathbf{r})\mathsf{E}_z\mathsf{U}^\dagger(\mathbf{r}) \tag{40}$$

(32) implies

$$\mathbf{E}\cdot\mathbf{r}|M(\mathbf{r})\rangle = 0 \tag{41}$$

Therefore $\mathbf{E}\cdot\mathbf{r}$ annihilates the transported basis states, that is, the transported basis lies in the subspace of eigenvectors of $\mathbf{E}\cdot\mathbf{r}$ with eigenvalue zero (indeed, this property could have been used to define the transported basis). Eigenvectors where this eigenvalue is not zero correspond to unphysical spin states with $S_1 \neq S_2$. The definition of transported operators (e.g. equations 14 and 15) ensure that these unphysical states never arise during quantum exchanges. The relation (41) also ensures that the transported operators are invariant under gauge transformations generated by exchange rotations about the z axis: if $\mathsf{U}(\mathbf{r})$ is preceded by the exchange rotation in (33), $\mathbf{P}(\mathbf{r})$ transforms as

$$\mathbf{P}(\mathbf{r}) \rightarrow \mathbf{P}(\mathbf{r}) + \hbar \nabla \alpha(\mathbf{r}) \mathsf{E} \cdot \mathbf{r} / r \tag{42}$$

and the additional term vanishes for physical states $|M(\mathbf{r})\rangle$.

We note that in ordinary spin rotation, generated by S, the states analogous to the transported spin basis are the m=0 eigenstates of the spin along $\mathbf{r}$, that is of $\mathsf{S} \cdot \mathbf{r}$; in the next section we will have more to say about this analogy.

5. Geometric phases

To see the relation between the way in which we treat quantum identity and the conventional way, consider the coefficents $\psi_{M,\text{fixed}}(\mathbf{r})$ (equation 12) of the fixed-basis wavefunction $|\Psi(\mathbf{r})\rangle_{\text{fixed}}$ (equation 11). We saw in §3 that these coefficients are the same as the coefficients $\psi_M(\mathbf{r})$ in the transported-basis wavefunction $|\Psi(\mathbf{r})\rangle$ (equation 6). According to (9) , then, $\psi_{M,\text{fixed}}(\mathbf{r})$ acquires the familiar Pauli sign under exchange of positions and spins, that is $\mathbf{r} \rightarrow -\mathbf{r}$, $M \rightarrow \overline{M}$; this is the usual formulation of spin and statistics. In the transported basis, however, the sign change of the coefficients is compensated by the sign change (5) of the transported basis, and the total wave $|\Psi(\mathbf{r})\rangle$ is singlevalued.

With this observation, the Pauli sign $(-1)^{2S}$ is revealed as a geometric phase factor, arising from parallel transport generated by exchange rotation. As with all geometric phases (Berry, 1984, Shapere and Wilczek, 1989), this sign is the result of dividing a system into two parts; here they are space and spin, and the space wavefunctions $\psi_M(\mathbf{r})$ inherit the phase factor from the transported spin kets, to keep the total state singlevalued. An analogous phenomenon is molecular pseudorotation (Longuet-Higgins et al., 1959, Delacrétaz et al., 1986), where a geometric sign change in electronic wavefunctions, when transported round a cycle of the nuclear coordinates that encloses an electronic degeneracy P, forces the nuclear wavefunctions to change sign when continued around P.

However, the geometric phase in particle exchange does not arise in the familiar way, from the line integral of a vector potential or the flux of a two-form, because these local quantities are zero (equation 4). Rather, the phase is of a different kind: global, and associated with noncontractible circuits in the doubly-connected (and nonorientable) configuration space. As we saw at the end of the last section, the transported basis states are analogous to the transported m=0 states in ordinary spin rotation, and these too are known to possess spin-dependent geometric sign changes $(-1)^j$, where j is the total spin (Robbins and Berry, 1994) for noncontractible circuits in the projective plane, with observable consequences.

Mathematically, the familiar geometric phase (produced for example by causing a spin to turn in a cycle) is associated with the Chern class, that is with monopole singularities of the two-form whose flux is the phase. On the other hand, the exchange phase is associated with the first Stiefel-Whitney class (Milnor and Stasheff, 1974, Nash and Sen, 1983, Nakahara, 1990) of the two-spin bundle for noncontractible loops in the doubly-connected space that incorporates identified states.

To explore the connection with the conventional formulation in more detail, we write the state in the transported basis explicitly for two spin-1/2 particles. Denoting quantum numbers ±1/2 by ±, (6) is

$$\begin{aligned}&|\Psi(\mathbf{r})\rangle = \\ &\psi_{++}(\mathbf{r})|++(\mathbf{r})\rangle + \psi_{+-}(\mathbf{r})|+-(\mathbf{r})\rangle + \psi_{-+}(\mathbf{r})|-+(\mathbf{r})\rangle + \psi_{--}(\mathbf{r})|--(\mathbf{r})\rangle\end{aligned} \tag{43}$$

As we have seen, the requirement that this be singlevalued leads to the conditions (9), which we can write as follows, using E to denote even functions and O to denote odd functions:

$$\begin{aligned}&\psi_{++}(\mathbf{r}) = -\psi_{++}(-\mathbf{r}) \equiv O_1(\mathbf{r}), \quad \psi_{--}(\mathbf{r}) = -\psi_{--}(-\mathbf{r}) \equiv O_{-1}(\mathbf{r}), \\ &\psi_{+-}(\mathbf{r}) = -\psi_{-+}(-\mathbf{r}) \equiv E(\mathbf{r}) + O_0(\mathbf{r}), \quad \psi_{-+}(\mathbf{r}) \equiv -E(\mathbf{r}) + O_0(\mathbf{r})\end{aligned} \tag{44}$$

Thus

$$|\Psi(\mathbf{r})\rangle = O_1(\mathbf{r})|++(\mathbf{r})\rangle + O_{-1}(\mathbf{r})|--(\mathbf{r})\rangle + O_0(\mathbf{r})(|+-(\mathbf{r})\rangle + |-+(\mathbf{r})\rangle) + E(\mathbf{r})(|+-(\mathbf{r})\rangle - |-+(\mathbf{r})\rangle) \qquad (45)$$

The conventional representation would employ the fixed basis (11), where the spin kets lack the **r** dependence. Since we have already seen that the coefficient functions, $\psi_{++}(\mathbf{r})$, etc, are the same in the fixed and transported bases, we can then recognise in (45), with transported kets replaced by fixed ones, the familiar decomposition into the triplet of states that are space-odd and spin-even, and the singlet state that is space-even and spin-odd. The traditional form of the exclusion principle follows from the vanishing of $|\Psi(\mathbf{r})\rangle$ when the two spin states are the same (++ or --) and the two positions are the same (**r**=0). It follows from the exchange relation (5) that the transformation from the fixed to the transported bases reverses the parity of the basis states under spin exchange, thereby restoring the singlevaluedness of the total wavefunction $|\Psi(\mathbf{r})\rangle$.

Because the states $|m_1 m_2(\mathbf{r})\rangle = |M(\mathbf{r})\rangle$ are not carried into themselves under $\mathbf{r} \to -\mathbf{r}$, but rather into $(-1)^{2S}|m_2 m_1(\mathbf{r})\rangle$, the Pauli geometric phase factor $(-1)^{2S}$ is nonabelian. The alternative transported basis $|j\mu(\mathbf{r})\rangle$ of eigenstates of total spin $(\mathbf{S}_1+\mathbf{S}_2)^2$ and its z component $(\mathbf{S}_1+\mathbf{S}_2)_z$ (e.g. the triplet and singlet states of (45)) abelianizes the basis $|m_1 m_2(\mathbf{r})\rangle$ that we have been using. The states $|j\mu(\mathbf{r})\rangle$ are related to $|m_1 m_2(\mathbf{r})\rangle$ by the Clebsch-Gordan coefficients in the usual way. Unlike m_1 and m_2, j and μ are good quantum numbers under exchange rotations (cf. 24), so that $|j\mu(\mathbf{r})\rangle$ is carried into itself under $\mathbf{r} \to -\mathbf{r}$. The geometric phase it acquires is just like the m=0 phase for ordinary spin; $|j\mu(\mathbf{r})\rangle$ is an eigenstate of $\mathbf{r}.\mathbf{E}$, with eigenvalue zero, and is also (unlike $|m_1 m_2(\mathbf{r})\rangle$) an eigenstate of $\mathbf{E}^2$, with eigenvalue $j(j+1)$ (cf. 25), and thus acquires a sign $(-1)^j$ under exchange.

The relation between $(-1)^j$ and the Pauli sign is provided by the Clebsch-Gordan coefficients, which under exchange of spins change by $(-1)^{2S-j}$. This latter sign was conjectured by Leinaas and Myrheim (1977) to be implicated in the spin-statistics relation, which was completed by regarding the additional sign $(-1)^j$ as arising from the parity under position exchange of the spherical harmonics describing the states of total spin.

6. More than two particles

For N identical particles with spin S, it is convenient to represent the positions, and the quantum numbers representing the spin components in the z direction, by (cf. 1)

$$\mathbf{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_N\}, \quad \mathbf{M} = \{m_1, \ldots, m_N\} \tag{46}$$

The elements in these lists (e.g. $\mathbf{r}_2$, m_3) denote particle properties, and places in the lists denote particle labels. The spin $\mathbf{S}_i$ of each particle is represented by the pair a_i and b_i of Schwinger oscillators, in terms of whose operators the state $|\mathbf{M}\rangle$ is, in an obvious notation (cf. 28),

$$|\mathbf{M}\rangle = C\prod_{i=1}^{N}\left(\mathsf{a}_i^\dagger\right)^{S+m_i}\left(\mathsf{b}_i^\dagger\right)^{S-m_i}|\mathbf{0}\rangle \tag{47}$$

where C is a normalization constant.

For N>2, we must consider general permutations of positions $\mathbf{R}$ and spins $\mathbf{M}$, not only two-particle exchanges. Permutations will be denoted by g, and we adopt the convention that properties of particle i are transferred to particle $g(i)$. Thus, for the three-particle permutation $g(1)=3$, $g(2)=1$, $g(3)=2$,

$$\left|g\{m_1, m_2, m_3\}\right\rangle = \left|m_{g^{-1}(1)}, m_{g^{-1}(2)}, m_{g^{-1}(3)}\right\rangle = \left|m_2, m_3, m_1\right\rangle \tag{48}$$

The general transformation is conveniently expressed in terms of permutation matrices, that is

$$|g\mathbf{M}\rangle = \left|P_{1j}(g)m_j, P_{2j}(g)m_j, \ldots\right\rangle, \quad \text{where } P_{ij}(g) \equiv \delta_{i,g(j)} \tag{49}$$

Every g can be factored into a product of exchanges, and has a parity $\varepsilon(g)$, defined, independently of the chosen sequence, as 0 or 1 if the number of exchanges is even (i.e. det P_{ij}=+1) or odd (i.e. det P_{ij}=-1).

As in §2 we represent the spin states with a transported basis $|\mathbf{M}(\mathbf{R})\rangle$ that depends on the positions, obtained (cf. 2) from the fixed basis $|\mathbf{M}\rangle$ by

$$|\mathbf{M}(\mathbf{R})\rangle = \mathsf{U}(\mathbf{R})|\mathbf{M}\rangle \tag{50}$$

The basis must be a singlevalued and smooth function of **R**, chosen so that the spins are permuted along with the positions, that is (cf. 3)

$$|g\mathbf{M}(g\mathbf{R})\rangle = (-1)^{K(g)}|\mathbf{M}(\mathbf{R})\rangle \tag{51}$$

It is possible to envisage more general phase factors (we do not consider parastatistics), but the restriction to a sign follows from the argument given after (3), and the fact that any g can be decomposed into exchanges. As also explained after (3), $K(g)$ is independent of **R** and **M**. The basis must be parallel-transported, that is (cf. 4)

$$\langle \mathbf{M}'(\mathbf{R})|\nabla_{\mathbf{R}}\mathbf{M}(\mathbf{R})\rangle = 0 \tag{52}$$

The unitary operator $\mathsf{U}(\mathbf{R})$ will be a 'permutation rotation', a generalization of exchange rotations to be defined later.

As we will show, any permutation rotation satisfying the conditions implies that the sign in (51) is

$$K(g) = 2S\varepsilon(g), \text{ i.e. } \quad |g\mathbf{M}(g\mathbf{R})\rangle = (-1)^{2S\varepsilon(g)}|\mathbf{M}(\mathbf{R})\rangle \tag{53}$$

from which will follow the spin-statistics relation. There remains the problem of finding an explicit general construction, for all N>2, of a $\mathsf{U}(\mathbf{R})$ that generates a smooth transported basis satisfying the above conditions, as we did for N=2 with the (26) and (27). This is a difficult problem (one reason being that permutations do not commute) and we have not solved it; in our view the difficulties are technical rather than fundamental. We do have an explicit construction for N=3 (it is rather elaborate), and we envisage several possibilities for the general case. These will be reported more fully elsewhere; below, we formulate the mathematical problem involved in such constructions.

Given the sign (53), we proceed as in §3, and represent any state $|\Psi(\mathbf{R})\rangle$ of the N particles by the $(2S+1)^N$-dimensional vector $\psi_{\mathbf{M}}(\mathbf{R})$ (cf. 6), where

$$|\Psi(\mathbf{R})\rangle = \sum_{\mathbf{M}} \psi_{\mathbf{M}}(\mathbf{R})|\mathbf{M}(\mathbf{R})\rangle \tag{54}$$

With this representation we must identify the configurations $\mathbf{R}$ and $g\mathbf{R}$, since these correspond to a permutation of spins as well as positions and so are indistinguishable; therefore the wavefunction must be singlevalued. Imposing this condition, and using (53), gives

$$\begin{gathered}|\Psi(\mathbf{R})\rangle = |\Psi(g\mathbf{R})\rangle = \sum_{\mathbf{M}} \psi_{\mathbf{M}}(g\mathbf{R})|\mathbf{M}(g\mathbf{R})\rangle = \\ \sum_{\mathbf{M}} \psi_{g\mathbf{M}}(g\mathbf{R})|g\mathbf{M}(g\mathbf{R})\rangle = (-1)^{2S\varepsilon(g)} \sum_{\mathbf{M}} \psi_{g\mathbf{M}}(g\mathbf{R})|\mathbf{M}(\mathbf{R})\rangle\end{gathered} \tag{55}$$

From (54) now follows

$$\psi_{g\mathbf{M}}(g\mathbf{R}) = (-1)^{2S\varepsilon(g)} \psi_{\mathbf{M}}(\mathbf{R}) \tag{56}$$

By an identical argument to that following (9), this can be interpreted as the spin-statistics connection in its familiar form.

To obtain the sign (53), we write each permutation g as a product over L exchanges e_l:

$$g = e_L e_{L-1} \dots e_1 \tag{57}$$

Applying these exchanges in sequence gives

$$\begin{aligned}|g\mathbf{M}(g\mathbf{R})\rangle &= |e_L(e_{L-1}\dots e_1\mathbf{M})(e_L(e_{L-1}\dots e_1\mathbf{R}))\rangle \\ &= (-1)^{K(e_L)}|e_{L-1}\dots e_1\mathbf{M}(e_{L-1}\dots e_1\mathbf{R})\rangle \\ &= (-1)^{K(e_L)+K(e_{L-1})+\dots K(e_1)}|\mathbf{M}(\mathbf{R})\rangle\end{aligned} \tag{58}$$

Where $K(e_l)$ is the sign associated with the l th exchange. From invariance under relabelling, all these signs are the same, so

$$(-1)^{K(g)} = (-1)^{\sum_{l=1}^{L} K(e_l)} = (-1)^{LK(e_1)} = (-1)^{K(e_1)\varepsilon(g)} \tag{59}$$

At the end of appendix C we show that $K(e_1)=2S$, which is the sign obtained in §4 for the exchange of two isolated particles; thence (53).

Before discussing the construction of the transported basis, we must define permutation rotations. Let $\mathbf{E}^{(ij)}$ be exchange angular momentum operators for the particle pairs i, j, defined (cf. 21 and 22) in terms of the Schwinger oscillators a_i, b_i for each of the spins $\mathbf{S}_i$ by

$$\mathsf{E}^{(ij)} = \mathsf{E}_a^{(ij)} + \mathsf{E}_b^{(ij)} \tag{60}$$

where

$$\mathsf{E}_a^{(ij)} = \tfrac{1}{2}\begin{pmatrix} \mathsf{a}_i^\dagger & \mathsf{a}_j^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} \mathsf{a}_i \\ \mathsf{a}_j \end{pmatrix}$$

$$\text{i.e. } \mathsf{E}_{az}^{(ij)} = \tfrac{1}{2}\left(\mathsf{a}_i^\dagger \mathsf{a}_i - \mathsf{a}_j^\dagger \mathsf{a}_j\right), \tag{61}$$

$$\mathsf{E}_{a+}^{(ij)} \equiv \mathsf{E}_{ax}^{(ij)} + \mathrm{i}\mathsf{E}_{ay}^{(ij)} = \mathsf{a}_i^\dagger \mathsf{a}_j, \quad \mathsf{E}_{a-}^{(ij)} \equiv \mathsf{E}_{ax}^{(ij)} - \mathrm{i}\mathsf{E}_{ay}^{(ij)} = \mathsf{a}_j^\dagger \mathsf{a}_i$$

and similarly for $\mathsf{E}_b^{(ij)}$. Permutation rotations are the unitary operators generated by the exchange angular momenta $\mathsf{E}^{(ij)}$, namely

$$\mathsf{U}(\mathbf{R}) = \exp\left\{-\mathrm{i} \sum_{\mu<\nu=1}^{N} \mathbf{c}_{\mu\nu}(\mathbf{R}) \cdot \mathsf{E}^{(\mu\nu)}\right\} \tag{62}$$

The parallel-transport requirement is automatically satisfied by the basis generated by this operator, as follows from an easy generalization of appendix B.

Naturally associated with $\mathsf{U}(\mathbf{R})$ is the NxN matrix

$$U_{ij} = \left[\exp\left(-\tfrac{1}{2}\mathrm{i} \sum_{\mu<\nu=1}^{N} \mathbf{c}_{\mu\nu}(\mathbf{R}) \cdot \boldsymbol{\sigma}^{(\mu\nu)}\right)\right]_{ij} \tag{63}$$

Here the $\boldsymbol{\sigma}^{(\mu\nu)}$ are generalized Pauli matrices, defined as vectors of NxN traceless hermitian matrices labelled by $\mu<\nu$, whose only nonzero elements are

$$\begin{aligned} \left[\boldsymbol{\sigma}^{(\mu\nu)}\right]_{\mu\mu} &= \boldsymbol{\sigma}_{11}, & \left[\boldsymbol{\sigma}^{(\mu\nu)}\right]_{\mu\nu} &= \boldsymbol{\sigma}_{12} \\ \left[\boldsymbol{\sigma}^{(\mu\nu)}\right]_{\nu\mu} &= \boldsymbol{\sigma}_{21}, & \left[\boldsymbol{\sigma}^{(\mu\nu)}\right]_{\nu\nu} &= \boldsymbol{\sigma}_{22} \end{aligned} \tag{64}$$

The $\boldsymbol{\sigma}^{(\mu\nu)}$ span the space of NxN hermitian matrices. Therefore permutation rotations can be regarded as a representation of SU(N) (the group of NxN unitary matrices with determinant unity), just as exchange rotations can be regarded as a representation of SU(2). In the Schwinger representation, the action of $\mathsf{U}(\mathbf{R})$ on the states $|\mathbf{M}\rangle$ is effected by the asociated matrix U_{ij} acting to the right on the creation operators, i.e. (cf. 35)

$$a_i^\dagger \to \left(a_i^\dagger\right)' = a_j^\dagger U_{ji} \tag{65}$$

and similarly for the b oscillators.

In appendix C we show that the permutation condition (51) implies the following relation for the associated matrices

$$U_{ij}(g\mathbf{R}) = U_{ik}(\mathbf{R}) Q_{kl}\left(g^{-1}\right) D_{lj}(g,\mathbf{R}) \tag{66}$$

Here, Q is a rephased permutation matrix with det Q=1, that is (cf 49)

$$Q_{ij}(g) = \delta_{i,g(j)} \exp\{\mathrm{i}\nu_i\}, \quad \text{where } \sum_{i=1}^{N} \nu_i = \pi\varepsilon(g) \tag{67}$$

and D is a diagonal matrix in SU(N). Therefore the construction of the transported basis reduces to finding a $U_{ij}(\mathbf{R})$ in SU(N) satisfying (66).

One possible construction is to take the columns of $U_{ij}(\mathbf{R})$ to be the eigenvectors $|n(\mathbf{R})\rangle$, written as column vectors $\langle i|n(\mathbf{R})\rangle$, of an NxN hermitian matrix $\mathsf{H}(\mathbf{R})$ depending smoothly on $\mathbf{R}$. Thus

$$\begin{aligned} \mathsf{H}(\mathbf{R}) &= \sum_{n=1}^{N} \lambda_n(\mathbf{R}) |n(\mathbf{R})\rangle\langle n(\mathbf{R})|, \quad \text{i.e.} \\ H_{ij}(\mathbf{R}) &= \sum_{n=1}^{N} \lambda_n(\mathbf{R}) U_{in}(\mathbf{R}) U^\dagger{}_{nj}(\mathbf{R}) \end{aligned} \tag{68}$$

where the eigenvalues $\lambda_n(\mathbf{R})$ are chosen to be symmetric under permutations g. Orthogonality and completeness of the $|n(\mathbf{R})\rangle$ guarantee that $U_{ij}(\mathbf{R})$ is unitary. Our hope that $\mathsf{H}(\mathbf{R})$ will be simpler than $U_{ij}(\mathbf{R})$ springs from the fact that for N=2, where $U_{ij}(\mathbf{R})$ is the matrix in (35) (cf. (26) and (27)), the expression (68) gives the simple formula

$$\mathsf{H}(\mathbf{R}) = \mathbf{r}\cdot\boldsymbol{\sigma} \tag{69}$$

with $\lambda_\pm(\mathbf{r}) = \pm r$.

Calculations based on (66) and (67) lead to the requirements

$$\left[\mathsf{H}(\mathbf{R}), \mathsf{H}(g\mathbf{R})\right] = 0 \tag{70}$$

showing that the matrices related by permutation of positions commute, and

$$\left|n(g\mathbf{R})\right\rangle\left\langle n(g\mathbf{R})\right| = \left|g^{-1}(n)(\mathbf{R})\right\rangle\left\langle g^{-1}(n)(\mathbf{R})\right| \tag{71}$$

showing that a permutation of positions leads the same permutation of the eigenstates $|n(\mathbf{R})\rangle$ (and possibly a change of phase). Another way to write this last result is

$$\mathsf{H}(g\mathbf{R}) = \sum_{n=1}^{N} \lambda_{g(n)}(\mathbf{R})\left|n(\mathbf{R})\right\rangle\left\langle n(\mathbf{R})\right| \tag{72}$$

- that is, permutation of positions changes the eigenvalue associated with each state $|n(\mathbf{R})\rangle$ from $\lambda_n(\mathbf{R})$ to its permuted counterpart $\lambda_{g(n)}(\mathbf{R})$. These general conclusions are illustrated by (69) where g is the exchange $\mathbf{r}\rightarrow -\mathbf{r}$, whose effect on H is to interchange either the two states $|\pm(\mathbf{r})\rangle$ or the two eigenvalues $\pm r$.

A consequence of (71), together with the orthogonality of the states $|n(\mathbf{R})\rangle$, is that for positions $\mathbf{R}=g\mathbf{R}$, where two or more particles coincide, the states $|n(\mathbf{R})\rangle$, and hence the matrix $U_{ij}(\mathbf{R})$, are singular, so $\mathsf{H}(\mathbf{R})$ is degenerate (as in (69) where $\mathbf{r}=0$). The number of degenerating eigenvalues must equal the number of coinciding particles. So, finding a nonsingular transported basis reduces to the mathematical problem of finding an NxN hermitian matrix $H_{ij}(\mathbf{R})$ with degeneracies only where $\mathbf{R}=g\mathbf{R}$; degeneracies at other points would lead to singularities in the basis constructed from $H_{ij}(\mathbf{R})$. Here the codimension of degeneracies gives cause for optimism. To make two particles coincide, it is necessary to vary three coordinates (to make the components of the interparticle vector vanish), and this is precisely the codimension of degeneracies (that is, the number of parameters in a generic complex hermitian matrix that must be varied in order to produce a degeneracy of two eigenvalues). To make N particles coincide, it is necessary to vary $3N$-3 coordinates. For $N>2$ this is less than the codimension N^2-N of a degeneracy of N eigenvalues, showing that the matrices we seek are special rather than generic, and there are insufficent parameters to lead us to expect unwanted degeneracies on higher-dimensional manifolds containing the points where $\mathbf{R}=g\mathbf{R}$.

7. Discussion

It is important to be precise about the sense in which we claim to have derived the connection between spin and statistics, embodied in the Pauli sign $(-1)^{2S}$. The form of quantum mechanics that we have used to describe identical particles, involving the transported spin basis and wavefunctions invariant under $\mathbf{r} \rightarrow -\mathbf{r}$, was not itself derived, but postulated as being closest in spirit to the more familiar quantum mechanics without exchange. This is physics, not mathematics, and so it can be tested by experiment. But since this quantum mechanics leads to the same physics (e.g. the exclusion principle) as more conventional formulations, the experiments have already been carried out, and support the theoretical predictions.

However, given this form of quantum mechanics, and our particular implementation in terms of exchange angular momentum, the Pauli principle is inevitable and we have derived it. The obvious question now is whether the result is unique. We are not claiming that the implementation is unique: there could be ways to define exchange angular momentum that do not involve harmonic oscillators, and, more generally, ways to augment spin so as to incorporate exchange without introducing exchange angular momentum. Nevertheless, we conjecture that the Pauli sign will emerge from any transported basis that satisfies the specified conditions. Certainly this is true for any transported basis generated by exchange rotations. More generally, we suggest that the relation between spin and statistics could be determined by the first Stiefel-Whitney class of the two-spin bundle, which in turn could be determined just by the conditions (a-c) in §2.

The main role of the Schwinger representation has been to provide a way of embedding the spin space in a larger Hilbert space (eigenstates of harmonic oscillators), to enable it to be parallel-transported. It is natural to ask whether the exchange rotation could be accomplished without this larger space, using the individual spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$ alone. It could not, at least while maintaining the requirement of parallel transport (the abandonment of which leads to the bizarre consequences explored in appendix D). An analogous enlargement of spin space has been noted before (Berry, 1987) in the interpretation of experiments on polarized light in a coiled optical fibre. Photons are spin-1 particles, which because of transversality are confined to the two-state subspace of spinors with spin quantum numbers $m=\pm1$ along the propagation direction. However, in order to accommodate within a fixed basis the changing propagation direction in a coiled fibre, three states are necessary. It is curious that in fibres the local $m=0$ state is excluded, whereas in spin exchange it is the $m\neq0$ states - unphysical because they correspond to particles with different spins - that are excluded. However, as J.H. Hannay has pointed out to us, linearly polarized light can be regarded as being in the $m=0$ state of the component of spin along the polarization, rather than propagation, direction.

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Appendix A. The transported basis is smooth

Using the Schwinger representation, we need to show that the transported function on the r.h.s. of (36) is a smooth function of **r** for $r>0$. Obviously it is necessary to examine only the neighbourhood of the poles. Near the north pole, we can set up local cartesian coordinates

$$\xi=\theta\cos\phi,\quad \eta=\theta\sin\phi\quad (\theta<<1) \tag{A1}$$

Then the local approximation to (36) is

$$\begin{aligned}&\left(\mathrm{a}_1^\dagger\right)^{n_{1a}}\left(\mathrm{a}_2^\dagger\right)^{n_{2a}}\left(\mathrm{b}_1^\dagger\right)^{n_{1b}}\left(\mathrm{b}_2^\dagger\right)^{n_{2b}}\rightarrow\\ &\qquad\left(\mathrm{a}_1^\dagger+\tfrac{1}{2}(\xi+\mathrm{i}\eta)\mathrm{a}_2^\dagger\right)^{n_{1a}}\left(\mathrm{a}_2^\dagger-\tfrac{1}{2}(\xi-\mathrm{i}\eta)\mathrm{a}_1^\dagger\right)^{n_{2a}}\times\\ &\qquad\left(\mathrm{b}_1^\dagger+\tfrac{1}{2}(\xi+\mathrm{i}\eta)\mathrm{b}_2^\dagger\right)^{n_{1b}}\left(\mathrm{b}_2^\dagger-\tfrac{1}{2}(\xi-\mathrm{i}\eta)\mathrm{b}_1^\dagger\right)^{n_{2b}}\end{aligned} \tag{A2}$$

which is obviously a smooth function of ξ and η.

Near the south pole, we can set up local cartesian coordinates

$$\xi=(\pi-\theta)\cos\phi,\quad \eta=(\pi-\theta)\sin\phi\quad ((\pi-\theta)<<1) \tag{A3}$$

Now the local approximation to (36) is

$$\left(\mathrm{a}_1^\dagger\right)^{n_{1a}}\left(\mathrm{a}_2^\dagger\right)^{n_{2a}}\left(\mathrm{b}_1^\dagger\right)^{n_{1b}}\left(\mathrm{b}_2^\dagger\right)^{n_{2b}} \to \left(\exp\{\mathrm{i}\phi\}\right)^{n_{1a}+n_{1b}}\left(-\exp\{\text{-}\mathrm{i}\phi\}\right)^{n_{2a}+n_{2b}}$$
$$\left(\mathrm{a}_2^\dagger + \tfrac{1}{2}(\xi - \mathrm{i}\eta)\mathrm{a}_1^\dagger\right)^{n_{1a}}\left(\mathrm{a}_1^\dagger - \tfrac{1}{2}(\xi + \mathrm{i}\eta)\mathrm{a}_2^\dagger\right)^{n_{2a}} \times \tag{A4}$$
$$\left(\mathrm{b}_2^\dagger + \tfrac{1}{2}(\xi - \mathrm{i}\eta)\mathrm{b}_1^\dagger\right)^{n_{1b}}\left(\mathrm{b}_1^\dagger - \tfrac{1}{2}(\xi + \mathrm{i}\eta)\mathrm{b}_2^\dagger\right)^{n_{2b}}$$

Smoothness is threatened by the phase factors, which can be written as

$$\left(\exp\{\mathrm{i}\phi\}\right)^{n_{1a}+n_{1b}}\left(-\exp\{\text{-}\mathrm{i}\phi\}\right)^{n_{2a}+n_{2b}} = (-1)^{2S_2}\exp\{2\mathrm{i}\phi(S_1 - S_2)\} \tag{A5}$$

However, the ϕ-dependence disappears for identical spins, so the basis depends smoothly on ξ and η near the south pole. Note the contrast with the more familiar transport of spin-1/2 states by ordinary spin rotation, where the dependence on $\theta/2$ leads to a singularity at the south pole. Exchange rotation this avoids this because the singularities cancel from the commuting parts $\mathbf{E}_a$ and $\mathbf{E}_b$ of $\mathbf{E}$ (equation 21) - but only for the physical states S_1=S_2.

Smoothness can also be demonstrated using the gauge invariance (33) under exchange rotations about the z axis, as follows. We have that

$$|M(\mathbf{r})\rangle = \mathrm{U}(\mathbf{r})|M\rangle = \mathrm{U}'(\mathbf{r})|M\rangle \tag{A6}$$

where U' is any operator that differs from U by an arbitrary gauge rotation, that is

$$\mathrm{U}'(\mathbf{r}) = \mathrm{U}(\mathbf{r})\exp\left(\mathrm{i}\alpha(\mathbf{r})\mathrm{E}_z\right) \tag{A7}$$

Now, we can choose U as in (27), whose smoothness near the north pole guarantees that $|M(\mathbf{r})\rangle$ is smooth there, and we can choose U' to be the alternative exchange rotation - from $\mathbf{e}_z$ to -$\mathbf{e}_z$ and then from -$\mathbf{e}_z$ to $\mathbf{r}$ -

$$\mathrm{U}'(\mathbf{r}) = \exp\{-\mathrm{i}(\pi - \theta)\mathbf{n}(\mathbf{r})\cdot\mathbf{E}\}\exp\{-\mathrm{i}\pi\mathrm{E}_y\} \tag{A8}$$

whose obvious smoothness near the south pole guarantees that $|M(\mathbf{r})\rangle$ is smooth there also.

Appendix B. The transported basis is parallel-transported

To prove the vanishing of the connection (4), we begin by writing

$$\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = \langle M|\mathsf{U}^{\dagger}(\mathbf{r})\nabla \mathsf{U}(\mathbf{r})|M\rangle \tag{B1}$$

Since $\mathsf{U}(\mathbf{r})$ and $\mathsf{U}(\mathbf{r}+\mathrm{d}\mathbf{r})$ are infinitesimally different exchange rotations,

$$\mathsf{U}^{\dagger}(\mathbf{r})\nabla \mathsf{U}(\mathbf{r}) = \mathbf{A}(\mathbf{r})\mathsf{E}_x + \mathbf{B}(\mathbf{r})\mathsf{E}_y + \mathbf{C}(\mathbf{r})\mathsf{E}_z \tag{B2}$$

Thus

$$\begin{aligned}&\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = \\ &\mathbf{A}(\mathbf{r})\langle M'|\mathsf{E}_x|M\rangle + \mathbf{B}(\mathbf{r})\langle M'|\mathsf{E}_y|M\rangle + \mathbf{C}(\mathbf{r})\langle M'|\mathsf{E}_z|M\rangle\end{aligned} \tag{B3}$$

The matrix element involving E_z vanishes by (32). Those involving E_x and E_y can be written in terms of $\mathsf{E}_{a+}, \mathsf{E}_{a-}, \mathsf{E}_{b+}, \mathsf{E}_{b-}$ defined in (21). These operators shift quanta from the (a_1,b_1) oscillator pair to the (a_2,b_2) oscillator pair and therefore change (S_1,S_2) to $(S_1\pm 1/2, S_2\mp 1/2)$, so that all matrix elements such as those in (B3), with $S_1=S_2$, vanish, thereby proving (4).

(To demonstrate (B2) explicitly, we can use the result, valid for any operator $\mathbf{C}(\mathbf{r})$, that

$$\exp[-\mathbf{C}(\mathbf{r})]\nabla \exp[\mathbf{C}(\mathbf{r})] = \nabla\mathbf{C} + \tfrac{1}{2!}[\nabla\mathbf{C},\mathbf{C}] + \tfrac{1}{3!}[[\nabla\mathbf{C},\mathbf{C}],\mathbf{C}] + \dots \tag{B4}$$

For the particular operator (27), a calculation gives

$$\mathsf{U}^{\dagger}(\mathbf{r})\nabla \mathsf{U}(\mathbf{r}) = -\mathrm{i}\frac{\mathbf{e}_{\theta}\mathbf{n}\cdot\mathbf{E}}{r} + \mathrm{i}\frac{\mathbf{e}_{\phi}}{r\sin\theta}\left(\frac{\mathbf{r}\cdot\mathbf{E}}{r} + (1-2\cos\theta)\mathsf{E}_z\right) \tag{B5}$$

which has the form (B2).)

Parallel transport can also be proved by a lengthy calculation based on the representation of Bargmann (1962), in which creation operators are replaced by complex variables and use is made of

$$\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle \propto$$

$$\iint \mathrm{d}a_1\mathrm{d}a_2 \iint \mathrm{d}b_1\mathrm{d}b_2 \exp\left\{-\left(|a_1|^2+|a_2|^2+|b_1|^2+|b_2|^2\right)\right\}\times \\ (a_1'*)^{S+m_1'}(a_2'*)^{S+m_2'}(b_1'*)^{S-m_1'}(b_2'*)^{S-m_2'}\times \\ \nabla(a_1'*)^{S+m_1}(a_2'*)^{S+m_2}(b_1'*)^{S-m_1}(b_2'*)^{S-m_2} \tag{B6}$$

where the primed variables are defined in terms of the unprimed variables by (35), and the integrations are over the complex planes of the variables, e.g.

$$\mathrm{d}a_1 \equiv \mathrm{dRe}a_1\mathrm{dIm}a_1 \tag{B7}$$

We do not give the details.

Appendix C. Pauli sign from general exchange rotation

We assume that $\mathsf{U}(\mathbf{r})$ is an exchange rotation satisfying (2) and (3). It follows that

$$\mathsf{U}^\dagger(\mathbf{r})\mathsf{U}(-\mathbf{r})|M\rangle = (-1)^K|\overline{M}\rangle \tag{C1}$$

$|\overline{M}\rangle$ can be expressed in terms of $|M\rangle$ through a fixed exchange rotation, e.g. an exchange rotation $\mathsf{R}_y(\pi)$ by π about y. Indeed employing the Schwinger representation in the form (35), with $\theta=\pi$, $\phi=0$, we obtain

$$\begin{pmatrix}\mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger\end{pmatrix} \to \begin{pmatrix}\mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger\end{pmatrix}(\text{-i}\sigma_y) = \begin{pmatrix}\mathsf{a}_1^\dagger & \mathsf{a}_2^\dagger\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\mathsf{a}_2^\dagger & -\mathsf{a}_1^\dagger\end{pmatrix} \tag{C2}$$

From (28) and (36), we obtain

$$|\overline{M}\rangle = (-1)^{2S}\mathsf{R}_y(\pi)|M\rangle \tag{C3}$$

Thus (C1) can be written as

$$\mathsf{D}(\mathbf{r})|M\rangle = (-1)^{K-2S}|M\rangle \tag{C4}$$

where

$$\mathsf{D}(\mathbf{r}) = \mathsf{R}_y^\dagger(\pi)\mathsf{U}^\dagger(\mathbf{r})\mathsf{U}(-\mathbf{r}) \tag{C5}$$

As $\mathsf{D}(\mathbf{r})$ is a product of exchange rotations, it is itself an exchange rotation, and thus may be expressed in the form $\exp\{-\mathrm{i}\mathbf{c}(\mathbf{r}).\mathbf{E}\}$. The eigenvector equation (C4) then implies (cf. 28 and 29) that

$$\begin{aligned}&\left(\mathsf{a}_1{}'^{\dagger}\right)^{S+m_1}\left(\mathsf{a}_2{}'^{\dagger}\right)^{S+m_2}\left(\mathsf{b}_1{}'^{\dagger}\right)^{S-m_1}\left(\mathsf{b}_2{}'^{\dagger}\right)^{S-m_2}=\\&(\text{phase factor})\times\left(\mathsf{a}_1{}^{\dagger}\right)^{S+m_1}\left(\mathsf{a}_2{}^{\dagger}\right)^{S+m_2}\left(\mathsf{b}_1{}^{\dagger}\right)^{S-m_1}\left(\mathsf{b}_2{}^{\dagger}\right)^{S-m_2}\end{aligned}\qquad\text{(C6)}$$

where, as in (35), the primed and unprimed oscillator pairs are related by

$$\left(\mathsf{a}_1{}'^{\dagger}\quad \mathsf{a}_2{}'^{\dagger}\right)=\left(\mathsf{a}_1{}^{\dagger}\quad \mathsf{a}_2{}^{\dagger}\right)\exp\{-\mathrm{i}\mathbf{c}(\mathbf{r})\cdot\boldsymbol{\sigma}\}\qquad\text{(C7)}$$

and similarly for b and b'. Since (C6) must hold for all M, we deduce that

$$\mathsf{a}_j{}'=(\text{phase factor})\times\mathsf{a}_j,\quad \mathsf{b}_j{}'=(\text{phase factor})\times\mathsf{b}_j\qquad\text{(C8)}$$

Thus $\exp\{-\mathrm{i}\mathbf{c}(\mathbf{r}).\sigma\}$ is diagonal, (that is $\mathbf{c}(\mathbf{r})$ is along the z direction), so

$$\mathsf{D}(\mathbf{r})=\exp\{-\mathrm{i}\alpha(\mathbf{r})\mathsf{E}_z\}\qquad\text{(C9)}$$

Then (33) implies that the eigenvalues of the states $|M\rangle$ in (C4) are in fact unity, and (5) follows. (It is not the case that the operator $\mathsf{D}(\mathbf{r})$ is equal to unity: its action on unphysical states, where the particles would have different spins, would introduce phases.)

This derivation of the Pauli sign generalizes to $N>2$ particles. As argued in §6, it suffices to consider the exchange e of a single pair, say particles 1 and 2. The permutation condition (51) implies

$$\mathsf{U}^{\dagger}(\mathbf{R})\mathsf{U}(e\mathbf{R})|\mathbf{M}\rangle=(-1)^{K(e)}|e\mathbf{M}\rangle\qquad\text{(C10)}$$

$|e\mathbf{M}\rangle$ is related to $|\mathbf{M}\rangle$ by, for example, the exchange rotation $\mathsf{R}_y^{(12)}(\pi)=\exp\left\{-\mathrm{i}\pi\mathsf{E}_y^{(12)}\right\}$, whose associated matrix (63) is

$$\exp\left\{-\tfrac{1}{2}\mathrm{i}\pi\sigma_y^{(12)}\right\}=\begin{pmatrix}-\mathrm{i}\sigma_y & 0\\ 0 & \mathrm{I}\end{pmatrix}\qquad\text{(C11)}$$

where I is the N-2-dimensional identity matrix. Its effect in the Schwinger representation (cf 65) is to replace $\left(\mathsf{a}_1^{\dagger}\quad \mathsf{a}_2^{\dagger}\right)$ by $\left(\mathsf{a}_2^{\dagger}\quad -\mathsf{a}_1^{\dagger}\right)$ while leaving the other a_j unchanged, and similarly for b_j. It follows that

$$|e\mathbf{M}\rangle = (-1)^{2S}\mathsf{R}_y^{(12)}(\pi)|\mathbf{M}\rangle \tag{C12}$$

Thus (C10) can be written as

$$\mathsf{D}(e,\mathbf{R})|\mathbf{M}\rangle = (-1)^{K(e)-2S}|\mathbf{M}\rangle \tag{C13}$$

where

$$\mathsf{D}(e,\mathbf{R}) = \mathsf{R}_y^{(12)\dagger}(\pi)\mathsf{U}^\dagger(\mathbf{R})\mathsf{U}(e\mathbf{R}) \tag{C14}$$

By an argument identical to that for N=2 (equations C6-C9), it follows that the matrix associated with D is diagonal, and thence,using the generalization

$$\mathsf{E}_z^{(\mu\nu)}|\mathbf{M}\rangle = 0 \tag{C15}$$

of (32), that $K(e)=2S$, which is what we wanted to show. Further, the relation (66) for a general permutation now follows by repeated application of (C14), in the form

$$\mathsf{U}(e\mathbf{R}) = \mathsf{U}(\mathbf{R})\mathsf{Q}(e)\mathsf{D}(e,\mathbf{R}) \tag{C16}$$

with $\mathsf{Q}(e) = \mathsf{R}_y^{(12)}$.

Appendix D. The counterconstruction: spin-statistics without parallel transport

Without the enlargement of the spin space forced by the parallel-transport requirement (4), the operator $\mathsf{U}(\mathbf{r})$ in (2) can be represented as a $(2S+1)\text{x}(2S+1)$ matrix. Then the exchange condition (3) can be written as

$$\mathsf{U}(-\mathbf{r}) = \mathsf{U}(\mathbf{r})(-1)^K\mathsf{P} \tag{D1}$$

where P is the permutation matrix, satisfying

$$\mathsf{P}|M\rangle = |\overline{M}\rangle \tag{D2}$$

that exchanges the labels of the $(2S+1)^2$ pairs of spin quantum numbers m_1 and m_2. It is not hard to show that

$$\det \mathsf{P} = (-1)^{S(2S+1)} \tag{D3}$$

(one way is to order the pairs so that those with the same spin quantum numbers are listed together, and those with different quantum numbers are adjacent to their exchanged partner pairs; then the matrix P consists of a $(2S+1)$-dimensional unit diagonal block, followed by $S(2S+1)$ unit off-diagonal 2x2 blocks, whose determinant is easy to calculate). Thus

$$\det \mathsf{U}(-\mathbf{r}) = \det \mathsf{U}(\mathbf{r})(-1)^{(K+S)(2S+1)} \tag{D4}$$

Now we incorporate the smoothness condition a of §2. It implies that $\mathsf{U}(\mathbf{r})$ can be defined on a closed loop and remain smooth as the loop is continuously contracted to a point only if det $\mathsf{U}(\mathbf{r})$ has zero winding number round the loop. Considering the loop to be any great circle in a sphere with constant $r=|\mathbf{r}|$, we see that this is impossible (i.e. winding is unavoidable) if the sign in (D4) is negative. Therefore

$$(K+S)(2S+1) \text{ is even} \tag{D5}$$

When this condition holds, $\mathsf{U}(\mathbf{r})$ can be constructed explicitly by exploiting the implication of (C1) that $(-1)^K \mathsf{P}$ is unitary with unit determinant, and so can be written as

$$(-1)^K \mathsf{P} = \exp\{\mathrm{i}\mathsf{H}\}, \quad \left[(-1)^K \mathsf{P}\right]^2 = \exp\{2\mathrm{i}\mathsf{H}\} = 1 \tag{D6}$$

where H is hermitian and traceless. Then we define $\mathsf{U}(\mathbf{r})$ on the equator $\theta=\pi/2$ (using polar coordinates for $\mathbf{r}$) as

$$\mathsf{U}\left(r, \tfrac{1}{2}\pi, \phi\right) = \exp\left\{\mathrm{i}\frac{\phi}{\pi}\mathsf{H}\right\} \tag{D7}$$

This is a continuous family of unitary matrices with unit determinant, satisfying (D1) and beginning (at $\phi=0$) and ending (at $\phi=2\pi$) at the identity. Since SU($2S+1$) is simply connected, the closed loop $\mathsf{U}(r,\pi/2,\phi)$ can be continuously contracted to the identity, and we can define $\mathsf{U}(\mathbf{r})$ in the northern hemisphere so that on circles of latitude it interpolates between the loop (D7) on the equator and the identity at the north pole. $\mathsf{U}(\mathbf{r})$ is then defined in the southern hemisphere by continuation using (D1).

For each S, we can examine the implications of the condition (D5) for K even (boson statistics) and K odd (fermion statistics). There are four cases.

(i) If S=0, 2, 4,..., (D4) requires K even, that is bose statistics.

(ii) If S=1, 3, 5,..., (D4) requires K odd, that is fermi statistics.

(iii) If S=1/2, 5/2, 9/2,..., (D4) cannot be satisfied for any integer K, so the counterconstruction is incompatible with any statistics.

(iv) If S=3/2, 7/2, 11/2,..., (D4) can be satisfied for any integer K, so the counterconstruction is compatible with both bose and fermi statistics.

The results of this counterconstruction are not only bizarre but also incoherent, in the sense that the construction cannot be carried out for case (iii) and does not lead to a unique spin-statistics relation in case (iv) (as well as giving the wrong relation for case (ii)).

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