Recent Developments in the Mathematics of Reactive Systems

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This paper is an extended abstract of an invited talk to the Eighth International Conference on Concurrency Theory held in Warsaw in July 1997. The time evolution of certain reactive systems can be modelled by the dynamics of self-maps of n-dimensional space which are nonexpansive in the supremum norm. The paper surveys recent progress in this area.

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Recent Developments in the Mathematics of Reactive Systems (Extended Abstract)

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A reactive system, when coupled with its environment, is an example of a *dynamical system.* That is, at any particular instant the system is in one of a collection of possible states and this state changes over time. Mathematicians have studied dynamical systems for nearly a century but have largely concentrated on systems arising from physics: planetary systems, fluids, elastic solids, etc, [11]. By constrast, much less is known about so-called *discrete event dynamical systems* (or reactive systems) which arise in computer science, communications, operations research, manufacturing, etc.

A number of models have been proposed for studying such systems, including (timed or stochastic) Petri nets, Jackson networks, Generalised Semi-Markov Processes and forms of process algebra. A good overview of such models appears in a special issue of the IEEE Proceedings, [12]. For the most part, these models are mathematically intractable in the following sense: it is hard to find any theorem about them which engineers would wish to learn as an aid to designing reactive systems.

In this talk I will discuss some recent ideas which take a different approach. They have emerged through the independent work of several people coming from different standpoints, [1, 2, 9, 13, 17]. A more detailed overview can be found in [6, §4].

Suppose that the set of states of the system can be represented by $\mathbf{R}^n$. For instance, if the system has n possible events labelled $1, \cdots, n$, then $(x_1, \cdots, x_n) \in \mathbf{R}^n$ might represent the times of occurrence of each event, relative to some arbitrary origin of time. Suppose further that the time evolution of the system is represented by a function $F : \mathbf{R}^n \to \mathbf{R}^n$ so that if the system is currently in state $\mathbf{x}$ then it evolves in the next time step to state $F(\mathbf{x})$. These may seem like absurdly restrictive assumptions but let us proceed with them for the moment.

What assumptions should we make about F? The following are very natural. Firstly, the origin of time should be irrelevant. Hence, F should be *homogeneous*: for all $\mathbf{x} \in \mathbf{R}^n$ and $h \in \mathbf{R}$,

$$F(\mathbf{x} + h) = F(\mathbf{x}) + h \ .$$

(We use here the convention that when a vector and a scalar appear together in a binary operation or relation then the operation is performed, or the relation is required to hold, on each component of the vector. Hence, $(\mathbf{x}+h)_i = x_i + h$ for all i, while $\mathbf{x} = h$ means that $x_i = h$ for all i.) Secondly, if we delay the times of occurrence of each event, then the next occurrences should not be faster than they were before. That is, F should be *monotonic*: for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$,

$$\mathbf{x} \leq \mathbf{y} \implies F(\mathbf{x}) \leq F(\mathbf{y}) \ ,$$

where $\mathbf{x} \leq \mathbf{y}$ denotes the product ordering on $\mathbf{R}^n$, $x_i \leq y_i$ for all i. This axiom is perhaps less immediately compelling than the first one but it has a clear intuition and can be seen to hold in practice in a wide variety of systems.

Functions with both properties are called *topical functions*, [9]. We want to understand the dynamics of the corresponding system, so we shall study the trajectories $\mathbf{x}, F(\mathbf{x}), F^2(\mathbf{x}), \cdots$. This emphasis will allow us, hopefully, to answer such questions as *"does the system attain an equilibrium or cycle indefinitely or blow up?"*, *"how sensitive is the system to a change of its parameters?"*, *"how fast does the system operate?"*, etc. Questions like these are often very much in the minds of engineers when designing reactive systems but, for the most part, they have not had the tools to answer them.

It turns out that there are real-life systems which can be represented even by the simple model above. For instance, the problem of *clock schedule verification* in synchronous digital circuits has been solved by finding the equilibrium points of a suitable topical function, [7, 16]. Furthermore, the model can be extended in various ways to accommodate nondeterminism, stochasticity and more complex states. These extensions suggest that some of the other models in current use can be incorporated within this framework, although our understanding of this important question is still rudimentary, [2, 4]. I will not discuss such extensions here, for fear of putting the cart before the horse. As we shall see, the horse has not yet been tamed and already presents us with some difficult problems.

The first remark to make about topical functions is that they are nonexpansive in the ℓ_∞ (or supremum) norm. Let $\|\mathbf{x}\| = \max_{1 \leq i \leq n} |x_i|$. This defines a norm on $\mathbf{R}^n$, so that $\|\mathbf{x} - \mathbf{y}\|$ is a metric. F is nonexpansive if

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\| \ .$$

This property has an important effect on the dynamics of F and constrains it in ways that are still not fully understood. For instance, it limits the extent of cyclic behaviour in the dynamics of F, [15]. Of course, if F was contractive then the Banach Contraction Principle would tell us that the dynamics of F were straightforward: there is an unique equilibrium point and all trajectories converge to it. When F is merely nonexpansive, its dynamics are much more subtle, [5].

The space of topical functions, $\mathsf{Top}(n, n)$, includes a number of important examples studied in optimal control, game theory, mathematical economics and operations research, [10]. In particular, nonnegative matrices can be considered as topical functions. The dynamics of such matrices has been extensively studied under the name of Perron-Frobenius theory, [3]. From this perspective, topical functions lead the way towards a nonlinear generalisation of Perron-Frobenius.

One of the interesting results to emerge from this is that any topical function can be approximated by so-called min-max functions. These latter functions are topical functions which are built recursively from the operations max, min and addition. The approximation is similar to the way in which polynomials approximate continuous functions but has the added feature that some of the dynamics of the topical function are inherited by its approximating min-max functions, [10]. For nonnegative matrices, these approximations are new and they probably would not have been found if not for the introduction of topical functions. It is a welcome development that topical functions are interesting both through their applications to reactive systems and through their intrinsic mathematical qualities. Perhaps this will encourage more mathematicians to think about the problems of reactive systems.

I will concentrate in the talk on two related questions. How do we measure the speed of the underlying system? When does the system have an equilibrium point? For the former question, the limit

$$\lim_{k\to\infty} F^k(\mathbf{x})/k$$

turns out to be the appropriate measure. It can be thought of as the asymptotic average slowness of each event. This limit does not exist for all topical functions—it is an important open problem to identify those for which it does—but if it does exist, it is independent of $\mathbf{x}$. Hence it associates to F a vector, called *the cycle time vector*, $\mathcal{X}(F) \in \mathbf{R}^n$. We are starting to understand the properties of $\mathcal{X}$ as a (partial) functional, $\mathcal{X} : \mathsf{Top}(n,n) \to \mathbf{R}^n$. These properties allow us to calculate $\mathcal{X}$ in the case of min-max functions and hence to estimate it when we do not know how to calculate it exactly. It turns out that the cycle time is closely related to the existence of equilibrium points. If F has an equilibrium point, so that $F(\mathbf{x}) = \mathbf{x}$, then it is easy to see that $\mathcal{X}(F) = \mathbf{0}$. Conversely, if F is a min-max function and $\mathcal{X}(F) = 0$, then F has an equilibrium point.

There are a number of unsolved conjectures and open problems in this area which I will try and point out.

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