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On the Measure of Entanglement for Pure States

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We point out a formal correspondence between thermodynamics and entanglement. By applying it to previous work, we show that *entropy of entanglement* is the unique measure of entanglement for pure states. 3.65.Bz, 89.70.+c

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Quantum entanglement is a most remarkable effect, and we have yet to develop all the tools we need for studying it. One tool we require is a measure of entanglement [1]. Among the tools in use for studying entanglement are information theory [2] and various definitions of entropy [3]. The use of information theory and entropy to analyze entanglement indicates a close correspondence between entanglement and thermodynamics. In part, the correspondence consists of common definitions, such as von Neumann entropy. In part, the correspondence is formal: a formal principle of thermodynamics may apply, *mutatis mutandis*, to the study of entanglement. The purpose of this note is to point out a simple, but useful, formal correspondence: a formal principle of thermodynamics, applied to previous work on entanglement [3], singles out a unique measure of entanglement for pure states.

When Einstein searched for a universal formal principle from which to derive a new mechanics (namely, special relativity) he took for inspiration a general principle of thermodynamics: The laws of nature are such that it is impossible to construct a *perpetuum mobile* [4]. This general principle (the second law) enabled Carnot to show that all reversible heat engines operating between given temperatures T_1 and T_2 are equally efficient. Consider two reversible heat engines; suppose that both absorb heat Q_1 at T_1 and expel heat Q_2 at T_2, but one does work W, and the other does work $W' > W$, per cycle. The first engine, if run in reverse, is a refrigerator—*absorbs* heat Q_2 at T_2 and *expels* heat Q_1 at T_1—and requires only work W per cycle. Thus the two engines together could provide $W' - W$ in work per cycle without changing their environment. Such a conclusion contradicts the second law, so both engines must do the same work: $W = W'$.

The formal correspondence with entanglement is as follows: The laws of nature are such that it is impossible to create entanglement by local operations. It is clear that quantum mechanics does not allow local operations to create entanglement, although they may preserve or destroy entanglement. This general principle is the analogue of the second law of thermodynamics. The analogue of a reversible heat engine is any reversible transformation, consisting only of local operations, that transforms one entangled state into another. Let two experimenters, Alice and Bob, share pairs of quantum systems in an entangled state.

One quantum system in each pair goes to Alice, and the other goes to Bob. In addition, each experimenter may have access to other quantum systems that are not initially entangled. Local operation include any measurements or unitary transformations that Alice performs on her systems, and that Bob performs on his. Alice and Bob can even exchange messages in the usual way, but Alice may not send Bob any system entangled with one that she keeps, nor may Bob send such a system to Alice; thus, they cannot create entanglement between their systems. In principle, however, this restriction still allows them to transform entanglement in one pair of systems into entanglement involving other systems. For example, they may, using local operations, transfer entanglement between two spins in a singlet state to two identical spins that initially were uncorrelated, *i.e.* in a product state, and the original spins will now be in a product state. This transformation is reversible: local operations can transfer the singlet entanglement back to the original spins, leaving the substitute spins uncorrelated.

We may consider more general transformations. Suppose that Alice and Bob share n pairs of systems in an entangled state, and that, by local operations only, they transform the entanglement to k pairs of systems in a different entangled state. Since Bob and Alice have access to other systems that are not initially entangled, n and k may be different. Even if $k > n$, there need be no contradiction with the general principle that it is impossible to create entanglement by local operations, because the state of the k pairs may be less entangled than the state of the original n pairs. If Alice and Bob can transform n pairs in one entangled state into k pairs in another entangled state *without destroying any entanglement*, then any measure of entanglement must assign the same entanglement to the n initial pairs and the k final pairs. But did they not destroy any entanglement? That is, a question arises with regard to the efficiency of the transformation: could Alice and Bob apply a different set of local operations to transform n' of the initial pairs into the same final state of k pairs, but with $n' < n$?

The answer is that they cannot, *if both transformations are reversible.* For if it were possible to transform n' of the initial pairs into k of the final pairs by a different transformation, Alice and Bob could then reverse the first transformation and transform the k pairs in the final state to n pairs in the initial entangled state. In doing so, they would have added $n' - n$ entangled pairs to their initial supply, contradicting the general principle that it is impossible to create entanglement by local operations. Thus $n' = n$.

Reversible local transformations lead naturally to a measure of entanglement that is intensive, in the thermodynamic sense. For suppose there exists a reversible transformation between n pairs in one state and k pairs in another state, both entangled. By repeating this transformation m times we could transform between mn pairs in the one state and mk pairs in the other state, and the repeated transformation, too, would be reversible. Since it would be reversible, no other transformation could be more efficient. We can now define the measure of entanglement with respect to a reference state, such as a singlet pair of spin-1/2 particles. If k entangled pairs prepared in a given entangled state transform by reversible local transformations into n copies of the reference state, then the measure of entanglement of the given state is n/k times the measure of entanglement of the reference state. This measure has a well defined limit as n and k become large.

The reversible local transformations we have assumed are, in fact, consistent with quantum mechanics. Bennett, Bernstein, Popescu and Schumacher [3] have shown that is possible, with local operations only, to transform n systems in one entangled state $|\Psi_{AB}\rangle$ into k systems in a different entangled state $|\Psi'_{AB}\rangle$. The transformation is reversible when the number of systems becomes arbitrarily large. That is, the ratio k/n tends to a constant in the limit $n \to \infty$. This constant is equal to $E(|\Psi_{AB}\rangle)/E(|\Psi'_{AB}\rangle)$, where $E(|\Psi_{AB}\rangle)$, the *entropy of entanglement* of the state $|\Psi_{AB}\rangle$, is the von Neumann entropy of the partial density matrix seen by either Alice or Bob (and equals the Shannon entropy of the squares of the coefficients of the entangled state in the Schmidt decomposition) [3]. The entropy of entanglement is zero for a pair of systems in a product state, and 1 for a pair of spin-1/2 particles in a singlet state; it is never negative. Note that the result of Bennett, Bernstein, Popescu and Schumacher [3] implies that if the measure of entanglement of one pair in a state $|\Psi_{AB}\rangle$ is $E(|\Psi_{AB}\rangle)$, then the measure of entanglement of n pairs in the same state is

$n \cdot E(|\Psi_{AB}\rangle)$.

Bennett, Bernstein, Popescu and Schumacher [3] argue that the entropy of entanglement is a good measure of entanglement for pure states, because local operations can interconvert states of equal entropy of entanglement with asymptotically perfect efficiency, but can never increase the entropy of entanglement. Here we present a stronger argument. The general principle that it is impossible to create entanglement by local operations leads to a unique measure of entanglement for pure states. According to the general principle, any reversible transformation between entangled states *must* preserve the measure of entanglement. The reversible transformations defined by Bennett, Bernstein, Popescu and Schumacher [3] then determine the measure to be the entropy of entanglement. For if we can transform n systems in an initial standard state, *e.g.* n pairs of spin-1/2 particles in a singlet state, into pairs of systems in any final state [3], then the ratio k/n of pairs in the final state to standard pairs defines the measure of entanglement of the final state. Thus, instead of arguing that the entropy of entanglement is a good measure of entanglement, we have demonstrated that it is the *unique* measure of entanglement for pure states (up to a constant factor).

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