Extension of Morphological Operations to Complete Semilattices and its Applications to Image and Video Processing

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Abstract

Mathematical Morphology is a nonlinear image processing theory, which is currently based on complete lattices. This work extends its scope to complete semilattices, which are more general. Specific morphological operators defined in complete semilattices are shown, by means of simulations, to be potentially useful in some video processing applications, like detection of fast motion, extraction of objects that appear in a sequence frame but not in its predecessor, and segmentation-based compression of sequences.

In this report, we first redefine the basic morphological operators in complete semilattices. Then, a few properties of morphological operators in complete lattices are proven to apply also to their semilattice counterparts. Next, some examples of semilattices and basic morphological operators defined on them are provided. Finally, the above applications are briefly described and demonstrated.

Keywords: Mathematical Morphology, complete semilattices, image processing, video processing, motion detection, innovation extraction, segmentation-based coding, skeleton, Top-Hat transform, morphological gradient.

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1 Introduction

Mathematical Morphology is a well-founded non-linear theory for Image Processing [1, 2, 3]. Its geometry-oriented nature provides a strong framework for addressing shape characteristics such as size, connectivity, and others, which are not easily accessed by the traditional linear approach. Morphology has been used in applications such as nonlinear filtering [4], sharpening [5], compression [6], shape analysis [7], segmentation [8], and others.

Mathematical Morphology is theoretically founded on complete lattices. Lattices are partially ordered sets in which every pair of elements have a least majorant (supremum) and a greatest minorant (infimum). For example, in the lattice of grayscale functions, a function f is said to be bigger than another function g, if $f(x) \geq g(x)$, for all x in their domain. Thus, the infimum and the supremum of two functions are the functions formed by the point-wise infimum and point-wise supremum of the original ones, respectively. The lattice is said *complete* when it contains a least and a greatest elements. In the case of the complete lattice of grayscale functions, for instance, the least and greatest elements are the all-($-\infty$) and the all-($+\infty$) functions, respectively. The lattice of grayscale functions is traditionally used for morphological image processing, where each image is a function in the lattice [2].

In many applications, the existence of a least and a greatest elements agrees with one's intuition, in which case Mathematical Morphology on the corresponding complete lattice proves to be very useful. The case where we consider bright (high grayscale valued) objects in an image to form its foreground, and the dark (low grayscale valued) objects to form its background, for example, agrees with the above choice of partial ordering, which leads to the least and greatest grayscale functions defined above.

There are cases, however, that the existence of both a least and a greatest element is not intuitive. For example, consider a real function, obtained by the difference between two other real functions, which could represent the error of a prediction operation. In traditional Mathematical Morphology, the difference function would be processed in the same way as any other function, that is, in the framework of the above lattice grayscale of functions, where the all-($-\infty$) function is the least element. However, in our opinion, it would be more useful to consider the *null function* to be the least difference-function, because it corresponds to the case when the two original functions are identical. In this case, there does not seem to exist one single greatest function, since both the all-($-\infty$) and the all-($+\infty$) functions could be considered equally "big".

In such situations, using a complete *semilattice* could be more useful and intuitive than using a complete lattice. An *inf semilattice* is a set where every pair of elements have an infimum (but not necessarily a supremum). It is complete when there exists a least element (but not necessarily a greatest element). A *sup semilattice* is defined dually. Thus, a lattice is an especial case of semilattices, where it is both an inf semilattice and a sup semilattice.

In this work, we propose the extension of many useful morphological image processing tools for complete inf semilattices (the results are naturally extended to complete sup semilattices too, by duality). These tools include erosion, opening, "top-hat" extraction, gradient, and the skeleton. In addition, some potential applications of these tools on complete semilattices for video processing are presented: Fast-motion detection, innovation extraction, and coding.

2 Theoretical Background: Morphology on Complete Lattices

This section provides a brief overview of Mathematical Morphology on complete lattices. Additional information can be found in [2].

2.1 Complete Semilattices and Lattices

A *partially ordered set* A is a set associated with a binary operator $\leq$, satisfying the following properties for any $x, y, z \in A$: reflexivity ($x \leq x$), anti-symmetry ($x \leq y, y \leq x \Rightarrow x = y$), and transitivity ($x \leq y, y \leq z \Rightarrow x \leq z$).

In a partially ordered set A, the least majorant $\vee X$ (also called *supremum*) of a subset $X \subseteq A$ is defined as an element $a_0 \in A$, such that: *i)* $x \leq a_0, \forall x \in X$, and *ii)* if there exists y, such that $x \leq y \leq a_0$ for all $x \in X$, then $y = a_0$. One defines the greatest minorant $\wedge X$ (also called *infimum*) of X, dually.

A partially ordered set $\mathcal{P}$ is an *inf semilattice* (resp. *sup semilattice*) if every two-element subset $\{X_1, X_2\}$ in $\mathcal{P}$ has an infimum $X_2 \wedge X_2$ (resp., a supremum $X_1 \vee X_2$) in $\mathcal{P}$. If $\mathcal{P}$ is both an inf and a sup semilattice, then it is called a *lattice*.

An inf semilattice (resp., sup semilattice) is *complete* when *every* non-empty subset $B \subset \mathcal{P}$ has an infimum $\wedge B$ (resp. supremum $\vee B$). In this case, there exists in the semilattice an unique element 0, called zero element, (resp., U, called universe), such that, for any $X \in \mathcal{P}$, $0 \wedge X = 0$ (resp. $U \vee X = U$). A complete lattice is a lattice which is both a complete inf and complete sup semilattice.

2.2 Basic Operations in Complete Lattices

2.2.1 Erosions and Dilations

In a complete lattice $(\mathcal{P}, \leq, \vee, \wedge)$, any function $\varepsilon : \mathcal{P} \longrightarrow \mathcal{P}$ which commutes with the infimum $\wedge$ and preserves the universe is called an *erosion*. In other words, $\varepsilon(\cdot)$ is an erosion iff for any collection $\{X_i\}$ of elements in $\mathcal{P}$:

$$\varepsilon\left(\bigwedge_i X_i\right) = \bigwedge_i \varepsilon\left(X_i\right), \tag{1}$$

and $\varepsilon(U) = U$.

Dually, a *dilation* is any function that commutes with the supremum and preserves the zero element, i.e., δ is a dilation iff:

$$\delta\left(\bigvee_i X_i\right) = \bigvee_i \delta\left(X_i\right), \tag{2}$$

and $\delta(0) = 0$.

Both dilation and erosion are increasing operations in the complete lattice, that is:

$$\forall X, Y \in \mathcal{P}, \quad X \leq Y \Rightarrow \begin{cases} \delta(X) \leq \delta(Y) \\ \varepsilon(X) \leq \varepsilon(Y) \end{cases} \tag{3}$$

For each dilation δ in a complete lattice there is a single erosion ε, satisfying:

$$\forall X, Y \in \mathcal{P}, \quad \delta(X) \leq Y \Leftrightarrow X \leq \varepsilon(Y). \tag{4}$$

Similarly, for each erosion there is a single dilation, such that (4) is satisfied. The pairs (δ, ε) satisfying the above duality are called *adjoint* or *dual*.

Given an erosion ε, its adjoint dilation is given, for all $X \in \mathcal{P}$, by:

$$\delta(X) = \bigwedge \{Y \in \mathcal{P} \mid \ X \leq \varepsilon(Y)\} \tag{5}$$

Adjoint erosions and dilations satisfy the following property, for all $X \in \mathcal{P}$:

$$\delta\varepsilon\delta(X) = \delta(X), \quad \varepsilon\delta\varepsilon(X) = \varepsilon(X). \tag{6}$$

2.2.2 Openings and Closings

An *algebraic opening* (or, simply, opening) γ in a complete lattice is an operator which is *idempotent* ($\gamma\gamma(X) = \gamma(X)$, $\forall X \in \mathcal{P}$), *increasing* ($X \leq Y \Rightarrow \gamma(X) \leq \gamma(Y)$, $\forall X, Y \in \mathcal{P}$), and *anti-extensive* ($\gamma(X) \leq X$, $\forall X \in \mathcal{P}$).

Similarly, an *algebraic closing* (or just closing) ϕ is an operator in $\mathcal{P}$ which is *idempotent*, *increasing*, and *extensive* ($\phi(X) \geq X$, $\forall X \in \mathcal{P}$).

Opening and closings are referred to as morphological filters, which remove "parts" of X that do not comply with a certain criterion. Often, this criterion is related to the notion of size [1].

Important particular cases of openings and closings are, respectively, the operators γ_ε and ϕ_ε, defined for $X \in \mathcal{P}$ by:

$$\gamma_\varepsilon(X) \stackrel{\triangle}{=} \delta\varepsilon(X), \quad \phi_\varepsilon(X) \stackrel{\triangle}{=} \varepsilon\delta(X), \tag{7}$$

where (δ, ε) is an adjoint pair. These operators are called, respectively, the *morphological opening* and the *morphological closing*, associated with the erosion ε.

Given an erosion ε, the morphological opening and closing associated with it are given, for all $X \in \mathcal{P}$, by:

$$\gamma_\varepsilon(X) = \bigwedge \{Y \in \mathcal{P} \mid \ \varepsilon(X) \leq \varepsilon(Y)\}, \tag{8}$$

$$\phi_\varepsilon(X) = \bigwedge \{\varepsilon(Y) \mid \ Y \in \mathcal{P}, X \leq \varepsilon(Y)\}. \tag{9}$$

From this point on, the index ε is removed from the notation of morphological openings and closings, that is, γ and ϕ will denote morphological opening and closing, respectively.

2.3 Some Image Processing Tools

2.3.1 The Lattice of Grayscale Images

The complete lattice of grayscale images $\mathcal{P}_f$ is the set of real-valued functions, with $\leq$, $\vee$, and $\wedge$ defined as follows, for all $f, g \in \mathcal{P}_f$:

$$f \leq g \Leftrightarrow f(x) \leq g(x), \forall x, \tag{10}$$
$$(f \vee g)(x) = \max\{f(x), g(x)\}, \forall x, \tag{11}$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \forall x. \tag{12}$$

In this lattice, the least element 0 is the function $0(x) \equiv -\infty$, and the universe U is the function $U(x) \equiv \infty$.

The following adjoint pair of dilation and erosion is of special interest in function lattices:

$$[\varepsilon(f)](x) \triangleq \bigwedge_{z \in B} f(x+z), \tag{13}$$
$$[\delta(f)](x) \triangleq \bigvee_{z \in B} f(x-z), \tag{14}$$

where B is a set of points in the domain of f, called *structuring element*. These operations, which are *translation-invariant* (TI), can be seen as nonlinear convolutions, where the structuring element B works as a moving window, and the traditional averaging of linear convolutions, performed inside the window for every x, is replaced by the infimum and supremum operations, respectively. They are called TI erosion/dilation by a *flat* structuring element (to differentiate them from the more general case, not reviewed here, where the structuring element is a function instead of a set). The above erosion (resp. dilation) usually causes object edges within images (functions) they operate upon to move, in such a way that bright regions shrink (resp. expand), and dark regions expand (resp. shrink).

In the next sections, we review some important morphological tools in $\mathcal{P}_f$. They are defined using *general* erosions ε and dilations δ, but most applications use specifically the *TI*, flat operators.

2.3.2 Gradients

Gradients are used mainly for extracting information about object edges in an image. They are useful for edge detection, segmentation [6], and sharpening [5].

There are three types of morphological gradients [3], associated to a given erosion ε: *i)* The *internal gradient* $\mathcal{G}_i$, which returns the internal boundary of the bright objects and the external boundary of the dark objects in f, *ii)* the *external gradient* $\mathcal{G}_e$, which returns the opposite result, and *iii)* the *total gradient* $\mathcal{G}$, which returns both the internal and external boundaries of the objects in f. They are defined, for any image $f \in \mathcal{P}_f$, by:

$$\mathcal{G}_i(f) \triangleq f - \varepsilon(f), \tag{15}$$
$$\mathcal{G}_e(f) \triangleq \delta(f) - f, \tag{16}$$
$$\mathcal{G}(f) \triangleq \delta(f) - \varepsilon(f). \tag{17}$$

2.3.3 Top-Hat Extraction

The Top-Hat transform or extraction is widely used for extracting fine details in an image (see [3]).

There are two kinds of Top-Hat transforms. The *white Top-Hat transform* $\mathcal{H}_w(f)$ (for extracting bright details), and the *black Top-Hat transform* $\mathcal{H}_b(f)$ (for extracting dark details), are defined with respect to a given opening or a given closing, by:

$$\mathcal{H}_w(f) \triangleq f - \gamma(f), \tag{18}$$

$$\mathcal{H}_b(f) \triangleq \phi(f) - f. \tag{19}$$

2.3.4 Skeleton

In the case of binary images (which can be seen as a particular case of grayscale image) the skeleton representation is usually used to produce a thin caricature of f, for analysis purposes. The skeleton has also been used for image compression, where it is considered as an efficient, size-oriented, decomposition of the image [2, 10, 9].

There are several definitions and generalizations of the skeleton decomposition [9]. We review in this section a grayscale version due to Maragos [10].

Given an erosion ε, let for all natural n:

$$\varepsilon^n \triangleq \begin{cases} I, & n = 0, \\ \varepsilon\varepsilon^{n-1}, & n > 0, \end{cases} \tag{20}$$

where I is the identity operator.

The *skeleton subsets* $\{s_n(f)\}$ of a given image $f \in \mathcal{P}_f$ are defined by:

$$s_n(f) \triangleq \varepsilon^n(f) - \gamma\varepsilon^n(f), \tag{21}$$

where γ is the morphological opening associated to ε.

If there exists N such that $\varepsilon^N(f) \equiv 0$, then the original image f can be recovered from its skeleton decomposition $\{s_n(f)\}$ by iteration, as follows:

$$\begin{cases} f_N \triangleq 0, \\ f_n \triangleq s_n(f) + \delta(f_{n+1}), \\ f = f_0, \end{cases} \tag{22}$$

where δ is the adjoint dilation of ε.

3 Morphology on Semilattices

In this section, we extend most of the notions and tools from Section 2 for semilattices. Without loss of generality, we restrict ourselves to *inf* semilattices, since all definitions and results are valid in sup semilattices as well, by duality.

3.1 Erosions and Openings

In this subsection we consider basic notions that are directly and naturally extendible from complete lattices to complete inf semilattices, namely, erosions and openings.

Definition 1 (Erosion): *A binary operator ε in an inf semilattice*[1] *$\mathcal{S}$ is an* erosion *iff, for all $\{X_i\} \subseteq \mathcal{S}$:*

$$\varepsilon\left(\bigwedge_i X_i\right) = \bigwedge_i \varepsilon\left(X_i\right). \tag{23}$$

Notice that, since there is no universe U in an inf semilattice, erosions are not required to preserve any element.

Proposition 1 *Erosions in inf semilattices are increasing.*

Proof

$$\begin{aligned} X \leq Y \quad &\Leftrightarrow \quad X \wedge Y = X \\ &\Rightarrow \quad \varepsilon(X \wedge Y) = \varepsilon(X) \Leftrightarrow \varepsilon(X) \wedge \varepsilon(Y) = \varepsilon(X) \\ &\Leftrightarrow \quad \varepsilon(X) \leq \varepsilon(Y). \end{aligned}$$

□

The extension of algebraic openings to semilattices is also straightforward:

Definition 2 (Algebraic Opening): *A binary operator γ in an inf semilattice $\mathcal{S}$ is an* algebraic opening *iff it is idempotent, increasing, and anti-extensive*[2]*.*

Compared to the above extensions, that of *morphological* opening, presented below, is a little less direct. This is because: *i)* In complete lattices, the morphological opening associated to an erosion is defined using its adjoint dilation, and *ii)* since there is no general definition of supremum in an inf semilattice, one cannot generally define dilations there[3]. Nevertheless, morphological opening can be completely extended to complete inf semilattices, by adapting equation (8), which provides the morphological opening associated to an erosion in a complete lattice, without the help of its adjoint dilation.

Definition 3 (Morphological Opening): *In a complete inf semilattice $\mathcal{S}$, the morphological opening γ_ε associated to an erosion ε is defined, for any $X \in \mathcal{S}$, by:*

$$\gamma_\varepsilon(X) \triangleq \bigwedge\{Y \in \mathcal{S} \mid \ \varepsilon(X) \leq \varepsilon(Y)\}. \tag{24}$$

[1] See definition of inf semilattices in Section 2.1, on page 2.

[2] See definitions in Section 2.2.2, on page 3.

[3] However, we do define limited versions of supremum and adjoint dilation in the sequel.

Proposition 2 *Given any erosion in a complete inf semilattice $\mathcal{S}$, the associated morphological opening $\gamma_\varepsilon(X)$ of any element $X \in \mathcal{S}$ exists in $\mathcal{S}$ and is unique.*

Proof *In any complete inf semilattice, the infimum $\wedge B$ of any non-empty set B exists inside the semilattice, and is unique. Therefore, all we have to prove is that the set $C_X \triangleq \{Y \in \mathcal{S} \mid\ \varepsilon(X) \leq \varepsilon(Y)\}$ is not empty. This is trivial, since $X \in C_X$.* □

From this point on, we remove the subscript ε from the morphological opening. That is, the symbol γ will represent the morphological opening associated to the erosion ε, unless otherwise stated.

Proposition 3 *For any erosion ε in a complete inf semilattice $\mathcal{S}$, $\varepsilon\gamma = \varepsilon$.*

Proof *For all X in the semilattice,*

$$\begin{aligned}
\varepsilon\gamma(X) &= \varepsilon\left(\bigwedge\{Y \in \mathcal{S} \mid\ \varepsilon(X) \leq \varepsilon(Y)\}\right) \\
&= \bigwedge\{\varepsilon(Y) \mid\ \varepsilon(X) \leq \varepsilon(Y)\} \\
&= \varepsilon(X).
\end{aligned}$$

□

Proposition 4 *The morphological opening in an complete inf semilattice is an algebraic opening.*

Proof *We have to prove that γ is idempotent, increasing, and anti-extensive.*

Idempotent:

$$\begin{aligned}
\gamma\gamma(X) &= \bigwedge\{Y \mid\ \varepsilon\gamma(X) \leq \varepsilon(Y)\} \\
&= \bigwedge\{Y \mid\ \varepsilon(X) \leq \varepsilon(Y)\} = \gamma(X).
\end{aligned}$$

Increasing:

$$\begin{aligned}
X \leq Y &\Rightarrow \varepsilon(X) \leq \varepsilon(Y) \\
&\Rightarrow \{Z \mid\ \varepsilon(X) \leq \varepsilon(Z)\} \supseteq \{Z \mid\ \varepsilon(Y) \leq \varepsilon(Z)\} \\
&\Rightarrow \bigwedge\{Z \mid\ \varepsilon(X) \leq \varepsilon(Z)\} \leq \bigwedge\{Z \mid\ \varepsilon(Y) \leq \varepsilon(Z)\} \\
&\Rightarrow \gamma(X) \leq \gamma(Y).
\end{aligned}$$

Anti-extensive:

$$\begin{aligned}
&\{Z \mid\ \varepsilon(X) \leq \varepsilon(Z)\} \supseteq \{X\} \\
\Rightarrow\ &\wedge\{Z \mid\ \varepsilon(X) \leq \varepsilon(Z)\} \leq \wedge\{X\} \\
\Rightarrow\ &\gamma(X) \leq X.
\end{aligned}$$

□

3.2 Supremum and Dilations

We now relate to basic morphological notions that are not directly extendible to inf semilattices, namely, supremum and dilation. Although impossible to define the above operations generally in inf semilattices, we provide limited versions of them.

Supremum

Unless it is a complete lattice, a complete inf semilattice does not have a well-defined supremum for all its subsets. Nevertheless, a supremum does exist for *some* subsets of the semilattice.

Definition 4 *Given a complete inf semilattice $\mathcal{S}$, define the set $\mathcal{U}_{\mathcal{S}}$ of upper-bounded subsets of $\mathcal{S}$ as follows:*

$$\mathcal{U}_{\mathcal{S}} \stackrel{\triangle}{=} \{\mathcal{S}' \subset \mathcal{S} \mid \ (\exists Y_0 \in \mathcal{S} \mid \ Y_0 \geq X, \forall X \in \mathcal{S}')\}. \tag{25}$$

In words, $\mathcal{U}_{\mathcal{S}}$ contains all the subsets of $\mathcal{S}$ that have a majorant Y_0. Supremum is defined only over elements of $\mathcal{U}_{\mathcal{S}}$.

Proposition 5 *The least majorant (supremum) $\vee B$ of a set $B \subset \mathcal{S}$ exists iff $B \in \mathcal{U}_{\mathcal{S}}$, in which case it is equal to:*

$$\vee B = \bigwedge \{Y \in \mathcal{S} \mid \ Y \geq X, \forall X \in B\}. \tag{26}$$

The above supremum is well defined over and only over $\mathcal{U}_{\mathcal{S}}$. This is because the set $\{Y \in \mathcal{S} \mid \ Y \geq X, \forall X \in B\}$ is non-empty iff $B \in \mathcal{U}_{\mathcal{S}}$.

In summary, if a subset of the complete inf semilattice has a majorant, then it has a least majorant, and this is the supremum.

Corollary 1 *Let $X, Y \in \mathcal{S}$. If $X \leq Y$, then $X \vee Y = Y$.*

Corollary 2 *Let $X, Y \in \mathcal{S}$. $(X \wedge Y) \vee Y = Y$ always, and $(X \vee Y) \wedge X = X$ if $X \vee Y$ exists.*

Corollary 2 says that the pair of operations $(\wedge, \vee)$ partially satisfies the *absorption laws* required by a pair (infimum,supremum) in a lattice.

Dilation

We opt not to define dilation generically (as an operator that commutes with the supremum), but we do define the *adjoint dilation* of a given erosion, restricted to a sub-domain of the semilattice. We are now inspired by equation (5) for this definition. But before presenting it, let us characterize the domain.

Definition 5 *Given an erosion ε in a complete inf semilattice $\mathcal{S}$, we define the set of* ε-bounded elements *of $\mathcal{S}$, symbolized by $E_{\mathcal{S}}(\varepsilon)$, as:*

$$E_{\mathcal{S}}(\varepsilon) \stackrel{\triangle}{=} \{Y \in \mathcal{S} \mid \ [\exists Z \in \mathcal{S} \mid \ Y \leq \varepsilon(Z)]\} \tag{27}$$

In words, $E_{\mathcal{S}}(\varepsilon)$ contains the elements in $\mathcal{S}$ that are smaller or equal to the erosion of some element in $\mathcal{S}$. This is the domain for the adjoint dilation of the erosion, as defined below.

Definition 6 (Adjoint Dilation): *Let $\mathcal{S}$ be a complete inf semilattice, and ε an erosion. If $X \in E_{\mathcal{S}}(\varepsilon)$, then the* adjoint dilation *of ε, denoted δ_ε, is defined as:*

$$\delta_\varepsilon(X) \stackrel{\triangle}{=} \bigwedge\{Y \in \mathcal{S} \mid \ X \leq \varepsilon(Y)\}. \tag{28}$$

Note that δ_ε is well-defined over (and only over) $E_{\mathcal{S}}(\varepsilon)$, since $\{Y \in \mathcal{S} \mid \ X \leq \varepsilon(Y)\}$ is non-empty iff $X \in E_{\mathcal{S}}(\varepsilon)$.

We drop from now on the index ε from the notation of its adjoint dilation, for simplification.

Proposition 6 *The adjoint dilation of ε is increasing in $E_{\mathcal{S}}(\varepsilon)$.*

Proof *For any $X, Y \in E_{\mathcal{S}}(\varepsilon)$:*

$$\begin{aligned} X \leq Y \ &\Rightarrow \ \{Z \mid \ X \leq \varepsilon(Z)\} \supseteq \{Z \mid \ Y \leq \varepsilon(Z)\} \\ &\Rightarrow \ \bigwedge\{Z \mid \ X \leq \varepsilon(Z)\} \leq \bigwedge\{Z \mid \ Y \leq \varepsilon(Z)\} \\ &\Rightarrow \ \delta(X) \leq \delta(Y). \end{aligned}$$

□

In Subsection 3.1 above, we define morphological opening in complete inf semilattices without the notion of dilation, because it is not a natural concept in semilattices. On the other hand, now that we have defined adjoint dilation, we are able to express the morphological opening also as the composition of an erosion with its adjoint dilation, which is the way it is traditionally defined in complete lattices:

Proposition 7 *For <u>all</u> $X \in \mathcal{S}$, $\gamma(X) = \delta\varepsilon(X)$.*

Proof $\delta[\varepsilon(X)] = \bigwedge\{Z \mid \ \varepsilon(X) \leq \varepsilon(Z)\} = \gamma(X)$. □

Note that Proposition 7 holds for *all* the elements of the semilattice. This is because $\varepsilon(X)$ is in the domain of δ, $\forall X \in \mathcal{S}$.

Proposition 8 *For a given X, if there exists Z such that $\varepsilon(Z) = X$, then $\delta(X) = \gamma(Z)$.*

Proof $\varepsilon(Z) = X \Rightarrow \delta\varepsilon(Z) = \delta(X) \Rightarrow \gamma(Z) = \delta(X)$. □

4 Examples of Semilattices

In this section, we present semilattices that are potentially useful for image processing, and show examples of basic morphological operations defined in them. In Section 5, we present a few applications of the operators defined in these semilattices.

4.1 Difference Semilattice

We say that a set $\mathcal{S}$ is a difference semilattice if: *i)* it is composed of functions $f : E \to R$, where E is an Euclidean space or a subset of it, and R is either $\mathbb{R} \cup \{-\infty, \infty\}$ (continuous case) or $\mathbb{Z} \cup \{-\infty, \infty\}$ (discrete case), and *ii)* it is associated with the the partial ordering $\leq$ given, for all f, g in the semilattice, by:

$$f \leq g \Leftrightarrow \forall x, \begin{cases} g(x) \geq f(x) \geq 0, & \text{if } g(x) \geq 0, \\ g(x) \leq f(x) \leq 0, & \text{if } g(x) < 0. \end{cases} \tag{29}$$

In a difference semilattice, the least element 0 is the function $0(x) \equiv 0$, and the infimum operator $\wedge$, is given by:

$$(f \wedge g)(x) = \begin{cases} \min\{f(x), g(x)\}, & \text{if } f(x), g(x) \geq 0, \\ \max\{f(x), g(x)\}, & \text{if } f(x), g(x) \leq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Figure 1(a) shows an example of applying the infimum operator $\wedge$ in a 1-D continuous difference semilattice. Notice that the infimum result is smaller (with respect to the above partial ordering $\leq$) than both its operands.

Let us consider the following operator, defined in a difference semilattice:

$$[\varepsilon(f)](x) \triangleq \bigwedge_{z \in B} f(x+z), \tag{31}$$

where B (which we call structuring element) is a pre-defined set of points in the Euclidean space E, i.e., $B \subset E$. It is easy to show that ε is an erosion, and also that it is translation invariant. This erosion is the difference-semilattice counterpart of the erosion by a flat structuring element, defined in grayscale lattices. Figure 1(b) shows the result of applying ε to a 1-D function.

It is easy to show that the erosions in difference semilattices as defined above are self-dual, in the sense that they satisfy: $\varepsilon(-f) = -\varepsilon(f)$. I.e., negative and positive components are dealt with in the same way. This property is also extended to all the other operators in a difference semilattice (openings, dilations, gradients, skeletons, etc).

As any morphological opening in a complete inf semilattice, the one associated with the above erosion is well defined by equation (24), and can be calculated as the composition of the erosion with its adjoint dilation. In this case, the adjoint dilation is given by:

$$[\delta(f)](x) = \bigvee_{z \in B} f(x-z), \tag{32}$$

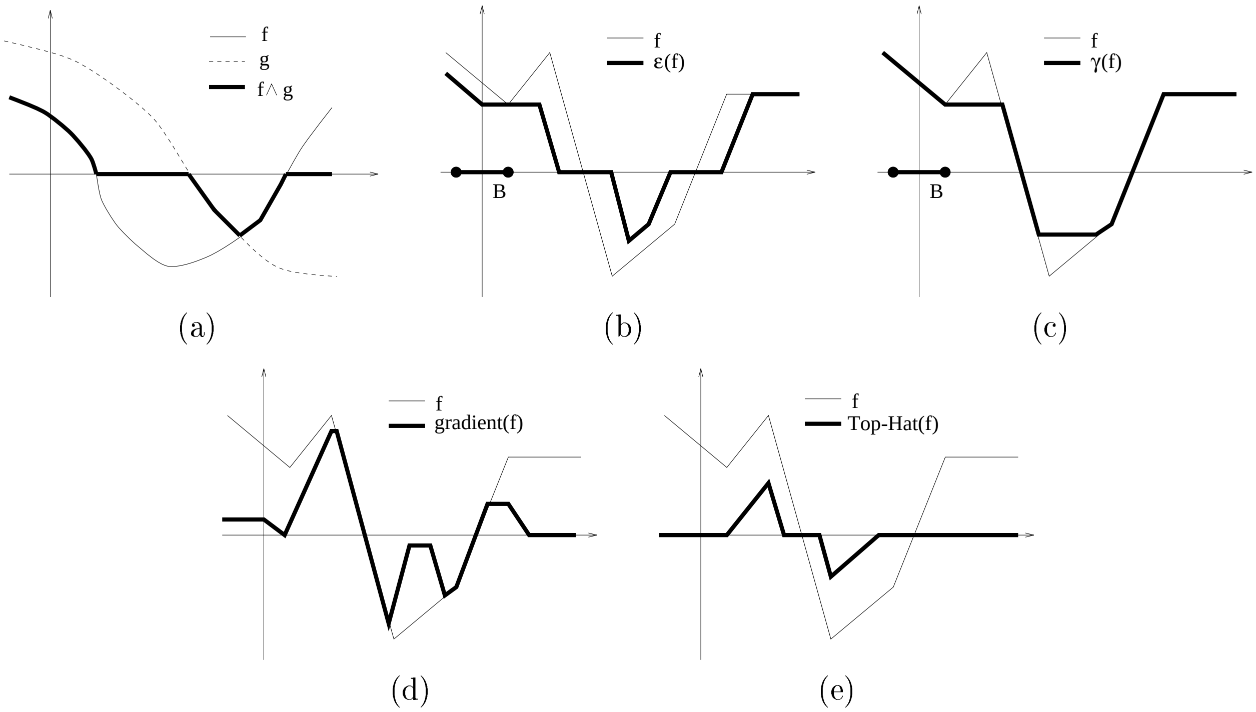


Figure 1: Operations in a difference semilattice. (a) Infimum, (b) erosion by a flat structuring element, (c) opening, (d) gradient, and (e) Top-Hat transform.

where supremum $\vee$ assumes here the following format:

$$(f \vee g)(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } f(x), g(x) \geq 0, \\ \min\{f(x), g(x)\}, & \text{if } f(x), g(x) \leq 0, \\ \text{Non existent}, & \text{otherwise}. \end{cases} \tag{33}$$

Figure 1(c) shows an example of morphological opening. Let us remark, once again, that only functions that are bounded by the erosion of some other function can be dilated. For example, the original function f in the example of Figures 1(b)-(e) cannot be dilated by the above adjoint dilation.

Usually, the morphological opening in lattices is used for filtering, from images, elements that are bright and either smaller or thinner than the structuring element. Similarly, the morphological opening in inf semilattices can be used for filtering small and thin elements of difference images, only that, due to the self-dual nature of the difference semilattice, it removes both positive (bright) and negative (dark) such elements.

Gradients, Top-Hat transforms and skeletons can also be defined in difference semilattices. However, only operators defined in terms of erosions and openings (and not dilations and closings) can be defined here. Therefore, from the three morphological gradients (see section 2.3.2 on page 4) and the two Top-Hat transforms (see section 2.3.3) defined in lattices, only the *internal* gradient and the *white* Top-Hat transform can be defined in the difference semilattice. The definition formulæ, on the other hand, remain the same, i.e., the gradient,

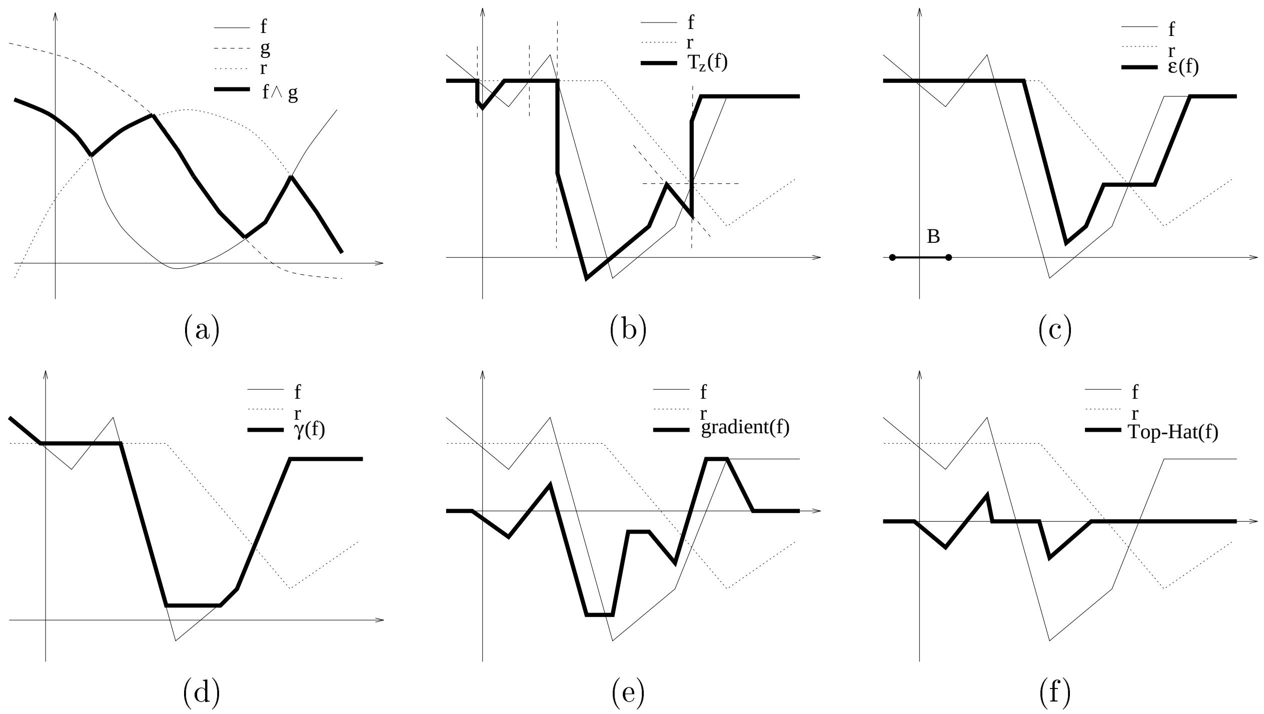


Figure 2: Operations in a reference semilattice. (a) Infimum, (b) translation by the operator $T_z(\cdot)$, (c) erosion by a flat structuring element, (d) opening, (e) gradient, and (f) Top-Hat transform.

Top-Hat, and skeleton are defined in the difference semilattice also by (15), (18), and (20)-(21), respectively, where now the symbols represent operators in the difference semilattice (not in the lattice). Figures 1(f) and (g) show examples of gradient operation and Top-Hat extraction.

4.2 Reference Semilattice

Another example of complete inf semilattice, which we call *reference semilattice*, consists of real functions, as the difference semilattice, but its partial ordering depends on a given function $r(x)$, called *reference function*. In a reference semilattice $\mathcal{S}_r$, the partial ordering $\leq$ is defined by:

$$f \leq g \Leftrightarrow \forall x, \begin{cases} g(x) \geq f(x) \geq r(x), & \text{if } g(x) \geq r(x), \\ g(x) \leq f(x) \leq r(x), & \text{if } g(x) < r(x). \end{cases} \tag{34}$$

The least element in $\mathcal{S}_r$ is the reference function $r(x)$, and the infimum is given by:

$$(f \wedge g)(x) = \begin{cases} \min\{f(x), g(x)\}, & \text{if } f(x), g(x) > r(x), \\ \max\{f(x), g(x)\}, & \text{if } f(x), g(x) < r(x), \\ r(x), & \text{otherwise.} \end{cases} \tag{35}$$

Figure 2(a) shows an example of applying the infimum operation in a 1-D reference semilattice.

The infimum in a reference lattice is identical to a morphological operation called *center* [2]; specifically, $f \wedge g$ in $\mathcal{S}_r$ is equal to the center of $f(x), g(x), r(x)$. The center is usually very useful for designing self-dual morphological filters [11].

Notice that, if we set $r(x) \equiv 0$, then the resulting reference semilattice is identical to a difference semilattice, which makes the latter a special case of the former.

Unfortunately, the operator given by eq. (31) is not an erosion in a reference semilattice (unless $r(x)$ is constant). This is because translation does not commute with the infimum operation in $\mathcal{S}_r$. To correct this, we define the following translation-like operator, which does commute with the infimum:

$$[T_z(f)](x) \triangleq \begin{cases} \max[f(x+z), r(x+z)], & \max[f(x+z), r(x+z)] > r(x) \text{ and } f(x) > r(x), \\ \min[f(x+z), r(x+z)], & \min[f(x+z), r(x+z)] < r(x) \text{ and } f(x) < r(x), \\ r(x), & \text{otherwise.} \end{cases} \tag{36}$$

This permits us to define the following erosion in $\mathcal{S}_r$:

$$\varepsilon(f) \triangleq \bigwedge_{z \in B} T_z(f), \tag{37}$$

Note that the translation $T_z(f)$ is itself an erosion, since it commutes with the infimum.

The adjoint dilation of the above erosion is given by:

$$\delta(f) = \bigvee_{z \in B} \tilde{T}_z(f), \tag{38}$$

where $\tilde{T}_z(f)$ is the adjoint dilation of $T_z(f)$, and is defined as follows:

$$\left[\tilde{T}_z(f)\right](x) \triangleq \begin{cases} f(x-z), & [f(x-z) > r(x) \text{ and } f(x-z) > r(x-z)] \text{ or} \\ & [f(x-z) < r(x) \text{ and } f(x-z) < r(x-z)] \\ r(x), & \text{otherwise.} \end{cases} \tag{39}$$

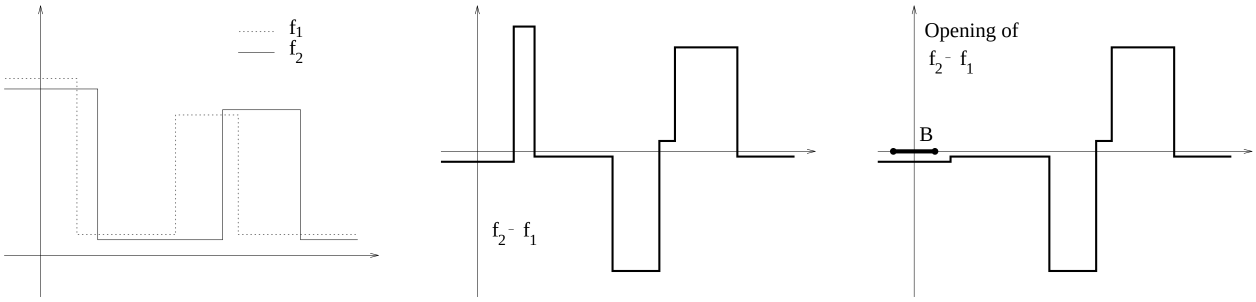
Here, the supremum $\vee$ assumes the format:

$$(f \vee g)(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } f(x), g(x) \geq r(x), \\ \min\{f(x), g(x)\}, & \text{if } f(x), g(x) \leq r(x), \\ \text{Non existent}, & \text{otherwise.} \end{cases} \tag{40}$$

As usual, the morphological opening associated with the above erosion ε is obtained according to: $\gamma = \delta\varepsilon$.

Figures 2(b)-(d) show examples of applying the above operations in a 1-D reference semilattice. Notice that, now, self-duality (in the strict sense of $\varepsilon(-f) = -\varepsilon(f)$) is lost. More insight regarding the characteristics and uses of the above operators is given in Section 5 below.

The definitions of gradient, Top-Hat transform, and skeleton, remain conceptually the same in reference semilattices. Figures 2(e),(f) show examples related to the erosion defined above.



(a) (b) (c)

Figure 3: Schematic example of fast-motion detection. (a) Two adjacent frames of a schematic 1-D sequence, (b) difference image, (d) result of the difference-semilattice scheme.

5 Applications

In this section, we refer to a few possible applications of the semilattices defined in Section 4. A more thorough examination of these applications (and others) is still required.

5.1 Fast-Motion Detection

Suppose we wish to detect *fast* motion in a video sequence, i.e., given 2 frames from a video sequence, we wish to detect the regions of the second frame that undertook a spatial displacement greater than a fixed number of pixels. This could be required for surveillance purposes, for instance.

We propose here two schemes that provide approximations to the desired result, one using a difference semilattice, and the other using a reference semilattice.

The first scheme consists of filtering the difference images (given by the pixel-by-pixel difference between adjacent frames) by means of an opening in the corresponding difference semilattice. In difference images, regions with high absolute value usually indicate motion or innovation (objects or regions that appear in only one of the two frames), and, in the former case, the width of the region is usually related to the amount of displacement (i.e., speed of motion). See a simple schematic example in Figures 3(a),(b). By filtering components with width smaller than a certain threshold out of the difference image, one gets an image which retain mainly innovation and fast motion (displacement higher than the filtering threshold). This kind of difference image filtering can be easily performed in a difference semilattice, by means of the opening by a disc-like flat structuring element, with diameter equal to the filtering threshold. The result of such an operation for the above schematic example is given in Figured 3(c). As another example, consider the two adjacent frames of the "Table tennis" sequence, shown in Figures 4(a) and (b). The difference image is shown in Figure 4(c). The opening of the difference image by a 3×3 square is given in Figure 4(d). The components of the opening result can be roughly interpreted as those image components with displacement greater than 3 pixels, or innovation.

As mentioned above, this scheme does not provide an exact solution to the proposed problem, i.e., there is not necessarily a one-to-one correspondence between the components in the filtered image and the fast-motion features in the sequence. This is because factors other

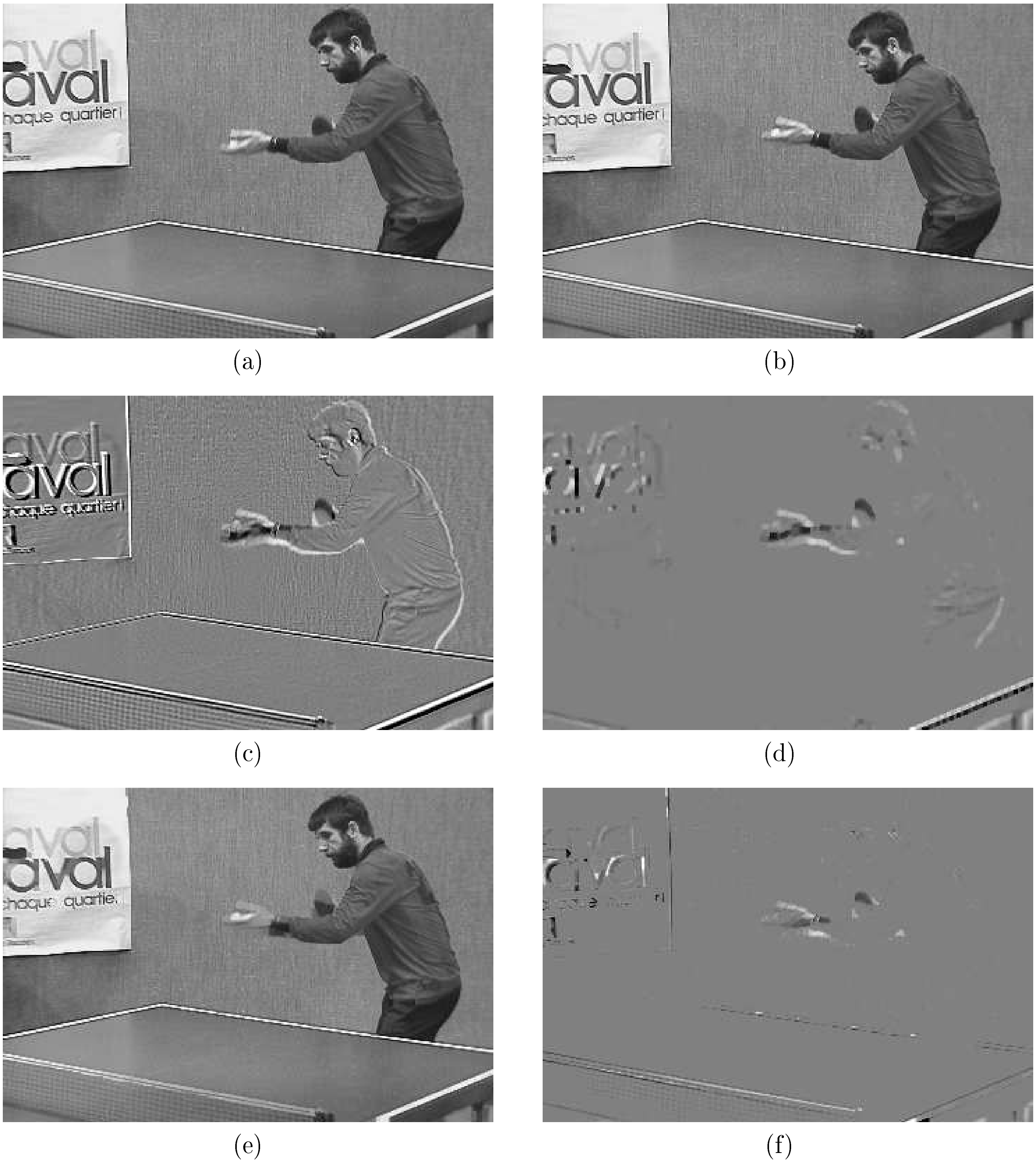
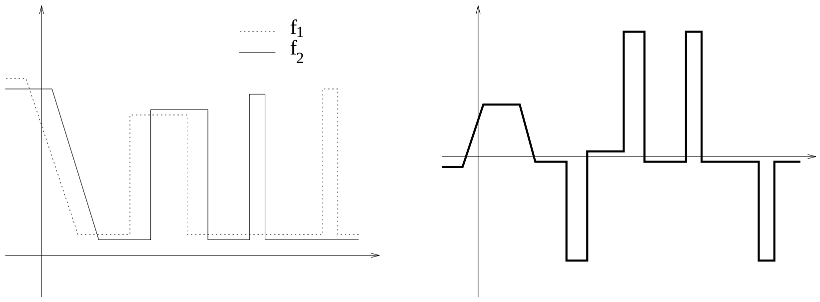


Figure 4: Example of fast-motion detection for a real sequence. (a) and (b) Two adjacent frames of the sequence “Table-Tennis”, (c) difference image, (d) final result of the difference-semilattice scheme: Opening of (c) in a difference semilattice, (e) Opening of (a) in the reference-semilattice defined by (b), (f) final result of the reference-semilattice scheme: Difference between (b) and (e).



(a) (b)

Figure 5: (a) Two adjacent frames of a schematic 1-D sequence, showing a blur and a sharp objects displaced by the same amount, and a thin object displaced by a greater amount, (b) difference image, showing that the width of the blur edge is greater than that of the sharp edges, and that the elements corresponding to the fast-moving thin object are thin.

than motion affect the width of the elements in the original difference image as well. For instance, a blur edge may produce a large component in the difference image even if it undertakes a small displacement, and thin objects usually produce thin components in the difference image even if they move fast (see schematic example in Figure 5). Therefore the above scheme sometimes erroneously retains slow-moving blur elements, and is usually not able to detect fast motion of thin objects.

The second scheme for fast-motion detection that we present is based on a reference semilattice. It usually overcomes the first of the above two disadvantages of the difference-semilattice scheme; i.e., it is robust to blurring. As for the second disadvantage, this is often manifested in the opposite way; the scheme usually detects fast moving-thin objects, but sometimes it also erroneously retains slow-moving ones as well.

The reference-semilattice scheme consists of the following steps (depicted schematically in Figure 6):

1. Given two adjacent frames f_1 and f_2, in this chronological order, create a reference semilattice where f_2 is the reference image[4].

2. In the above semilattice, calculate:

$$e \triangleq \varepsilon^n(f_1) = \underbrace{\varepsilon \cdots \varepsilon}_{n \text{ times}}(f_1), \tag{41}$$

where the structuring element B is now a *unitary* disc-like element, and n is the filtering threshold. Each iteration of the above series of erosions causes the edges in f_1 to "migrate" one pixel in the direction of the edges in f_2. The overall operation is also an erosion, causing a total displacement of up to n pixels. Some edges in f_2 are reached by the migrating edges of f_1; these are considered to correspond to slow-moving objects (with respect to the threshold n). Thus, the edges of f_2 that are not reached correspond to fast-moving objects.

[4]Note that, in many applications, including MPEG, the term "reference image" refers to f_1, but this situation is inverted here.

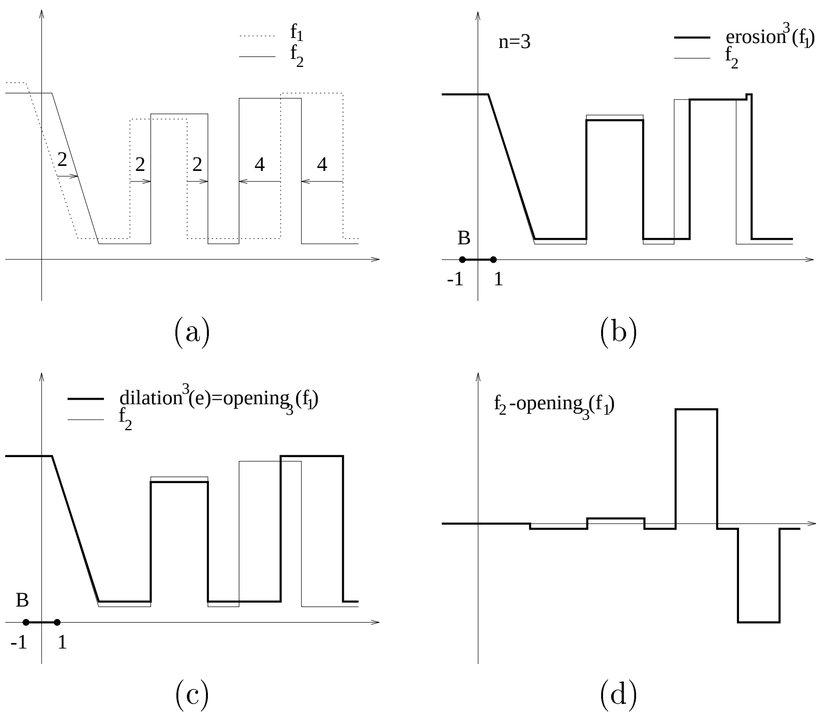


Figure 6: Schematic example of fast-motion detection using a reference semilattice. (a) Two adjacent frames of a schematic 1-D sequence, (b) three iterations of erosion, causing migration of the edges in f_1 towards f_2, (c) opening obtained after three iterations of dilations on (b). This causes the edges if f_1 that did not reach those in f_2 in the previous step to return to their original position, (d) the final result: $f_2 - \text{opening}_3(f_1)$, containing those edges in f_2 corresponding to fast-moving objects.

3. Calculate: $\delta^n(e)$. This inverts the direction of the migration. The edges of f_1 that did not reach an edge in f_2 return to their original position. On the other hand, those edges of f_1 that did reach an edge in f_2 are retained and do not move back. The overall operation $\delta^n(e) = \delta^n \varepsilon^n(f_1)$ consists of a morphological opening, which we denote by $\gamma_n(f_1)$. According to the above explanation, this opening causes slow-moving objects to "move" to their corresponding positions in f_2, while fast-moving objects remain still.

4. Subtract the result from the reference image, i.e., calculate: $g \triangleq f_2 - \gamma_n(f_1)$. The resulting image is expected to approximately retain the elements in f_2 which are related to fast-moving objects, while slow-moving objects are removed.

Figure 4(d) shows the result of the above operation on the "Table-Tennis" sequence. The structuring element used here is a 3×3 square, and the number of iterations is $n = 3$, so that, approximately, edges displaced less than 3 pixels are removed, leaving fast-moving edges and innovation.

In order to the edges of an object in f_1 to migrate to the edges of the corresponding object in f_2, there must exist an overlapping between these objects. In this case, the intersection between an object in f_1 and its counterpart in f_2 serves as a "source," which expands by means of the above series of erosions. For n sufficiently large, ε^n will then cause this source to ultimately "fill" the whole object in f_2. On the other hand, if the motion of an object is such that there is no intersection between its versions in the two frames, then there will be no source to fill it in f_2. This is many times the case with moving thin objects, in which case they are retained in the final result, even if the motion is relatively small.

5.2 Innovation Extraction

Suppose now that we are interested in retaining innovation only. This could be required in applications such as inspection.

Consider, for example, the following specific application: A physician wishes to obtain an image showing the blood vessels in a certain region of the body of a patient. For this purpose, two X-ray images (f_1 and f_2) of the corresponding region are taken, respectively before and after a contrasting agent is injected and spread in the blood vessels (see Figures 7(a),(b)). The desired image consists of the innovation only. Notice that a simple difference between the images does not provide a good solution, because it contains not only the desired vessels, but also artifacts related to slight movements of the patient (see Figure 7(c)).

Similarly to before, we propose a reference semilattice as framework, with f_2 as the reference image, and an opening γ_n associated to a series of n erosions, ε^n, by a unitary disc-like structuring element. Now, we choose n larger than the larger expected (or assumed) displacement, so that the above opening causes all (or most) edges in f_1 to stick to their counterpart f_2. The above process can be roughly seen as an image registration process. As before, we subtract the result from f_2, obtaining the innovation only (see Figure 7(d)).

5.3 Compression of Segmented Sequences

Recently, segmentation-based coding has been emerging as an efficient approach to very high compression of video sequences [6]. It consists of, first, partitioning each frame into

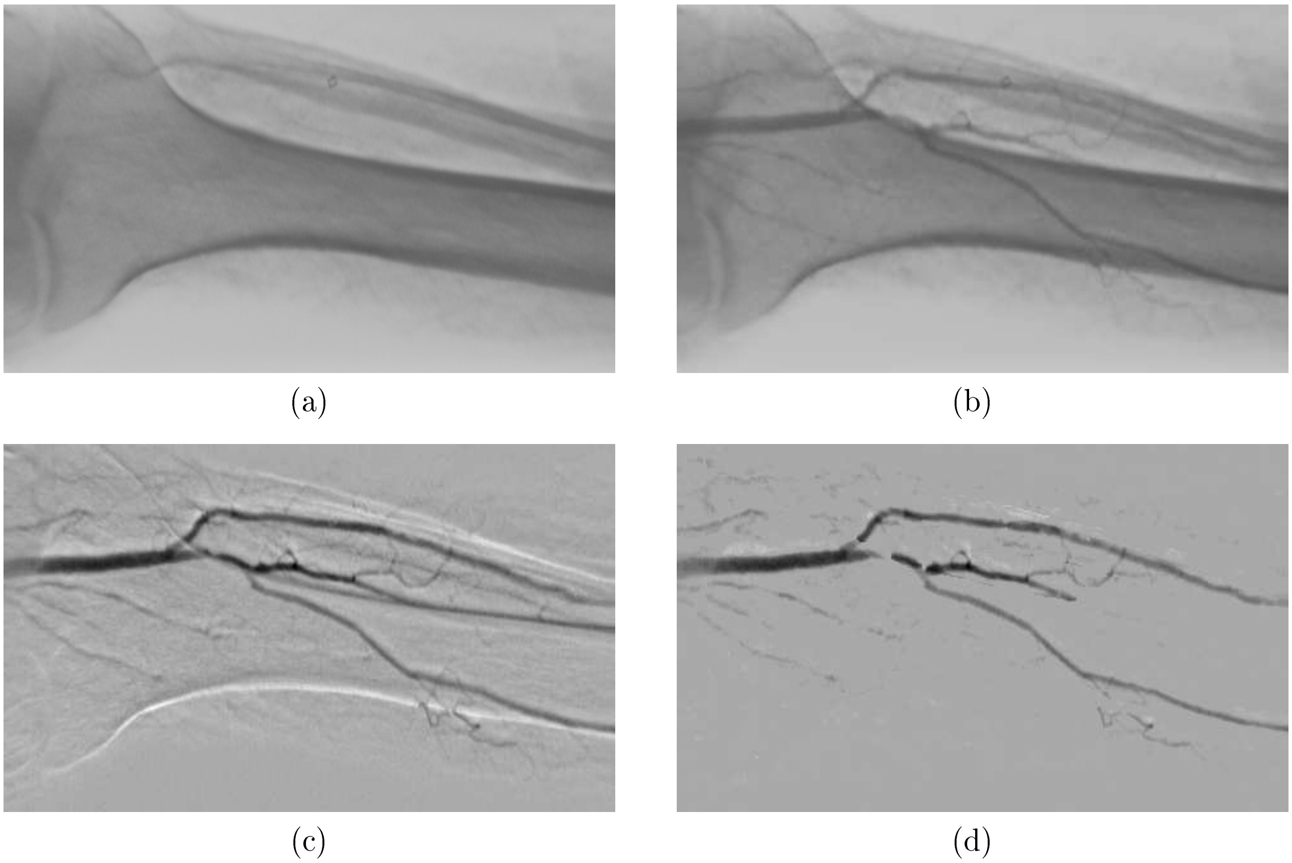


Figure 7: Innovation extraction. (a),(b) X-ray pictures, taken respectively before and after injection of a contrasting agent, (c) difference image, containing the desired blood vessels, and undesired edge artifacts, (d) final result of the proposed scheme: Difference between (b) and the opening of (a) in the reference semilattice defined by (b).

uniform segments (segmentation), where uniformity means that each segment contains only smooth transitions, or texture of a fixed pattern. Then, the segments of each frame are labeled, in such a way that the same label is assigned to segments corresponding to the same region in two adjacent frames. Finally, the segment contents and borders of each frame are coded separately. For instance, the contents of smooth segments are lossy coded using some linear approach, like parametric function matching, and texture is coded by some statistical method, whereas the segment borders are coded by chain code or morphological skeleton decomposition. The coding procedure takes into account the correlation between corresponding segments in adjacent frames (see [6] for details).

We propose here a skeleton decomposition in a reference semilattice, aimed as a first step for coding segment borders. This decomposition seems (in synthetic simulations) to efficiently represent the borders, taking into consideration correlation between frames.

We assume that we are given two frames of a sequence of images, f_1 and f_2, contained labeled segments only, without their texture or slow-transition contents (see Figures 8(a),(b)).

All one needs to fully characterize a skeleton decomposition is a semilattice, and an erosion defined in it. The erosion determines its associated opening, and the decomposition family (see eq. (20)), which are used in the recursive formula (21). This procedure provides the skeleton representation $\{s_n\}$.

In the present case, we choose as semilattice the reference semilattice defined by f_1 (differently from before, where we choose f_2 as the reference image). In this semilattice, we use the same erosion as in the previous applications, i.e., the one defined by eq. (37). The resulting skeleton decomposition is then applied to f_2. Figure 8(c) shows the positions of the skeleton points, related to the representation of Figure 8(b) in the reference semilattice defined by Figure 8(a). For simplicity, the radii of the skeleton points are not indicated, but they are needed for (perfect) reconstruction of 8(b). As one can observe in Figure 8(d), most of the skeleton points are attached to edges of f_1. For this reason, and because we assume a coding procedure where both the coder and the decoder already know f_1 when coding f_2, we expect to be able to efficiently code the positions of most of the skeleton points of f_2 (for instance, by an adapted version of the algorithm in [9]).

6 Conclusion

The basic operators of Mathematical Morphology are extended from complete lattices to complete semilattices. We prove that many of the properties enjoyed by these operators in the traditional lattice theory are also enjoyed by their semilattice counterparts.

Two examples of semilattices, which are potentially useful for image processing, are provided, namely, difference and reference semilattices, where the former is a special case of the latter. In each of these semilattices, a set of morphological basic operators (erosion, adjoint dilation, and opening) is defined.

Finally, three applications are briefly described, in which the above semilattices (the reference semilattice especially) seem to be useful, according to preliminary simulations. Namely, the applications are "fast-motion detection," "innovation extraction," and "compression of sequences."

Future work will follow two directions: Theoretical – broadening of operator properties in general complete semilattices, study of reference semilattices, and definition and analysis of morphological operators in them, – and practical – thorough investigation of the applications presented here, and search for other relevant applications.

Acknowledgment

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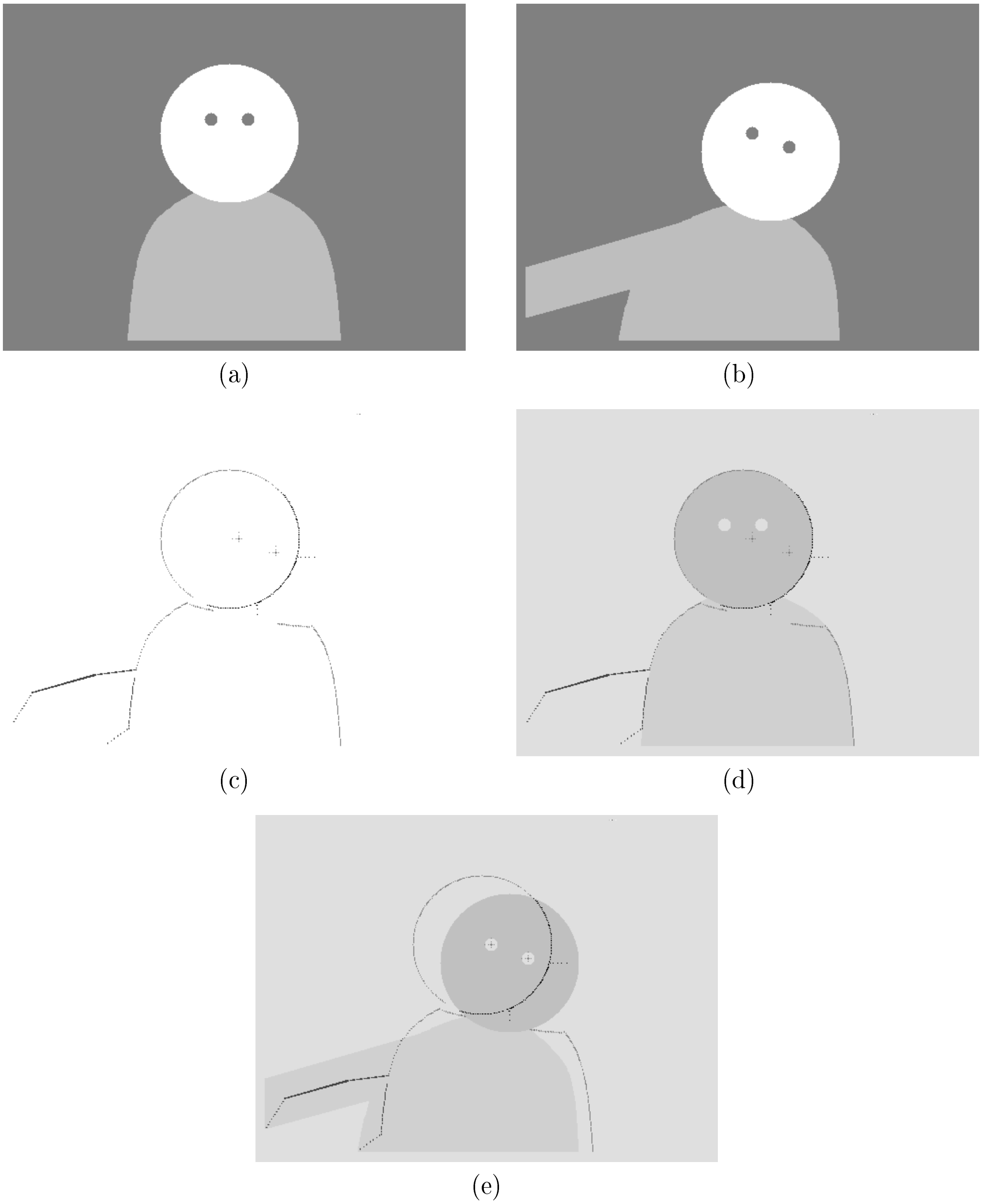

Figure 8: Skeleton representation in a reference semilattice. (a),(b) Two frames of a synthetic sequence of labeled segments, (c) the skeleton decomposition (skeleton position only, radii omitted) of (b) in the reference semilattice defined by (a), (d) the skeleton superposed to (a), (e) the skeleton superposed to (b).

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