A Symbolic Derivation of Beta-splines of Arbitrary Order

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Beta-splines are a class of splines with applications in the construction of curves and surfaces for computer-aided geometric design. One of the salient features of the Beta-spline is that the curves and surfaces thus constructed are *geometrically continuous*, a more general notion of continuity than the one used in ordinary B-splines. The basic building block for Beta-splines of order k is a set of *Beta-polynomials* of degree k-1, which are used to form the Beta-spline basis functions. The coefficients of the Beta-polynomials are functions of certain *shape parameters* $\beta_{s,j}$. In this paper, we present a symbolic derivation of the Beta-polynomials as polynomials over the field $\mathbf{K}_n$ of real rational functions in the indeterminates $\beta_{s,j}$. We prove, constructively, the existence and uniqueness of Beta-polynomials satisfying the design objectives of geometric continuity, minimum spline order, invariance under translation, and linear independence, and we present an explicit symbolic procedure for their computation. The initial derivation, and the resulting procedure, are valid for the general case of discretely-shaped Beta-splines of arbitrary order, over uniform knot sequences. By extending the field $\mathbf{K}_n$ with symbolic indeterminates z_s representing the lengths of the parametric intervals, the result is generalized to discretely-shaped Beta-splines over non-uniform knot sequences.

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1 Introduction

Beta-splines are a class of piecewise polynomial splines with applications in the construction of curves and surfaces for computer graphics and computer-aided geometric design. The original cubic Beta-spline was first developed in Barsky (1981). Beta-splines combine the power of a general mathematical formulation with an intuitive specification mechanism based on *shape parameters*. The shape parameters arise from the extra degrees of freedom that are liberated by constructing curves and surfaces using a more relaxed form of continuity called *geometric continuity* (Barsky & DeRose, 1984). The Beta-spline representation is sufficiently general and flexible so as to be capable of modeling irregular curved surface objects, and the presence of the shape parameters helps to make this technique quite useful in the modeling of complex curved shapes, such as those occurring in automotive bodies, aircraft fuselages, ship hulls, and turbine blades. The widely used B-splines (de Boor, 1972; de Boor, 1978; Riesenfeld, 1973; Schumaker, 1981) are a special case of Beta-splines, obtained by a specific setting of the shape parameters.

Subsequent to the original development of the Beta-spline, various aspects of this representation have been pursued, and numerous articles on Beta-splines and geometric continuity have appeared in the literature. For a survey of work in this area, including recent results and extensive bibliographies, the reader is invited to consult some of the tutorial articles and books on the subject, such as Barsky (1981), Barsky (1988), Barsky (1989) and Barsky & DeRose (1990) for Beta-splines, and such as Barsky & DeRose (1989), Barsky & DeRose (1990), Gregory (1989), Herron (1987) and Hoellig (1986) for geometric continuity. In addition, Bartels *et al.* (1987) is a comprehensive reference for both subjects.

Most of the results in the literature deal with *cubic* Beta-splines (order $k=4$). Generalization of Beta-splines to higher order was first discussed in Barsky & DeRose (1984), where the conditions for geometric continuity of arbitrary order, called the *Beta-constraints*, were developed. The existence of Beta-splines of arbitrary order satisfying these constraints was established in Dyn & Micchelli (1988) and in Goodman (1985). An explicit method for the computation of basis functions in the case of *uniformly-shaped* Beta-splines (where the same set of shape parameters is used in all parts of the curve) is presented in Dyn *et al.* (1987), together with explicit examples for cubic and quartic basis functions. However, there has been no general procedure for deriving the more general case of *discretely-shaped* Beta-splines of arbitrary order (where different sets of shape parameters are used in different parts of the curve, as will be precisely defined in Sections 2 and 3), except on a case-by-case basis. In a recent report (Seidel, 1990), explicit geometric constructions and knot insertion algorithms for Beta-splines of arbitrary order are provided by computing the representation of a Beta-spline basis function as a piecewise Bézier polynomial (Bézier, 1972). These results, as well as the results of Dyn *et al.* (1987), Dyn & Micchelli (1988) and Goodman (1985), are based on the total positivity of certain connection matrices related to the Beta-constraints, and use a classical approximation-theoretic approach to Beta-splines.

The basic building block for Beta-splines of order k is a set of *Beta-polynomials* of degree k-1, which are used to form the Beta-spline basis functions. The coefficients of the Beta-polynomials are symbolic functions of certain shape parameters $\beta_{s;i}$, for ranges of s and i to be specified elsewhere in the paper. While Beta-splines were recognized as symbolic entities from their onset, the problem of their general derivation has not been approached symbolically in the literature. In this paper, we present a purely symbolic derivation of Beta-spline basis functions of arbitrary order. Our derivation treats geometric continuity as a set of symbolic constraints on the coefficients of the Beta-polynomials. The latter are regarded as polynomials over the field $\mathbf{K}_n$ of real rational functions in the *indeterminates* $\beta_{s;i}$, and all computations and proofs are carried out over that field. We prove the existence and uniqueness of Beta-polynomials of arbitrary order, and we present explicit expressions and symbolic procedures for their computation. The method used for the derivation is similar to the one used in Lempel & Seroussi (1990) for spline bases in more general function spaces. Initially, the derivation and the resulting expressions are valid for the general case of discretely-shaped Beta-splines over uniform knot sequences. By further extending the field $\mathbf{K}_n$ with indeterminates z_s representing the lengths of the parametric intervals, the result is then generalized to discretely-shaped Beta-splines over non-uniform knot sequences.

The remainder of this paper is organized as follows. In Section 2 we describe the basic geometric setting for the construction of piecewise continuous parametric curves. We then present, first, the concept of parametric continuity on which traditional representations such as B-splines are based. Then, we define the more relaxed notion of geometric continuity, and present the Beta-constraints as the conditions for geometric continuity of a parametric curve. In Section 3 we formally define the Beta-polynomials and Beta-splines, and set four design objectives to be satisfied by them: geometric continuity, minimum spline order, invariance under translation, and linear independence. It is later shown in the paper that these design objectives uniquely determine the Beta-polynomials. In Section 4, we transform the design objectives into a set of linear equations in the coefficients of the Beta-polynomials. In Section 5 we explicitly solve these equations, show that the solution is unique, and present a symbolic procedure for the computation of Beta-polynomials over uniform knot sequences. In Section 6 we discuss local control, and determine the span of influence of the shape parameters $\beta_{s;i}$. In Section 7 we extend the results of Sections 5 and 6 to Beta-splines over non-uniform knot sequences. We also generalize to arbitrary order a well known result for the cubic case, showing an equivalence between uniform and non-uniform Beta-splines. Finally, in Section 8 we discuss end conditions and multiple knots in Beta-spline curves, and in Section 9 we present some concluding remarks.

2 Geometric Continuity and Beta-constraints

Let k and m be positive integers, with $m \geq k$, and let $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_m$ be a sequence of points in $\mathbb{R}^d$. The following describes a standard way of constructing an *order k, piecewise polynomial, parametric curve* $\mathbf{q}(u)$ in $\mathbb{R}^d$: Let $u_0 < u_1 < \cdots < u_{m-k+1}$ be a monotonically

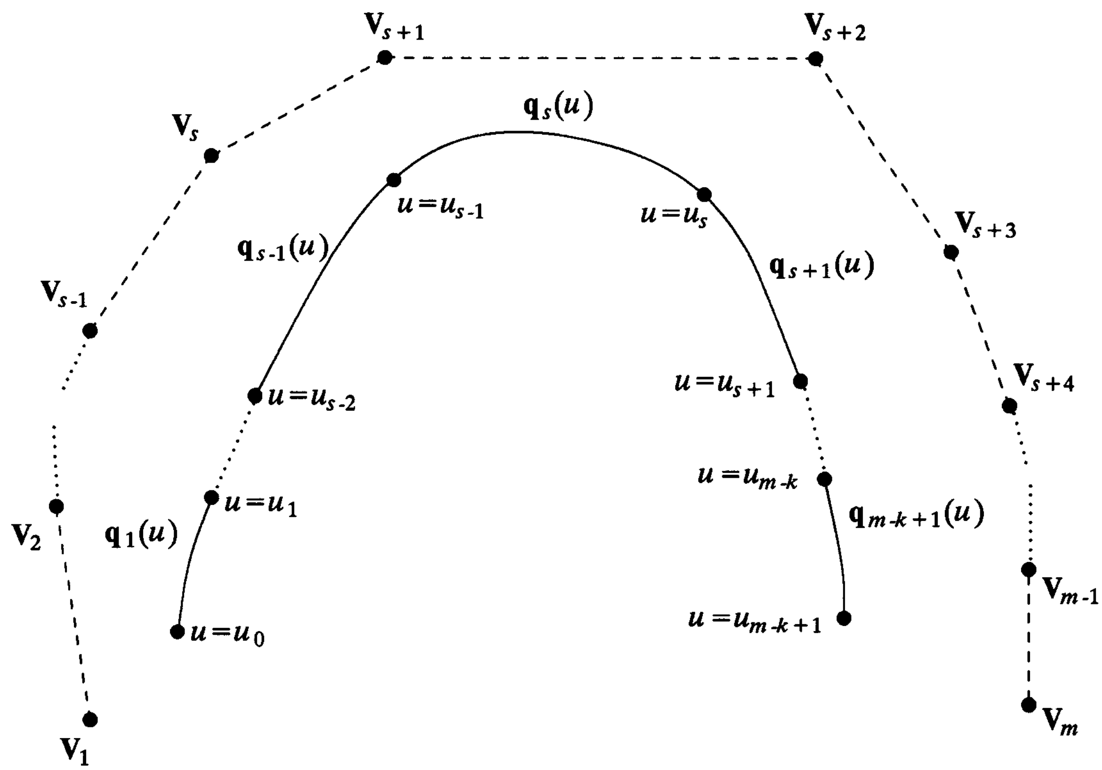

Figure 1: A piecewise continuous curve $\mathbf{q}(u)$

increasing[1] sequence of real numbers (known as *knots*). Then, $\mathbf{q}(u)$ is defined over the domain $u_0 \le u < u_{m-k+1}$ as a concatenation of *polynomial curve segments* $\mathbf{q}_1(u)$, $\mathbf{q}_2(u)$, $\cdots$, $\mathbf{q}_{m-k+1}(u)$ as follows:

$$\mathbf{q}(u) = \mathbf{q}_s(u)\,, \quad u_{s-1} \le u < u_s\,, \quad 1 \le s \le m-k+1\,, \tag{2.1}$$

where

$$\mathbf{q}_s(u) = \sum_{i=0}^{k-1} \mathbf{V}_{s+i}\, b_{s;i}(u-u_{s-1})\,, \quad u_{s-1} \le u \le u_s\,, \quad 1 \le s \le m-k+1\,. \tag{2.2}$$

Here, the $b_{s;i}(u)$ are scalar-valued polynomials[2] of degree at most k-1 in u. Notice that in (2.2) we define $\mathbf{q}_s(u)$ over the closed interval $[u_{s-1}, u_s]$, even though in (2.1) we use it only over the semi-open interval $[u_{s-1}, u_s)$.

The points $\mathbf{V}_i$ are called *control vertices*, and they form the *control polygon* of $\mathbf{q}(u)$. A typical curve $\mathbf{q}(u)$ and its control polygon are depicted in Figure 1. Strictly speaking, $\mathbf{q}(u)$ is just one of many possible parametrizations of the represented curve. We assume that

[1] This will be relaxed to *monotonically non-decreasing* in Section 8.

[2] We use a semicolon to emphasize the distinction between the segment index s and the element index i. We shall use this convention throughout this paper for polynomials, vectors and matrices. For example, $A_{s;ij}$ will denote the $(i,j)^{th}$ entry of a matrix A_s, related to the segment with index s.

$\mathbf{q}(u)$ is a *regular parametrization*; i.e. its first derivative vector never vanishes. Clearly, being built of polynomial segments, $\mathbf{q}(u)$ is continuous and infinitely differentiable at any internal parametric value u in the range $u_{s-1}<u<u_s$, for $1 \le s \le m-k+1$, but potential discontinuities may occur at the knots, where the curve segments meet. We say that $\mathbf{q}(u)$ is *n^{th} degree parametrically continuous*, denoted C^n, at $u=u_s$, if

$$\mathbf{q}_{s+1}^{(j)}(u)\,|_{u=u_s} = \mathbf{q}_s^{(j)}(u)\,|_{u=u_s}, \quad 0 \le j \le n, \quad 1 \le s \le m-k, \tag{2.3}$$

where $g^{(j)}(u)\,|_{u=x}$ denotes the j^{th} derivative of a function $g(u)$ with respect to u, evaluated at $u=x$. In this case, we also say that $\mathbf{q}_s(u)$ *meets* $\mathbf{q}_{s+1}(u)$ with n^{th} degree parametric continuity at $u=u_s$.

The concept of parametric continuity is used in computer-aided geometric design to capture the notion of "smoothness" of a piecewise parametric curve. In particular, the widely used B-splines (see, for instance, de Boor, 1978; Bartels *et al.*, 1987) form a basis for a very general family of parametrically continuous curves. However, this notion of continuity is sometimes too restrictive, as it reflects a property of the specific parametrization used, rather than of the curve itself.

For example, consider the 2-dimensional curve $(x(u), y(u))$ defined by the following parametrization:

$$(x(u), y(u)) = \begin{cases} (u^2-u,\ u^2+u) & 0 \le u < 1\,, \\ (4u^2-6u+2,\ 4u^2-2u) & 1 \le u < 2\,. \end{cases}$$

It can be readily verified that this parametrization has a first derivative discontinuity at $u=1$ (the derivative vector is (1,3) to the left of $u=1$, and (2,6) to the right). However, it can also be readily verified that for all u in the range $0 \le u < 2$, $x(u)$ and $y(u)$ satisfy the quadratic equation $x^2+y^2-2xy-2x-2y = 0$ (a parabola), and hence, the curve does not have any discontinuities in the (x,y) plane.

We now define a more general notion of continuity for parametric curves, called *geometric continuity*, which was first presented in the computer aided geometric design literature in Barsky & DeRose (1984); a similar concept called *contact of order n* was described in the German geometry literature in Geise (1962) and Scheffers (1910).[3]

We say that $\mathbf{q}(u)$ is *n^{th} degree geometrically continuous*, denoted G^n, at $u=u_s$ if there exists a continuous, n times differentiable *reparametrization* function $u=f_s(\tilde{u})$,

[3] Notice, however, that contact of order n refers to the tangential contact of two curves rather than the piecewise joining of curve segments.

$f_s : [\tilde{u}_{s-1}, \tilde{u}_s] \to [u_{s-1}, u_s]$, such that at the end points of the interval, we have $f_s(\tilde{u}_{s-1}) = u_{s-1}$, $f_s(\tilde{u}_s) = u_s$, and $\mathbf{q}_s(f_s(\tilde{u}))$ meets $\mathbf{q}_{s+1}(u)$ with n^{th} degree parametric continuity at $u = u_s$; i.e., we have

$$\mathbf{q}_{s+1}^{(j)}(u)\,|_{u=u_s} = \mathbf{q}_s^{(j)}(f_s(\tilde{u}))\,|_{\tilde{u}=\tilde{u}_s}, \quad 0 \le j \le n, \quad 1 \le s \le m\text{-}k\ , \tag{2.4}$$

where the j^{th} derivative at the righthand side of (2.4) is taken with respect to $\tilde{u}$. In computer-aided geometric design applications, the reparametrization function f_s is required to be *orientation preserving*; i.e. $f_s^{(1)}(\tilde{u}) > 0$. This guarantees that the new parametrization is regular, and also, that $\mathbf{q}_s(f_s(\tilde{u}))$, $\tilde{u}_{s-1} \le \tilde{u} \le \tilde{u}_s$, traces the same curve in $\mathbb{R}^d$ as $\mathbf{q}_s(u)$, $u_{s-1} \le u \le u_s$, without backtracking. Notice also that in practical applications, we do not need to actually compute the reparametrized curve segment. Instead, the curve is manipulated using the original parametrization, and the existence of a parametrically continuous reparametrization guarantees geometric continuity, and hence, the "smoothness" of the curve.

Let

$$\beta_{s;j} = f_s^{(j)}(\tilde{u})\,|_{\tilde{u}=\tilde{u}_s}\ , \quad 1 \le j \le n,$$

and let $\boldsymbol{\beta}_s = (\beta_{s;1}\ \beta_{s;2}\ \cdots\ \beta_{s;n})$. Then, using the chain rule for derivatives to expand the righthand side of (2.4), the latter is transformed into

$$\mathbf{q}_{s+1}^{(j)}(u_s) = \sum_{r=0}^{j} M_{s;jr}\mathbf{q}_s^{(r)}(u_s), \quad 0 \le j \le n, \tag{2.5}$$

where the coefficients $M_{s;jr}$ are polynomials in $\beta_{s;1}, \beta_{s;2}, \cdots, \beta_{s;n}$ derived from the application of the chain rule. These polynomials are given by *Faa di Bruno's formulas*, as given on page 50 of Knuth (1973):

$$M_{s;jr} = \sum_{\substack{k_1+k_2+\cdots+k_j = r \\ k_1+2k_2+\cdots+jk_j = j \\ k1,k2,\cdots,k_j \ge 0}} \frac{j!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots k_j!(j!)^{k_j}}\ \beta_{s;1}^{k_1}\beta_{s;2}^{k_2}\cdots\beta_{s;j}^{k_j}\ , \tag{2.6}$$

$$0 \le j \le n,\ 0 \le r \le n.$$

(In the above equation, we assume that empty sums are equal to zero. Also, the range of r has been extended to $0 \le r \le n$, but it is easily verified that $M_{s;jr} = 0$ for $r > j$). Define the $(n+1)\times(n+1)$ matrix $M_s(\boldsymbol{\beta}_s) = (M_{s;jr})$, $0 \le j \le n$, $0 \le r \le n$. Then, $M_s(\boldsymbol{\beta}_s)$ is lower triangular, and it has the general form

$$M_s(\boldsymbol{\beta}_s) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \beta_{s;1} & 0 & 0 & \cdots & 0 \\ 0 & \beta_{s;2} & \beta_{s;1}^2 & 0 & \cdots & 0 \\ 0 & \beta_{s;3} & 3\beta_{s;1}\beta_{s;2} & \beta_{s;1}^3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \beta_{s;n} & M_{s;n2} & M_{s;n3} & \cdots & \beta_{s;1}^n \end{bmatrix} . \tag{2.7}$$

$M_s(\boldsymbol{\beta}_s)$ is called a *Beta-connection matrix*. The following properties of $M_s(\boldsymbol{\beta}_s)$ are readily derived from (2.6):

(M1) $M_{s;00}=1$, $M_{s;0r}=M_{s;j0}=0$ for $1 \le j \le n$, $1 \le r \le n$.

(M2) $M_{s;j1}=\beta_{s;j}$ for $1 \le j \le n$.

(M3) $M_{s;jj}=\beta_{s;1}^j$ for $0 \le j \le n$.

(M4) $M_{s;jr}$ does not depend on n, as long as $j \le n$ and $r \le n$. For example, the 4×4 matrix at the upper left corner of $M_s(\boldsymbol{\beta}_s)$ in (2.7) is the Beta-connection matrix for $n=3$.

(M5) $M_s([1\,0\,0 \cdots 0])=I$, the identity matrix of order $n+1$.

In the remainder of the paper, we shall denote the Beta-connection matrix by M_s, the dependence on $\boldsymbol{\beta}_s$ being understood from the context.

Using the definition of M_s, Equation (2.5) can be rewritten in matrix form as

$$\begin{bmatrix} \mathbf{q}_{s+1}^{(0)}(u_s) \\ \mathbf{q}_{s+1}^{(1)}(u_s) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{q}_{s+1}^{(n)}(u_s) \end{bmatrix} = M_s \begin{bmatrix} \mathbf{q}_{s}^{(0)}(u_s) \\ \mathbf{q}_{s}^{(1)}(u_s) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{q}_{s}^{(n)}(u_s) \end{bmatrix} . \tag{2.8}$$

These are known as the *Beta-constraints* (Barsky & DeRose, 1984; Barsky & DeRose, 1989; Goldman & Barsky, 1989; Goodman, 1985) for n^{th} degree geometric continuity at $u=u_s$. Notice that, by property M5, when $\boldsymbol{\beta}_s=(1\,0\,0 \cdots 0)$, (2.8) reduces to the usual parametric continuity constraints.

Let $z_s = u_s - u_{s-1}$. We shall now assume that the knots u_s are *uniformly spaced*, and that the parametric intervals have unit length, namely, $z_s=1$ for $1 \le s \le m-k+1$. Later, in Section 7, we shall prove that there is no loss of generality in making this assumption, and we shall see that changing the parametric interval lengths is equivalent to a transformation of the parameters $\beta_{s;i}$, and scaling of the variable u.

3 Beta-splines

Under the uniform knot spacing assumption, the curve segment equation (2.2) can be rewritten as follows:

$$\mathbf{q}_s(u) = \sum_{i=0}^{k-1} \mathbf{V}_{s+i} b_{s;i}(u-u_{s-1}), \quad u_{s-1} \le u \le u_{s-1}+1=u_s, \quad 1 \le s \le m-k+1\,, \tag{3.1}$$

where $b_{s;0}(u), b_{s;1}(u), \cdots, b_{s;k-1}(u)$ are polynomials of degree at most k-1 in u. Notice that in (3.1), $b_{s;i}(u)$ is evaluated in the interval $0 \le u < 1$. The main result of this paper is the derivation of explicit symbolic expressions for polynomials $b_{s;i}(u)$ satisfying the following *design objectives*:

(i) $\mathbf{G}^n$ **Continuity**. For a given continuity degree $n \ge 0$, and for any choice of control vertices $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_m$, the Beta-constraints (2.8) are satisfied.

(ii) **Minimum spline order**. The order of the curve is the least integer k for which objective (i) can be satisfied.

(iii) **Shape preservation under translation**. For any fixed translation vector $\mathbf{Q} \in \mathbb{R}^d$, the following holds:

$$\mathbf{Q} + \mathbf{q}_s(u) = \sum_{i=0}^{k-1} (\mathbf{Q}+\mathbf{V}_{s+i}) b_{s;i}(u-u_{s-1}), \quad u_{s-1} \le u < u_s, \quad 1 \le s \le m-k+1.$$

Hence, a translation of all control vertices by a fixed vector results in the translation of all points of the curve by the same vector.

(iv) **Linear independence**. The polynomials $b_{s;0}(u), b_{s;1}(u), \cdots, b_{s;k-1}(u)$ are linearly independent and thus, they form a basis for the k-dimensional linear space of polynomials of degree at most k-1 in u.

Polynomials $b_{s;i}(u)$ satisfying objectives (i)-(iv) will be called *Beta-polynomials*, and the curves constructed according to (2.1)-(2.2) will be called called *Beta-spline curves*. The parameters $\beta_{s;i}$ are referred to as *shape parameters*. A short discussion of terminology is in order at this time. Beta-polynomials are also known in the literature as *Beta-spline basis segments* (Bartels *et al.*, 1987), and the corresponding piecewise functions $F_s(u)$, defined over the domain $[u_0, u_{m-k+1}]$ by

$$F_s(u) = \begin{cases} 0 & u<u_{s-k} \text{ or } u \geq u_s \\ & & 1 \leq s \leq m\,. \\ b_{s-i;i}(u-u_{s-i-1}) & u_{s-i-1} \leq u < u_{s-i}, \quad k-1 \geq i \geq 0\,, \end{cases}$$

are known as the *Beta-spline basis functions* for the given knot sequence. This is justified by the fact that (2.1) and (2.2) are equivalent to

$$\mathbf{q}(u) = \sum_{s=1}^{m} \mathbf{V}_s F_s(u)\,.$$

The relation between the Beta-polynomials and the basis functions is depicted in Figure 2.

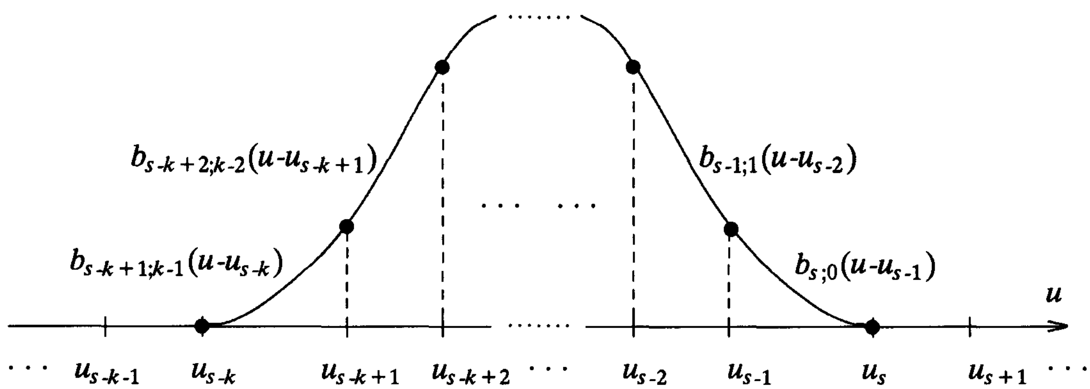

Figure 2: Basis function $F_s(u)$.

In general, however, different mathematical objects are given the name *basis function* in the literature. In the computer graphics literature, some authors refer to the Beta-polynomials as the basis functions (these polynomials are a special basis for the space of polynomials of degree $\leq k$-1). Moreover, another ambiguity arises from the fact that the term "Beta-spline" is sometimes used to refer to the basis functions F_s and other times to the curves $\mathbf{q}(u)$ constructed from them. In an attempt to avoid confusion, we shall use the term Beta-polynomials to refer to the polynomials $b_{s;i}(u)$. Furthermore, since the segment-by-segment approach appears more suited for symbolic treatment, our derivation does not deal directly with the piecewise functions $F_s(u)$. Clearly, deriving the Beta-polynomials is equivalent to deriving the functions $F_s(u)$.

As we shall see, design objectives (i)-(iv) uniquely determine the Beta-polynomials. The existence of Beta-polynomials, and other properties of Beta-splines of arbitrary order, had been established in Dyn & Micchelli (1988) and Goodman (1985), based on the total positivity of the matrix M_s. The main result of this paper is a purely symbolic and constructive proof of the existence and uniqueness of Beta-polynomials of arbitrary order, and the derivation of explicit expressions for their symbolic computation. Although the results of the paper are presented for geometric continuity based on Beta-constraints, they can be readily extended to the framework of Frenet-frame continuity (Dyn & Micchelli, 1988), where more general connection matrices are used.

We now present the formal algebraic framework for our symbolic derivation. While the $b_{s;i}(u)$ are polynomials in u, their coefficients are, in general, functions of parameters $\beta_{l;1}, \beta_{l;2}, \cdots, \beta_{l;n}$, for integers l in a range to be determined in the derivation. More formally, for integers $r \geq 0$ and $n \geq 1$, let

$$\mathbf{K}_{r;n} = \mathbf{R}(\beta_{-r;1}, \cdots, \beta_{-r;n}, \beta_{-r+1;1}, \cdots, \beta_{-r+1;n}, \cdots \quad \cdots, \beta_{r;1}, \cdots, \beta_{r;n})$$

denote the field of symbolic rational functions over $\mathbf{R}$, in the indeterminates $\beta_{l;i}$ for $-r \leq l \leq r$ and $1 \leq i \leq n$. Also, let

$$\mathbf{K}_n = \bigcup_{r=0}^{\infty} \mathbf{K}_{r;n}$$

denote the field of rational functions in the indeterminates $\beta_{l;i}$ for $1 \leq i \leq n$ and all integers l. Then, for $0 \leq i \leq k\text{-}1$, $b_{s;i}(u)$ will be a polynomial of degree at most $k-1$ with coefficients in $\mathbf{K}_n$. In the remainder of the paper, we deal with polynomials, vectors and matrices with coefficients in $\mathbf{K}_n$ and, unless explicitly stated otherwise, algebraic properties such as linear independence and nonsingularity will be understood to be defined over that field (for a treatment of extension fields and rational function fields see, for instance, Herstein, 1975).

A particular case of interest occurs when we make the substitution $\beta_{s;i} = \beta_i$ for $1 \leq i \leq n$ and all s; i.e. we use the same set of parameters $\boldsymbol{\beta} = (\beta_1 \ \beta_2 \ \cdots \ \beta_n)$ for the Beta-constraints at all the knots. The splines thus constructed are called *uniformly-shaped Beta-splines*, while the splines in the general case described above are referred to as *discretely-shaped Beta-splines* (Bartels *et al.*, 1987). Making a further restriction, we obtain another case of interest by setting $\beta_{s;1} = 1$, and $\beta_{s;2} = \beta_{s;3} = \cdots = \beta_{s;n} = 0$ for all s. In this case, the Beta-polynomials form the usual uniform B-splines, and $\mathbf{q}(u)$ is a uniform B-spline curve. If the uniform knot spacing assumption is removed, we obtain the non-uniform B-splines.

4 Beta-constraints Revisited

We now proceed to transform the design objectives (i)-(iv) to a set of algebraic equations which will later be solved for the polynomials $b_{s;i}(u)$. In the derivation, we will ignore end conditions, namely, we will derive the Beta-polynomials $b_{s;i}(u)$ for a generic segment $\mathbf{q}_s(u)$ "far away" from the ends of the curve (or, in a different interpretation, we will assume that $b_{s;i}(u)$ and all the related parameters are defined for all integers s). End conditions will be dealt with in Section 8.

Let $b_{s;ij}$ denote the coefficients of $b_{s;i}(u)$; i.e.

$$b_{s;i}(u) = \sum_{i=0}^{k-1} b_{s;ij} u^j , \quad 0 \le i \le k-1,$$

and define the $k \times k$ matrix $B_s = (b_{s;ij})$, $0 \le i,j \le k-1$. Also, let $\mathbf{b}_{s;i} = (b_{s;i0}\ b_{s;i1} \cdots b_{s;i,k-1})$. Thus, we have

$$\begin{bmatrix} b_{s;0}(u) \\ b_{s;1}(u) \\ \cdot \\ \cdot \\ \cdot \\ b_{s;k-1}(u) \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{s;0} \\ \mathbf{b}_{s;1} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_{s;k-1} \end{bmatrix} \begin{bmatrix} 1 \\ u \\ \cdot \\ \cdot \\ \cdot \\ u^{k-1} \end{bmatrix} = B_s \begin{bmatrix} 1 \\ u \\ \cdot \\ \cdot \\ \cdot \\ u^{k-1} \end{bmatrix}. \tag{4.1}$$

Clearly, B_s, $(b_{s;0}(u), b_{s;1}(u), \cdots, b_{s;k-1}(u))$ and $(\mathbf{b}_{s;0}, \mathbf{b}_{s;1}, \cdots, \mathbf{b}_{s;k-1})$ are different representations of the same mathematical object, and, in the remainder of the paper, we shall switch freely between these representations.

Consider the j^{th} order Beta-constraint (2.5), and substitute the expressions for $\mathbf{q}_s^{(j)}(u)$ and $\mathbf{q}_{s+1}^{(j)}(u)$ derived from (3.1) into (2.5). Recalling that $u_s - u_{s-1} = 1$ and that $M_{s;jr} = 0$ for $r > j$, we obtain

$$\sum_{i=0}^{k-1} \mathbf{V}_{s+i+1} b_{s+1;i}^{(j)}(0) - \sum_{r=0}^{n} M_{s;jr} \sum_{i=0}^{k-1} \mathbf{V}_{s+i} b_{s;i}^{(r)}(1) = 0, \quad 0 \le j \le n. \tag{4.2}$$

(Recall that we assume that the geometric continuity degree n is given, and we have yet to establish a relation between n and k).

Since (4.2) must hold for all choices of control vertices, the coefficient of $\mathbf{V}_{s+i}$ in the lefthand side of (4.2) must be identically zero for $0 \le i \le k$. This leads to the following system of equations, equivalent to the Beta-constraints (2.8). To emphasize the link with (4.2), we label each equation with the control vertex whose coefficient is being equated to zero.

$$\mathbf{V}_s : \quad -\sum_{r=0}^{n} M_{s;jr}\, b_{s;0}^{(r)}(1) = 0, \quad 0 \le j \le n, \tag{4.3.a}$$

$$\mathbf{V}_{s+i+1} : \quad b_{s+1;i}^{(j)}(0) - \sum_{r=0}^{n} M_{s;jr}\, b_{s;i+1}^{(r)}(1) = 0, \quad 0 \le i \le k-2, \quad 0 \le j \le n, \tag{4.3.b}$$

$$\mathbf{V}_{s+k} : \quad b_{s+1;k-1}^{(j)}(0) = 0, \quad 0 \le j \le n. \tag{4.3.c}$$

Before we further investigate Equations (4.3), we present a few more definitions. Let $S(k,u)$ be the $k \times k$ matrix with $(i,j)^{th}$ entry defined by

$$S_{ij}(k,u) \triangleq \frac{d^j(u^i)}{d\,u^j} = \binom{i}{j} j!\,u^{i-j}\,, \qquad 0 \le i,j \le k\text{-}1\,, \tag{4.4}$$

It follows immediately from (4.4) that $S(k,u)$ is a lower triangular matrix, with main diagonal entries of the form $S_{ii}(k,u) = i!$ for $0 \le i \le k$-1. Let $S_j(k,u)$ denote the j^{th} column of $S(k,u)$, $0 \le j \le k$-1, and let

$$\bar{S}(k,u) \triangleq [S_0(k,u)\ S_1(k,u)\ \cdots\ S_{k\text{-}2}(k,u)] \tag{4.5}$$

denote the $k \times (k\text{-}1)$ matrix obtained by deleting the last column from $S(k,u)$. For any row vector $\mathbf{v}=(v_0\ v_1\ \cdots\ v_r)$, let $v(u)$ denote the polynomial $v(u) = \sum_{i=0}^{r} v_i u^i$, and, conversely, given a polynomial $v(u)$, let $\mathbf{v}$ denote its vector of coefficients. It follows from (4.4) that for any polynomial $v(u)$, of degree k-1, we have

$$v^{(j)}(u) = \mathbf{v} S_j(k,u)\,. \tag{4.6}$$

LEMMA 1: $S(k,u)$ is nonsingular for all u.

PROOF. This follows immediately from the fact that $S(k,u)$ is a lower triangular matrix with no zeroes on its main diagonal. •

We are now ready to address the "minimum spline order" design objective (ii).

THEOREM 1. The spline order k must satisfy $k \ge n+2$.

PROOF. Assume, by contradiction, that $k \le n+1$. Then, it follows from (4.3.c) that $b_{s;k\text{-}1}^{(j)}(0)=0$ for $0 \le j \le k$-1. Together with (4.6), this implies that $\mathbf{b}_{s;k\text{-}1} S(k,0) = \mathbf{0}$. Since, by Lemma 1, $S(k,0)$ is nonsingular, we must have $\mathbf{b}_{s;k\text{-}1}=\mathbf{0}$, contradicting the linear independence of $\mathbf{b}_{s;0}, \mathbf{b}_{s;1}, \cdots, \mathbf{b}_{s;k\text{-}1}$ required by design objective (iv). Hence, we must have $k \ge n+2$. •

In the sequel, we assume $k=n+2$, thus meeting the "minimum spline order" requirement. Clearly, this also implies that $k \ge 2$.

We now resume our investigation of the Beta-constraints (4.3). Using (4.6), we observe that, for $0 \le i \le k$-1 and $0 \le j \le n$, we have

$$\sum_{r=0}^{n} M_{s;jr}\, b_{s;i}^{(r)}(u) = \sum_{r=0}^{n} M_{s;jr}\, \mathbf{b}_{s;i} S_r(k,u) = \mathbf{b}_{s;i} \sum_{r=0}^{n} M_{s;jr} S_r(k,u)\,.$$

Setting $k=n+2$, and recalling the definition of $\bar{S}(k,u)$ in (4.5), it follows from the last equation that

$$\sum_{r=0}^{k-2} M_{s;jr}\, b_{s;i}^{(r)}(u) = \mathbf{b}_{s;i}\, \bar{S}(k,u)\, M_{s;j}^T\,, \qquad 0 \le i \le k\text{-}1,\ \ 0 \le j \le k\text{-}2, \tag{4.7}$$

where $M_{s;j}$ is the j^{th} row of M_s, and superscript T denotes transposition. Using (4.6) and (4.7), the Beta-constraints (4.3) can be rewritten as follows:

$$\mathbf{b}_{s;0}\ \bar{S}(k,1)\, M_{s;j}^T = 0, \qquad 0 \le j \le k\text{-}2, \tag{4.8.a}$$

$$\mathbf{b}_{s+1;i}\ S_j(k,0) - \mathbf{b}_{s;i+1}\ \bar{S}(k,1)\, M_{s;j}^T = 0, \qquad 0 \le i \le k\text{-}2,\ \ 0 \le j \le k\text{-}2, \tag{4.8.b}$$

$$\mathbf{b}_{s+1;k\text{-}1}\ S_j(k,0) = 0, \qquad 0 \le j \le k\text{-}2. \tag{4.8.c}$$

Equation (4.8.a) represents k-1 scalar equations, each involving one column $M_{s;j}^T$. These can be combined into one vector equation involving the matrix M_s^T. A similar transformation can be applied to (4.8.b) and (4.8.c). Thus, we obtain the following equivalent version of the Beta-constraints:

$$\mathbf{b}_{s;0}\bar{S}(k,1)M_s^T = \mathbf{0}\,, \tag{4.9.a}$$

$$\mathbf{b}_{s+1;i}\bar{S}(k,0) - \mathbf{b}_{s;i+1}\bar{S}(k,1)M_s^T = \mathbf{0}\,, \qquad 0 \le i \le k\text{-}2, \tag{4.9.b}$$

$$\mathbf{b}_{s+1;k\text{-}1}\bar{S}(k,0) = \mathbf{0}\,. \tag{4.9.c}$$

The zero vectors at the right hand sides of (4.9.a)-(4.9.c) are of dimension k-1.

At this point, we can also address design objective (iii), "shape preservation under translation", and express it in terms of the unknown vectors $\mathbf{b}_{s;i}$. As is well known (see, for example, page 191 of Bartels *et al.*, 1987), objective (iii) is satisfied if and only if the Beta-polynomials sum to unity, namely

$$\sum_{i=0}^{k-1} b_{s;i}(u) = 1.$$

The above equation is interpreted as a polynomial identity, and can be transformed into the following vector form, which, together with (4.9.a)-(4.9.c) form our set of basic constraints.

$$\sum_{i=0}^{k-1} \mathbf{b}_{s;i} = (1\,0\ \cdots\ 0). \tag{4.9.d}$$

5 Solving for the Beta-polynomials

We now solve Equations (4.9) for the vectors $\mathbf{b}_{s;i}$. Let $\bar{\mathbf{b}}_{s;i}$ denote the vector obtained by deleting the last entry of $\mathbf{b}_{s;i}$; i.e.

$$\bar{\mathbf{b}}_{s;i} = (b_{s;i0}\ b_{s;i1}\ \cdots\ b_{s;i,k-2}),\quad 0 \le i \le k-1,$$

and let $\gamma_{s;i}$ denote the last entry of $\mathbf{b}_{s;i}$; i.e.

$$\gamma_{s;i} = b_{s;i,k-1},\quad 0 \le i \le k-1.$$

Let $\mathbf{y}(k,u)$ denote the last row of the matrix $\bar{S}(k,u)$ defined in (4.5). It follows from (4.4) and (4.5) that

$$\bar{S}(k,u) = \begin{bmatrix} S(k-1,u) \\ \\ \mathbf{y}(k,u) \end{bmatrix}. \tag{5.1}$$

Now, using the definitions of $\bar{\mathbf{b}}_{s;i}$ and $\gamma_{s;i}$, and (5.1), it follows from (4.9.c) (with index s decremented by one) that

$$\bar{\mathbf{b}}_{s;k-1}\, S(k-1,0) + \gamma_{s;k-1}\mathbf{y}(k,0) = \mathbf{0}.$$

Since $\mathbf{y}(k,0)=\mathbf{0}$, and $S(k-1,0)$ is nonsingular, this implies that we must have

$$\bar{\mathbf{b}}_{s;k-1} = \mathbf{0}. \tag{5.2}$$

One consequence of (5.2) is that $b_{s;k-1}(u)$ is always of the form $\gamma_{s;k-1}u^{k-1}$ for some $\gamma_{s;k-1} \in \mathbf{K}_n$. Using, again, (5.1) and the fact that $\mathbf{y}(k,0)=\mathbf{0}$, it follows from (4.9.b) that

$$\begin{aligned} \bar{\mathbf{b}}_{s+1;i}\, S(k-1,0) &= \mathbf{b}_{s;i+1}\, \bar{S}(k,1) M_s^T \\ &= \bar{\mathbf{b}}_{s;i+1}\, S(k-1,1) M_s^T + \gamma_{s;i+1}\, \mathbf{y}(k,1) M_s^T. \end{aligned} \tag{5.3}$$

Let

$$A_s = S(k-1,1)\, M_s^T\, S(k-1,0)^{-1}. \tag{5.4}$$

It follows from the definition of $S(k,u)$ in (4.4), and from the general form of M_s in (2.7) that A_s is a nonsingular $(k-1)\times(k-1)$ matrix of the form

$$A_s = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ \cdot & & & & & \\ \cdot & & & D_s & & \\ \cdot & & & & & \\ 1 & & & & & \end{bmatrix}, \tag{5.5}$$

where D_s is a $(k\text{-}2)\times(k\text{-}2)$ matrix with entries in $\mathbf{K}_n$. Define the $(k\text{-}1)$-dimensional vector

$$\mathbf{a}_s = \mathbf{y}(k,1)M_s^T S(k\text{-}1,0)^{-1}. \tag{5.6}$$

Multiplying both sides of (5.3) to the right by $S(k\text{-}1,0)^{-1}$, and using (5.4) and (5.6), we obtain

$$\bar{\mathbf{b}}_{s+1;i} = \bar{\mathbf{b}}_{s;i+1}A_s + \gamma_{s;i+1}\,\mathbf{a}_s\,.$$

Now, decrementing the index s by one yields

$$\bar{\mathbf{b}}_{s;i} = \bar{\mathbf{b}}_{s\text{-}1;i+1}A_{s\text{-}1} + \gamma_{s\text{-}1;i+1}\,\mathbf{a}_{s\text{-}1}\,, \quad 0\le i\le k\text{-}2\,. \tag{5.7}$$

Decrementing s again, and incrementing i, yields the following expression for $\bar{\mathbf{b}}_{s\text{-}1;i+1}$:

$$\bar{\mathbf{b}}_{s\text{-}1;i+1} = \bar{\mathbf{b}}_{s\text{-}2;i+2}A_{s\text{-}2} + \gamma_{s\text{-}2;i+2}\,\mathbf{a}_{s\text{-}2}\,, \quad 0\le i\le k\text{-}3\,.$$

Substituting this expression for $\bar{\mathbf{b}}_{s\text{-}1;i+1}$ in (5.7), we obtain

$$\begin{aligned}\bar{\mathbf{b}}_{s;i} &= \left(\bar{\mathbf{b}}_{s\text{-}2;i+2}A_{s\text{-}2} + \gamma_{s\text{-}2;i+2}\,\mathbf{a}_{s\text{-}2}\right)A_{s\text{-}1} + \gamma_{s\text{-}1;i+1}\,\mathbf{a}_{s\text{-}1} \\ &= \bar{\mathbf{b}}_{s\text{-}2;i+2}A_{s\text{-}2}A_{s\text{-}1} + \gamma_{s\text{-}2;i+2}\mathbf{a}_{s\text{-}2}A_{s\text{-}1} + \gamma_{s\text{-}1;i+1}\mathbf{a}_{s\text{-}1}, \quad 0\le i\le k\text{-}3.\end{aligned}$$

We can now shift indices again in (5.7), and obtain an expression for $\bar{\mathbf{b}}_{s\text{-}2;i+2}$, which we can then substitute in the above equation. This procedure can be iterated a total of $k\text{-}i\text{-}2$ times, yielding

$$\begin{aligned}\bar{\mathbf{b}}_{s;i} = {} & \bar{\mathbf{b}}_{s\text{-}(k\text{-}i\text{-}1);k\text{-}1}\,A_{s\text{-}(k\text{-}i\text{-}1)}A_{s\text{-}(k\text{-}i\text{-}2)}\cdots A_{s\text{-}1} \\ & + \gamma_{s\text{-}(k\text{-}i\text{-}1);k\text{-}1}\mathbf{a}_{s\text{-}(k\text{-}i\text{-}1)}\,A_{s\text{-}(k\text{-}i\text{-}2)}A_{s\text{-}(k\text{-}i\text{-}3)}\cdots A_{s\text{-}1} + \cdots + \\ & + \gamma_{s\text{-}3;i+3}\mathbf{a}_{s\text{-}3}\,A_{s\text{-}2}A_{s\text{-}1} + \gamma_{s\text{-}2;i+2}\mathbf{a}_{s\text{-}2}\,A_{s\text{-}1} + \gamma_{s\text{-}1;i+1}\mathbf{a}_{s\text{-}1}\,.\end{aligned} \tag{5.8}$$

Applying (5.2) to index $s\text{-}(k\text{-}i\text{-}1)$ instead of s, we have $\bar{\mathbf{b}}_{s\text{-}(k\text{-}i\text{-}1);k\text{-}1}=\mathbf{0}$. Hence, the first term on the righthand side of (5.8) vanishes. To abbreviate notation, let $A(s;j)$ denote the matrix

$$A(s;j) = \begin{cases} I, & j=0, \\ A_{s-j+1}A_{s-j+2} \cdots A_s, & j \geq 1. \end{cases} \tag{5.9}$$

Then, (5.8) can be rewritten as

$$\bar{\mathbf{b}}_{s;i} = \sum_{j=0}^{k-i-2} \gamma_{s-j-1;i+j+1}\, \mathbf{a}_{s-j-1}\, A(s-1;j)\,, \quad 0 \leq i \leq k-2. \tag{5.10}$$

Define

$$\bar{B}_s = \begin{bmatrix} \bar{\mathbf{b}}_{s;0} \\ \bar{\mathbf{b}}_{s;1} \\ \cdot \\ \cdot \\ \cdot \\ \bar{\mathbf{b}}_{s;k-2} \end{bmatrix}. \tag{5.11}$$

$\bar{B}_s$ is the $(k-1)\times(k-1)$ matrix at the upper left corner of B_s. Together with (5.2) and the definition of $\gamma_{s;i}$, we have

$$B_s = \begin{bmatrix} & & & \gamma_{s;0} \\ & & & \gamma_{s;1} \\ & \bar{B}_s & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & \gamma_{s;k-2} \\ 0 \;\; 0 & \cdots & 0 & \gamma_{s;k-1} \end{bmatrix}. \tag{5.12}$$

Define also the $(k-1)\times(k-1)$ matrices

$$\Gamma_s = \begin{bmatrix} \gamma_{s-1;1} & \gamma_{s-2;2} & \cdots & \gamma_{s-k+2;k-2} & \gamma_{s-k+1;k-1} \\ \gamma_{s-1;2} & \gamma_{s-2;3} & \cdots & \gamma_{s-k+2;k-1} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_{s-1;k-2} & \gamma_{s-2;k-1} & \cdots & 0 & 0 \\ \gamma_{s-1;k-1} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{5.13}$$

and

$$E_s = \begin{bmatrix} \mathbf{a}_{s-1} \\ \mathbf{a}_{s-2} A(s-1;1) \\ \mathbf{a}_{s-3} A(s-1;2) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_{s-k+1} A(s-1;k-2) \end{bmatrix} \tag{5.14}$$

Then, (5.10) can be rewritten in matrix form as

$$\bar{B}_s = \Gamma_s E_s \,. \tag{5.15}$$

Notice that E_s is defined in terms of the known vectors $\mathbf{a}_{s-j-1}$ and matrices $A(s-1;j)$, $0 \le j \le k-2$. Hence, to make (5.15) an explicit expression for the matrix $\bar{B}_s$, we need to determine the entries of Γ_s, namely $\gamma_{s-j-1;i+j+1}$ for $0 \le i \le k-2$, $0 \le j \le k-i-2$. If we also determine the parameters $\gamma_{s;i}$ for $0 \le i \le k-1$ then, by (5.12), the matrix B_s of coefficients of the Beta-polynomials will be completely determined.

Equation (5.15) was derived from (4.9.b)-(4.9.c). We still need to take into account the constraints imposed by (4.9.a) and (4.9.d). To achieve this goal, we present a few definitions and a series of lemmas based on substitution arguments.

We define the *restriction* operator $* : \mathbf{K}_n \to \mathbf{R}$ as follows: Let τ be an element of $\mathbf{K}_n$. Then, the restriction τ^* of τ is obtained by effecting the substitution $\beta_{l;1}=1,\ \beta_{l;2}=0,\ \cdots,\ \beta_{l;n}=0$ in τ, for all l; i.e.

$$\tau^* \triangleq \tau \,\big|_{\beta_{l;1}=1,\ \beta_{l;2}=0,\ \cdots,\ \beta_{l;n}=0}\,, \quad \text{all } l,$$

provided the denominator of τ does not vanish under the above substitution (if this condition holds, we say that τ^* is *well-defined*). The restriction operator extends naturally to a matrix or polynomial T over $\mathbf{K}_n$ as the component-wise restriction of its entries (respectively, coefficients). T^* is undefined if the restriction of any of its entries (respectively, coefficients) is undefined. The following are straightforward properties of the restriction operator, and are presented below without proof. Unless stated otherwise, T and U represent matrices or polynomials over $\mathbf{K}_n$.

(R1) If T^* is well-defined, and $T^* \neq 0$, then $T \neq 0$ (the converse is not true).

(R2) Assume T^* and U^* are well-defined. Then, for $\bigcirc \in \{+,-,\times\}$, $(T \bigcirc U)^* = T^* \bigcirc U^*$.

(R3) Let T be a square matrix such that T^* is well-defined. Then $|T|^* = |T^*|$.

(R4) Let T be a square matrix such that T^* is well-defined. If T^* is nonsingular then T is nonsingular, and $(T^{-1})^* = (T^*)^{-1}$.

LEMMA 2. Let A_s be the matrix defined in (5.4). Then, A_s^* is well-defined, and we have

$$A_s^* = \left[\binom{i}{j} \right], \quad 0 \le i,j \le k\text{-}2\,. \tag{5.16}$$

Also, $\mathbf{a}_s^*$ is well-defined, and we have

$$\mathbf{a}_s^* = \left[1 \ \binom{k\text{-}1}{1} \ \binom{k\text{-}1}{2} \ \cdots \ \binom{k\text{-}1}{k\text{-}2} \right]. \tag{5.17}$$

PROOF. By property (M5) of M_s, we have $M_s^* = I$. Hence, it follows from (5.4) and from property (R2) above that

$$A_s^* = S(k\text{-}1,1)(M_s^T)^* S(k\text{-}1,0)^{-1} = S(k\text{-}1,1) S(k\text{-}1,0)^{-1}.$$

Similarly,

$$\mathbf{a}_s^* = \mathbf{y}(k,1) S(k\text{-}1,0)^{-1}.$$

The claims of the lemma now follow from the definitions of $S(k,u)$ in (4.4), and of $\mathbf{y}(k,u)$ in (5.1). ●

Notice that neither A_s^* nor $\mathbf{a}_s^*$ depends on s, and we shall write $A_s^* = A^*$ and $\mathbf{a}_s^* = \mathbf{a}^*$. Also, it follows from the definition of $A(s;j)$ in (5.9) and from property (R2) that

$$A(s;j)^* = (A^*)^j. \tag{5.18}$$

For any square matrix T, let $g_T(x) = |\,xI - T\,|$ denote the characteristic polynomial of T (see, for instance, Herstein, 1975). Also, for any polynomial $v(x)$, let $v(T)$ denote the square matrix obtained by evaluating v with T as its argument.

LEMMA 3. Let E_s be the matrix defined in (5.14). Then, E_s is nonsingular over $\mathbf{K}_n$.

PROOF. First, we notice that, by the definition of E_s in (5.14), and by Lemma 2, property (R2) and (5.18), E_s^* is well-defined, and we have

$$E_s^* = \begin{bmatrix} \mathbf{a}^* \\ \mathbf{a}^* \cdot A^* \\ \mathbf{a}^* \cdot (A^*)^2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}^* \cdot (A^*)^{k-2} \end{bmatrix} . \tag{5.19}$$

We claim that E_s^* is nonsingular over $\mathbb{R}$. Assume that $\mathbf{v}E_s^* = \mathbf{0}$ for some row vector $\mathbf{v} \in \mathbb{R}^{k-1}$. Then, we have

$$\mathbf{v}E_s^* = \mathbf{a}^* \nu(A^*) = \mathbf{0} . \tag{5.20}$$

It follows readily from (5.16) that the characteristic polynomial of A^* is

$$g_{A^*}(x) = (x-1)^{k-1}. \tag{5.21}$$

Factor $\nu(x)$ as $\nu(x) = (x-1)^j h(x)$, where $0 \le j \le k$-2 and either h is identically zero, or $h(1) \ne 0$. If $h(1) \ne 0$, then, by (5.21), $h(x)$ is relatively prime to the characteristic polynomial of A^*, and, therefore, $h(A^*)$ is nonsingular. Thus, it follows from (5.20) and the above factorization of $\nu(x)$ that

$$\mathbf{a}^* \cdot (A^* - I)^j = \mathbf{0} .$$

Hence, in (5.20) we can assume without loss of generality that either $\mathbf{v} = \mathbf{0}$, or $\nu(x) = (x-1)^j$ for some j, $0 \le j \le k$-2. It follows from (5.16) that $(A^* - I)$ is a $(k$-1$) \times (k$-1$)$ lower triangular matrix with zeroes on its main diagonal, and strictly positive entries below the main diagonal. Hence, if $\mathbf{v} \ne \mathbf{0}$, then

$$\nu(A^*) = (A^* - I)^j$$

contains at least one strictly positive entry for $0 \le j \le k$-2, and all its nonzero entries are positive. Thus, since by (5.17) all the entries of $\mathbf{a}^*$ are strictly positive, $\mathbf{a}^* \nu(A^*)$ cannot vanish unless $\mathbf{v} = \mathbf{0}$. Therefore, E_s^* is nonsingular and, by property (R4), E_s is nonsingular. ●

We are now ready to address constraint (4.9.d). It follows from (5.12) and (5.15) that

$$\sum_{i=0}^{k-1} \mathbf{b}_{s;i} = [\, \mathbf{1}\bar{B}_s \quad \sum_{j=0}^{k-1} \gamma_{s;j} \,] = [\, \mathbf{1}\Gamma_s E_s \quad \sum_{j=0}^{k-1} \gamma_{s;j} \,] , \tag{5.22}$$

where $\mathbf{1}$ is a row vector of (k-1) ones, and $[\mathbf{v} \;\; x]$ denotes the vector obtained by appending x to $\mathbf{v}$. Hence, for (4.9.d) to hold, we must have

$$\mathbf{1}\Gamma_s E_s = (1\, 0 \, \cdots \, 0) , \tag{5.23}$$

and

$$\sum_{j=0}^{k-1} \gamma_{s;j} = 0. \tag{5.24}$$

By Lemma 3, (5.23) is equivalent to

$$\mathbf{1}\Gamma_s = (1 0 \cdots 0)E_s^{-1}. \tag{5.25}$$

Let

$$\mathbf{w}_s \triangleq (w_{s;0}\, w_{s;1} \cdots w_{s;k-2}) = (1 0 \cdots 0)E_s^{-1}. \tag{5.26}$$

Then, (5.23) is equivalent to

$$\mathbf{1}\Gamma_s = \mathbf{w}_s. \tag{5.27}$$

Recalling the definition of Γ_s in (5.13), Equation (5.27) is equivalent to

$$\sum_{i=0}^{k-j-2} \gamma_{s-j-1;i+j+1} = w_{s;j}, \quad 0 \leq j \leq k-2. \tag{5.28}$$

Equation (5.28) can be used to solve for the parameters $\gamma_{s-j;j}$. By direct application of (5.28), one can readily verify that

$$\gamma_{s-j;j} = \begin{cases} w_{s;j-1} - w_{s+1;j} & 1 \leq j \leq k-2, \\ \\ w_{s;k-2} & j = k-1. \end{cases} \tag{5.29}$$

Translated to matrix form, (5.29) gives an expression for Γ_s. Consider the $(k-2)\times(k-2)$ matrix

$$W_s = \begin{bmatrix} w_{s+1;1} & w_{s+1;2} & \cdots & w_{s+1;k-2} & 0 \\ w_{s+2;2} & w_{s+2;3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ w_{s+k-3;k-3} & w_{s+k-3;k-2} & \cdots & 0 & 0 \\ w_{s+k-2;k-2} & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5.30}$$

Then, it follows from (5.13) and (5.29) that

$$\Gamma_s = \begin{bmatrix} \mathbf{w}_s \\ \\ W_s \end{bmatrix} - \begin{bmatrix} W_s \\ \\ \mathbf{0} \end{bmatrix}. \tag{5.31}$$

Since E_s, E_s^{-1}, and $\mathbf{w}_s$ are known for all s, equation (5.29) uniquely determines the parameters $\gamma_{s;j}$, for all s and $1 \le j \le k\text{-}1$. Hence, the matrices Γ_s and $\bar{B}_s$ in (5.15), are uniquely determined, and so is the last column of B_s in (5.12), except for the first row entry $\gamma_{s;0}$. The latter is determined by (5.24), which dictates

$$\gamma_{s;0} = -\sum_{j=1}^{k-1} \gamma_{s;j} \,. \tag{5.32}$$

The preceding discussion is summarized in the following lemma.

LEMMA 4. There is one and only one solution B_s to equations (4.9.b)-(4.9.d). The solution is given by equations (5.12)-(5.15), (5.26), and (5.29)-(5.32). ●

Notice that the solution B_s was uniquely determined from (4.9.b)-(4.9.d), without taking into account (4.9.a). Hence, the system (4.9) appears to be over-determined, and we can only hope that the solution found is consistent with (4.9.a). The following lemma shows that this is actually the case.

LEMMA 5. The solution B_s of Lemma 4 is consistent with (4.9.a).

PROOF. Multiplying both sides of (4.9.a) by $S(k\text{-}1,0)^{-1}$ (to the right), and using (5.1) and the definitions of A_s and $\mathbf{a}_s$, we obtain the following equivalent equation:

$$\bar{\mathbf{b}}_{s;0} A_s + \gamma_{s;0}\, \mathbf{a}_s = \mathbf{0}\,. \tag{5.33}$$

Now, by (5.15), (5.30) and (5.31), the vector $\bar{\mathbf{b}}_{s;0}$ in our solution satisfies

$$\bar{\mathbf{b}}_{s;0} = \mathbf{w}_s E_s - (w_{s+1;1}\, w_{s+1;2} \, \cdots \, w_{s+1;k\text{-}2}\, 0) E_s \,.$$

By the definition of $\mathbf{w}_s$ in (5.26), this implies that

$$\bar{\mathbf{b}}_{s;0} = (1\,0 \, \cdots \, 0) - (w_{s+1;1}\, w_{s+1;2} \, \cdots \, w_{s+1;k\text{-}2}\, 0) E_s \,. \tag{5.34}$$

Multiplying both sides of (5.34) to the right by A_s, and recalling that the first row of A_s is $(1\,0 \, \cdots \, 0)$, we obtain

$$\bar{\mathbf{b}}_{s;0}A_s = (1\,0 \cdots 0) - (w_{s+1;1}\, w_{s+1;2} \cdots w_{s+1;k-2}\, 0)\, E_s A_s \,. \tag{5.35}$$

Now, recalling the definitions of E_s and $A(s;j)$, it follows from (5.35) that

$$\begin{aligned} \bar{\mathbf{b}}_{s;0}A_s &= (1\,0 \cdots 0) - \sum_{j=1}^{k-2} w_{s+1;j}\, \mathbf{a}_{s-j} A(s-1;j-1) A_s \\ &= (1\,0 \cdots 0) - \sum_{j=1}^{k-2} w_{s+1;j}\, \mathbf{a}_{s-j} A(s;j) = \\ &= (1\,0 \cdots 0) - (\, \mathbf{w}_{s+1} E_{s+1} - w_{s+1;0}\, \mathbf{a}_s \,)\,. \end{aligned} \tag{5.36}$$

Since $\mathbf{w}_{s+1} E_{s+1} = (1\,0 \cdots 0)$, (5.36) implies that

$$\bar{\mathbf{b}}_{s;0}A_s = w_{s+1;0}\, \mathbf{a}_s \,. \tag{5.37}$$

Finally, by (5.13) and (5.27), we have $w_{s+1;0} = \sum_{j=1}^{k-1} \gamma_{s;j}$, which, together with (5.32) and (5.37) implies that

$$\bar{\mathbf{b}}_{s;0}A_s = -\gamma_{s;0}\, \mathbf{a}_s \,. \tag{5.38}$$

Clearly, (5.38) is consistent with (5.33) and, hence, with (4.9.a). ●

We now prove that the matrix B_s in the solution is nonsingular, i.e. the polynomials $b_{s;0}, b_{s;1}, \cdots, b_{s;k-1}$ are linearly independent, as required by design objective (iv). To this end, we first prove the following lemma.

LEMMA 6. $\gamma_{s;k-1} \neq 0$ for all s.

PROOF. By (5.29), $\gamma_{s;k-1} = w_{s+k-1;k-2}$. Hence, $\gamma_{s;k-1} \neq 0$ for all s if and only if $w_{s;k-2} \neq 0$ for all s. By (5.26), we have $\mathbf{w}_s = (1\,0\,0 \cdots 0) E_s^{-1}$, and, by the argument in the proof of Lemma 3, E_s^* is well-defined and nonsingular. Thus, we can write $\mathbf{w}_s^* = (1\,0\,0 \cdots 0)(E_s^*)^{-1}$, or, equivalently, $\mathbf{w}_s^* E_s^* = (1\,0\,0 \cdots 0)$. By the form of E_s^* in (5.19), this implies

$$\mathbf{a}^*\, w_s^*(A^*) = (1\,0\,0 \cdots 0)\,. \tag{5.39}$$

Let

$$v(x) = w_s^*(x) \cdot (x-1)\,. \tag{5.40}$$

Since $w_s^*(x)$ is not identically zero (by virtue of (5.39)), neither is $v(x)$, and the degree of $v(x)$ is at most k-1. It follows from (5.39), (5.40) and from the fact that the first row of A^* is $(1\,0 \cdots 0)$ that

$$\mathbf{a}^* v(A^*) = \mathbf{0}. \tag{5.41}$$

By an argument similar to the one used in the proof of Lemma 3, since $v(x)$ is not identically zero, it must be of the form

$$v(x) = v_{k-1} \cdot (x-1)^{k-1} ,$$

for some real constant $v_{k-1} \neq 0$. Thus, by (5.40), we have

$$w_s^*(x) = v_k \cdot (x-1)^{k-2} ,$$

and $w^*_{s;k-2} = v_k \neq 0$. By property (R1), this implies that $w_{s;k-2} \neq 0$. Since this holds for all s, we have $\gamma_{s;k} \neq 0$ for all s. •

LEMMA 7. B_s is nonsingular.

PROOF. It follows from the result of Lemma 6, and from the form of Γ_s in (5.15) that Γ_s is nonsingular. By Lemma 3, E_s is nonsingular. Hence, by (5.15), $\bar{B}_s = \Gamma_s E_s$ is nonsingular. Now, using again Lemma 6, it follows from (5.12) that B_s is nonsingular. •

To summarize the results of this section, we present the following theorem.

THEOREM 2. There is one and only one set of Beta-polynomials $b_{s;0}(u), b_{s;1}(u), \cdots, b_{s;k-1}(u)$, with coefficients in $\mathbf{K}_n$, satisfying the design objectives (i)-(iv). The following procedure outlines the symbolic computation of the matrix of coefficients B_s of these polynomials:

PROCEDURE β1: **Computation of Beta-polynomials for uniform knot spacing.**

1. Compute M_s, using either the chain rule for derivatives or (2.6).

2. Compute $S(k\text{-}1,0)$ and $S(k\text{-}1,1)$ using (4.4), and A_s using (5.4). Compute $\mathbf{a}_s$ using (5.6). Notice that once A_s and $\mathbf{a}_s$ are computed for a given index s, A_t and $\mathbf{a}_t$ are obtained for any index t by substituting $\beta_{t;j}$ for $\beta_{s;j}$ in A_s (respectively $\mathbf{a}_s$) for $1 \leq j \leq n$, without having to repeat the computation in (5.4) (respectively (5.6)). Notice also that A_s and $\mathbf{a}_s$ depend only on $\boldsymbol{\beta}_s$.

3. Compute E_s using (5.9) and (5.14).

4. Compute $\mathbf{w}_s = (1 0 \cdots 0) E_s^{-1}$. Once $\mathbf{w}_s$ is known for a given index s, $\mathbf{w}_t$ is obtained, for any t, by substituting $\beta_{l+t-s;j}$ for $\beta_{l;j}$ in $\mathbf{w}_s$, for all l and $1 \leq j \leq n$.

5. Compute Γ_s using (5.30) and (5.31).

6. Compute $\bar{B}_s$ using (5.15).

7. Obtain $\gamma_{s;1}, \gamma_{s;2}, \cdots, \gamma_{s;k-1}$ from the first column of Γ_{s+1} (see (5.13)), and $\gamma_{s;0}$ from (5.32).

8. Compute B_s using (5.12).

For the sake of clarity, Procedure $\beta 1$ follows the derivation leading to Theorem 2 in a straightforward manner, and is not necessarily optimal from a computational point of view. The procedure was programmed in the symbolic manipulation system Mathematica™ (release 1.5),[4] and Beta-polynomials for various orders were computed. The results for $k=4$ are presented in Appendix A.

6 Local Control

In the example in Appendix A we can observe that the matrix B_s, for $k=4$, depends on the sets of parameters $\boldsymbol{\beta}_{s-2}$, $\boldsymbol{\beta}_{s-1}$, $\boldsymbol{\beta}_s$, and $\boldsymbol{\beta}_{s+1}$. We now formalize this result, and generalize it for arbitrary order k. To formalize the concept of dependence, we say that a matrix X over $\mathbf{K}_n$ is *independent* of $\boldsymbol{\beta}_l$ if its entries do not contain any occurrence of $\beta_{l;j}$ for any $1 \le j \le n$.[5] We say that X *depends only on* $\boldsymbol{\beta}_r, \boldsymbol{\beta}_{r+1}, \cdots, \boldsymbol{\beta}_{r+t}$ if X is independent of $\boldsymbol{\beta}_l$ for all $l<r$ and all $l>r+t$ (hence, the entries of X may, but not necessarily do, contain indeterminates from $\boldsymbol{\beta}_r, \boldsymbol{\beta}_{r+1}, \cdots, \boldsymbol{\beta}_{r+t}$).

LEMMA 8. E_s depends only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s-1}$.

PROOF. This follows by straightforward observation of equations (5.4), (5.6), (5.9) and (5.14): E_s is given in terms of the vectors $\mathbf{a}_{s-j-1}$ and the matrices $A(s-1,j)$ for $0 \le j \le k-2$. These vectors and matrices are computed from matrices with real entries, and from $M_{s-k+1}, M_{s-k+2}, \cdots, M_{s-1}$, which, in turn, depend only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s-1}$. Thus, E_s depends only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s-1}$, as claimed. •

LEMMA 9. $\mathbf{w}_s$ depends only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s-2}$.

Proof: Let $\mathbf{v}=\mathbf{y}(k,1)S(k-1,1)^{-1}$ (notice that $\mathbf{v} \in \mathbf{R}^{k-1}$). Then, it follows from the definitions of A_s and $\mathbf{a}_s$ that $\mathbf{a}_s = \mathbf{v}A_s$. Substituting $\mathbf{v}A_{s-j-1}$ for $\mathbf{a}_{s-j-1}$, $0 \le j \le k-2$, in the definition (5.14) of E_s, and using the definition (5.9) of $A(s,j)$, we obtain

$$E_s = \begin{bmatrix} \mathbf{v}A_{s-1} \\ \mathbf{v}A_{s-2}A(s-1;1) \\ \mathbf{v}A_{s-3}A(s-1;2) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v}A_{s-k+1}A(s-1;k-2) \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{v}A(s-2;1) \\ \mathbf{v}A(s-2;2) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v}A(s-2;k-2) \end{bmatrix} A_{s-1} \triangleq \hat{E}_s A_{s-1} \,. \tag{6.1}$$

Now, by the definition of $\mathbf{w}_s$ in (5.27), and by the form of A_s in (5.5), it follows from (6.1)

[4] Mathematica is a trademark of Wolfram Research Inc.

[5] We assume the entries of X are reduced rational functions.

that

$$\mathbf{w}_s = (10 \cdots 0)E_s^{-1} = (10 \cdots 0)A_{s-1}^{-1}\hat{E}_s^{-1} = (10 \cdots 0)\hat{E}_s^{-1}. \tag{6.2}$$

By an argument similar to the one used in the proof of Lemma 8, it can be readily verified that $\hat{E}_s$ depends only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s-2}$. Hence, by (6.2), so does $\mathbf{w}_s$. ●

THEOREM 3. Assume $k \geq 3$. Then,[6]
(a) B_s depends only on $\boldsymbol{\beta}_{s-k+2}, \boldsymbol{\beta}_{s-k+3}, \cdots, \boldsymbol{\beta}_{s+k-3}$.
(b) Conversely, a given set of shape parameters $\boldsymbol{\beta}_l$ affects only the matrices B_s for $l-k+3 \leq s \leq l+k-2$.

PROOF. In (5.30)-(5.31), Γ_s is expressed in terms of the entries of $\mathbf{w}_{s+i}$ for $0 \leq i \leq k-2$. Hence, by Lemma 9, Γ_s depends only on $\boldsymbol{\beta}_{s-k+1}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s+k-4}$. Since, by Lemma 8, this includes the range of dependency of E_s for $k \geq 3$, we conclude that $\bar{B}_s = \Gamma_s E_s$ depends only on $\boldsymbol{\beta}_l$ for $s-k+1 \leq l \leq s+k-4$. The only index in this range that is outside the claim of the theorem is $l = s-k+1$ on the low end. We claim that $\bar{B}_s$ is independent of $\boldsymbol{\beta}_{s-k+1}$. Since, by (5.30) and Lemma 9, W_s is independent of $\boldsymbol{\beta}_{s-k+1}$, it follows from (5.14), (5.15) and (5.31) that there are only two possible sources of indeterminates from $\boldsymbol{\beta}_{s-k+1}$ in $\bar{B}_s$: the first row of Γ_s, due to its dependence on $\mathbf{w}_s$, and the last row of E_s, due to its dependence on $\mathbf{a}_{s-k+1}$. Hence, since the only nonzero entry in the last column of Γ_s is in its first row, only the first row of $\bar{B}_s$, $\bar{\mathbf{b}}_{s;0}$, may depend on $\boldsymbol{\beta}_{s-k+1}$. However, by (5.37) and the nonsingularity of A_s, we have

$$\bar{\mathbf{b}}_{s;0} = w_{s+1;0}\, \mathbf{a}_s A_s^{-1} .$$

By Lemma 9, $w_{s+1;0}$ is independent of $\boldsymbol{\beta}_{s-k+1}$, and, by their definitions, $\mathbf{a}_s$ and A_s depend only on $\boldsymbol{\beta}_s$. Hence, $\bar{\mathbf{b}}_{s;0}$ is independent of $\boldsymbol{\beta}_{s-k+1}$, and so is $\bar{B}_s$, as claimed. To complete the proof of the theorem, it remains to check the last column of B_s, given by $(\gamma_{s;0}\ \gamma_{s;1}\ \cdots\ \gamma_{s;k-1})^T$. It follows from (5.13) that $(\gamma_{s;1}\ \gamma_{s;2}\ \cdots\ \gamma_{s;k-1})^T$ is the first column of Γ_{s+1}. By the discussion on Γ_s above, this column depends only on $\boldsymbol{\beta}_{s-k+2}, \boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s+k-3}$, as claimed by the theorem. Finally, $\gamma_{s;0}$ is given by (5.32) in terms of $\gamma_{s;1}, \gamma_{s;2}, \cdots, \gamma_{s;k-1}$, and, therefore, it depends on the same parameters. Part (b) of the theorem is an immediate consequence of part (a). ●

The property expressed by Theorem 3 is referred to as *local control* of the shape parameters $\beta_{s;i}$ with respect to the Beta-spline curve: each curve segment depends on $2(k-2) = 2n$ sets of parameters, and conversely, each parameter affects $2(k-2)$ curve segments, regardless of the number of segments in the curve. This local control property is also shown in Goodman (1985). Notice that a similar property is enjoyed by the control vertices, by virtue of the curve construction procedure (each segment depends on k control vertices, and each control vertex affects k curve segments).

[6] If $k=2$, then $n=0$, and there are no shape parameters.

7 Non-uniform Knot Spacing

The derivation leading to Theorem 2, and the resulting procedure for constructing the Beta-polynomials for uniform knot spacing can be carried out with very little change for the case of non-uniform knot spacing as well. The main difference in the set of basic constraints (4.9) is that, in the case of non-uniform knot spacing, we substitute $\bar{S}(k, z_s)$ for $\bar{S}(k, 1)$ in (4.9.a)-(4.9.c), where $z_s = u_s - u_{s-1}$ is the length of the s^{th} parametric interval. Thus, the basic constraints for non-uniform knot spacing are given by

$$\mathbf{b}_{s;0}^{NU}\,\bar{S}(k,z_s)M_s^T = \mathbf{0}\,, \tag{7.1.a}$$

$$\mathbf{b}_{s+1;i}^{NU}\,\bar{S}(k,0) - \mathbf{b}_{s;i+1}^{NU}\,\bar{S}(k,z_s)M_s^T = \mathbf{0}\,, \quad 0 \le i \le k\text{-}2, \tag{7.1.b}$$

$$\mathbf{b}_{s+1;k-1}^{NU}\,\bar{S}(k,0) = \mathbf{0}\,, \tag{7.1.c}$$

$$\sum_{i=0}^{k-1} \mathbf{b}_{s;i}^{NU} = (1\,0\,\cdots\,0), \tag{7.1.d}$$

where we use the superscript *NU* to denote vectors, matrices, and polynomials derived for the non-uniform case. To simplify terminology, we shall call the Beta-polynomials for the case of non-uniform knot spacing *non-uniform Beta-polynomials*, as opposed to the *uniform Beta-polynomials* derived in Section 5. The derivation of a solution to Equations (7.1) is completely analogous to the derivation of the solution to (4.9) leading to Theorem 2. Thus, in analogy to (5.4) and (5.6), we define

$$A_s^{NU} = S(k\text{-}1, z_s)\,M_s^T\,S(k\text{-}1, 0)^{-1}, \tag{7.2}$$

$$\mathbf{a}_s^{NU} = \mathbf{y}(k, z_s)\,M_s^T\,S(k\text{-}1, 0)^{-1}\,, \tag{7.3}$$

and E_s^{NU}, $\mathbf{w}_s^{NU}$, W_s^{NU}, Γ_s^{NU}, $\bar{B}_s^{NU}$, B_s^{NU}, and $b_{s;i}^{NU}(u)$ are computed accordingly, following Procedure $\beta 1$. Similar to the parameters $\beta_{s;i}$, we treat the parameters z_s as symbolic indeterminates, and we operate in the rational function field $\mathbf{K}_n^{NU}$ defined by extending $\mathbf{K}_n$ with the indeterminates z_s for all integers s. The definition of the restriction operator used in the proofs of existence and uniqueness in Section 5 is extended to $\mathbf{K}_n^{NU}$ by setting the indeterminates z_s to 1.

It follows from the preceding discussion that one way to obtain explicit symbolic expressions for the non-uniform Beta-polynomials is to follow Procedure $\beta 1$ using the vectors and matrices with the superscript *NU*. However, as we shall now show, the non-uniform Beta-polynomials are more easily obtained from the uniform ones by effecting a simple symbolic substitution. This is a generalization of a result shown in Goodman & Unsworth (1986) for the case of cubic Beta-splines ($k=4$; see also pp. 344-346 of Bartels *et al.*, 1987).

For any scalar, matrix, or polynomial T over $\mathbf{K}_n$, let T^+ denote the corresponding scalar, matrix, or polynomial over $\mathbf{K}_n^{NU}$ obtained by substituting $\beta_{s;i}^+$ for $\beta_{s;i}$ in T, where

$$\beta^+_{s;i} = \frac{z^i_{s+1}}{z_s}\,\beta_{s;i}\,, \quad 1 \le i \le n\,, \quad \text{all integers } s\,. \tag{7.4}$$

Also, define the diagonal matrix

$$Z_s(k) = \operatorname{diag}(1,\, z_s,\, z_s^2,\, \cdots,\, z_s^{k-1})\,. \tag{7.5}$$

LEMMA 10. $M^+_s = Z_{s+1}(k\text{-}1)\, M_s\, Z_s(k\text{-}1)^{-1}$.

PROOF. By (2.6), a typical entry $M_{s;jr}$ in M_s is a sum of monomials of the form $c\,\beta^{k_1}_{s;1}\beta^{k_2}_{s;2}\cdots\beta^{k_j}_{s;j}$, for some constants c, and where $k_1+k_2+\cdots+k_j = r$, and $k_1+2k_2+\cdots+jk_j = j$. Substituting $\beta^+_{s;j}$, as defined in (7.4), for $\beta_{s;j}$ in the above monomial yields the corresponding monomial in $M^+_{s;jr}$, namely

$$c\,(\beta^+_{s;1})^{k_1}(\beta^+_{s;2})^{k_2}\cdots(\beta^+_{s;j})^{k_j} = c\,\beta^{k_1}_{s;1}\beta^{k_2}_{s;2}\cdots\beta^{k_j}_{s;j}\,\frac{z_{s+1}^{k_1+2k_2+\cdots+jk_j}}{z_s^{k_1+k_2+\cdots+k_j}}$$

$$= c\,\beta^{k_1}_{s;1}\beta^{k_2}_{s;2}\cdots\beta^{k_j}_{s;j}\,\frac{z^j_{s+1}}{z^r_s}\,.$$

Hence, $M^+_{s;jr} = z^j_{s+1}\, M_{s;jr}\, z_s^{-r}$ for $0 \le j,r \le k\text{-}2$, and the claim of the lemma follows. •

THEOREM 4. The matrix of coefficients B^{NU}_s of the non-uniform Beta-polynomials is given by

$$B^{NU}_s = B^+_s\; Z_s(k)^{-1}\,,$$

where B_s is the matrix of coefficients of the uniform Beta-polynomials. Equivalently, we have

$$b^{NU}_{s;i}(u) = b^+_{s;i}(u/z_s)\,, \quad 0 \le i \le k\text{-}1, \quad \text{all integers } s\,. \tag{7.6}$$

PROOF. It can be readily verified that, by the definitions of $\bar{S}(k,\, u)$ in (4.5) and of $Z_s(k)$ in (7.5), we have

$$\bar{S}(k, z_s) = Z_s(k)\,\bar{S}(k,\, 1)\, Z_s(k\text{-}1)^{-1}\,. \tag{7.7}$$

Also, since $S(k,0)$ and $Z_{s+1}(k)$ are diagonal matrices of order k, they commute, and consequently, deleting the last column of their product, we obtain

$$\bar{S}(k,0)\, Z_{s+1}(k\text{-}1) = Z_{s+1}(k)\,\bar{S}(k,0)\,. \tag{7.8}$$

Multiply both sides of Equations (7.1.a)-(7.1.c) to the right by the nonsingular matrix $Z_{s+1}(k\text{-}1)$, and Equation (7.1.d) by $Z_s(k)$. Applying (7.7) and (7.8), and noting that $(1\,0\,\cdots\,0)\,Z_{s+1}(k) = (1\,0\,\cdots\,0)$, (7.1.a)-(7.1.d) are transformed into the following equivalent equations:

$$\mathbf{b}_{s;0}^{NU}\, Z_s(k)\,\bar{S}(k,1)\, Z_s(k\text{-}1)^{-1}\, M_s^T\, Z_{s+1}(k\text{-}1) = \mathbf{0}\,, \qquad (7.9.a)$$

$$\mathbf{b}_{s+1;i}^{NU}\, Z_{s+1}(k)\,\bar{S}(k,0) - \mathbf{b}_{s;i+1}^{NU}\, Z_s(k)\,\bar{S}(k,1)\, Z_s(k\text{-}1)^{-1}\, M_s^T\, Z_{s+1}(k\text{-}1) = \mathbf{0}\,, \quad 0 \le i \le k\text{-}2, \qquad (7.9.b)$$

$$\mathbf{b}_{s+1;k-1}^{NU}\, Z_{s+1}(k)\,\bar{S}(k,0) = \mathbf{0}\,, \qquad (7.9.c)$$

$$\sum_{i=0}^{k-1} \mathbf{b}_{s;i}^{NU}\, Z_s(k) = (1\,0\,\cdots\,0). \qquad (7.9.d)$$

Finally, using the result of Lemma 10, (7.9.a)-(7.9.d) are equivalent to

$$[\mathbf{b}_{s;0}^{NU}\, Z_s(k)]\,\bar{S}(k,1)\,(M_s^+)^T = \mathbf{0}\,, \qquad (7.10.a)$$

$$[\mathbf{b}_{s+1;i}^{NU}\, Z_{s+1}(k)]\,\bar{S}(k,0) - [\mathbf{b}_{s;i+1}^{NU}\, Z_s(k)]\,\bar{S}(k,1)\,(M_s^+)^T = \mathbf{0}\,, \quad 0 \le i \le k\text{-}2, \qquad (7.10.b)$$

$$[\mathbf{b}_{s+1;k-1}^{NU}\, Z_{s+1}(k)]\,\bar{S}(k,0) = \mathbf{0}\,. \qquad (7.10.c)$$

$$\sum_{i=0}^{k-1} [\mathbf{b}_{s;i}^{NU}\, Z_s(k)] = (1\,0\,\cdots\,0). \qquad (7.10.d)$$

Now, we recognize that Equations (7.10) are equivalent to Equations (4.9), with the Beta-connection matrix M_s^+ substituted for M_s, and $\mathbf{b}_{s;i}^{NU}\, Z_s(k)$ substituted for $\mathbf{b}_{s;i}(u)$ for all s, and $0 \le i \le k\text{-}1$. Since, by Theorem 2, Equations (4.9) with Beta-connection matrix M_s^+ have a unique solution $\mathbf{b}_{s;0}^+, \mathbf{b}_{s;1}^+, \cdots, \mathbf{b}_{s;k-1}^+$, we must have

$$\mathbf{b}_{s;i}^{NU}\, Z_s(k) = \mathbf{b}_{s;i}^+\,, \quad 0 \le i \le k\text{-}1, \quad \text{all } s\,, \qquad (7.11)$$

or, equivalently, $B_s^{NU} = B_s^+\, Z_s(k)^{-1}$, as claimed by the theorem. The equivalent version of the theorem, given in (7.6), follows from the fact that postmultiplying $\mathbf{b}_{s;i}^+$ by $Z_s(k)^{-1}$ is equivalent to multiplying the j^{th} coefficient of $b_{s;i}^+(u)$ by z_s^{-j} for $0 \le i \le k\text{-}1$ and $0 \le j \le k\text{-}1$. This, in turn, is equivalent to changing the variable u to u/z_s. ●

Theorem 4 can be interpreted in two different ways. From a symbolic computation point of view, it offers a more efficient way of computing the symbolic Beta-polynomials for non-uniform knot spacing:

PROCEDURE $\beta 2$: **Computation of Beta-polynomials for non-uniform knot spacing.**

1. Compute the matrix of coefficients B_s of the *uniform* Beta-polynomials, using Procedure $\beta 1$.

2. Substitute $\beta_{t;j}^+$, as defined in (7.4), for $\beta_{t;j}$ in B_s, for all t and $1 \le j \le n$, obtaining the matrix B_s^+ over $\mathbf{K}_n^{NU}$.

3. Postmultiply B_s^+ by $Z_s(k)^{-1}$, obtaining the matrix of coefficients B_s^{NU} of the *non-uniform* Beta-polynomials.

From a curve construction point of view, Theorem 4 says the following: A given Beta-spline with non-uniform knot spacing defined by interval lengths z_s can be transformed to a Beta-spline with uniform knot spacing, by changing the shape parameters $\beta_{s;i}$ (of the non-uniform spline) to shape parameters $\beta^+_{s;i}$ (of the uniform spline) according to (7.4). Notice that in (7.6), as the argument u of the non-uniform Beta-polynomial $b^{NU}_{s;i}$ varies from 0 to z_s, the argument u/z_s of the uniform Beta-polynomial $b^+_{s;i}$ varies from 0 to 1, as expected.

The following corollary is an immediate consequence of Theorem 3 and 4.

COROLLARY 1. B^{NU}_s depends only on $\boldsymbol{\beta}_{s-k+2}, \boldsymbol{\beta}_{s-k+3}, \cdots, \boldsymbol{\beta}_{s+k-3}$ and $z_{s-k+2}, z_{s-k+3}, \cdots, z_{s+k-2}$.

8 End Conditions

By Corollary 1, the Beta-polynomials for a generic curve segment with index s depend on the sets of shape parameters $\boldsymbol{\beta}_{s-k+2}, \cdots, \boldsymbol{\beta}_{s+k-3}$, and on the interval lengths $z_{s-k+2}, z_{s-k+3}, \cdots, z_{s+k-2}$. However, when the segment is close to one of the ends of the curve, these parameters might correspond to "phantom" knots or parametric intervals that lay outside the definition domain of the curve (e.g. $\boldsymbol{\beta}_{-k+3}, \cdots, \boldsymbol{\beta}_{-1}$, or $z_{-k+3}, z_{-k+4}, \cdots, z_0$ for $s=1$), or to knots that do not have continuity requirements (e.g. $\boldsymbol{\beta}_0$ for the knot at $u=u_0$). In those cases, these "extra" parameters can be used to obtain desired end conditions. One widely used convention is to have the curve interpolate its first control vertex, and be tangential to the control polygon at that point. This can be achieved by making the "phantom" interval lengths $z_{-k+3}, z_{-k+4}, \cdots, z_0$ tend to zero (here we are relaxing the assumption of a monotonically increasing knot sequence to that of a monotonically nondecreasing one). This is known in the literature as having a $(k-1)$-fold *multiple knot* at $u=u_0$. It is interesting to notice that many times, the assumption made in the literature is that of a k-fold multiple knot at the beginning (or end) of the curve. By the result in Corollary 1, however, the first segment of the curve depends on the placement of just $k-1$ knots to the left of and including u_0, and is independent of the placement of the k^{th} knot. A similar argument applies to the other end of the curve. For uniform Beta-splines, by virtue of Theorem 4, the operation of making interval lengths tend to zero can be "simulated" by taking limits on the shape parameters $\beta_{s;i}$. The effect of taking such limits (to 0 and ∞) on the shape parameters is shown, for cubics, in Barsky (1988) and Goodman & Unsworth (1986).

9 Conclusion

We have presented a symbolic derivation, with a proof of existence and uniqueness, of discretely-shaped Beta-spline bases of arbitrary order, over non-uniform knot sequences. The derivation led to explicit procedures for the symbolic computation of the Beta-polynomials and, hence, of the Beta-spline basis functions. We have also shown how certain properties of Beta-splines, such as local control and the equivalence between uniform and non-uniform knot spacings, follow naturally from the symbolic framework. Other properties involving inequalities, such as the convex hull property, or the variation-diminishing property are less amenable to symbolic treatment, and are better dealt with using classical approximation theory methods (Dyn & Micchelli, 1988; Goodman, 1985). It should be noted, however, that the Beta-spline basis functions derived in this paper exist and are unique for all values of the shape parameters that do not make the denominators of the rational functions in the matrix B_s vanish, including values of the shape parameters for which the Beta-connection matrix M_s is not totally positive and hence, for which the results of Dyn & Micchelli (1988) and Goodman (1985) do not guarantee the variation-diminishing or convex hull properties. Some examples of the behavior of cubic Beta-splines for such values of the shape parameters can be found in Chapters 17 and 18 of Barsky (1988).

The derivation and proofs leading to Theorem 2 and Procedure $\beta 1$ were carried out for geometric continuity based on the Beta-constraints (2.8) defined by the Beta-connection matrix $M_s(\boldsymbol{\beta}_s)$. However, a closer examination of the proofs reveals that only the following properties of the connection matrix M_s are required:

1. The existence of a substitution of the symbolic indeterminates which transforms M_s into an identity matrix (this substitution defines the restriction operator used in the proofs of Lemmas 2, 3 and 6), and
2. The fact that the first column of M_s is $(1\,0\,\cdots\,0)^T$, which implies that the first row of A_s is $(1\,0\,\cdots\,0)$ (a necessary condition for the consistency proof in Lemma 5).

Therefore, the results presented in this paper are valid for a more general class of connection matrices, and in particular for Frenet-frame connection matrices of the form

$$C_s = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & c_{s;1,1} & 0 & 0 & \cdots \\ 0 & c_{s;2,1} & c_{s;1,1}^2 & 0 & \cdots \\ 0 & c_{s;3,1} & c_{s;3,2} & c_{s;1,1}^3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

(Dyn & Micchelli, 1988; Seidel, 1990). The results of Section 7 are also readily extended to Frenet-frame continuity using, instead of (7.4), the change of parameters

$$c_{s;i,j}^{+} = \frac{z_{s+1}^{i}}{z_s^{j}}\, c_{s;i,j}\,, \qquad 1 \le i,j \le n, \quad \text{all integers } s.$$

Appendix A

An Example: Cubic Beta-splines

Procedure $\beta 1$ was programmed in the symbolic manipulation system Mathematica™ (release 1.5) on a Hewlett-Packard 9000/370 workstation, and Beta-polynomials for various orders were computed. The results for $k=4$, with uniformly spaced knots, are printed below. To simplify notation, the matrix B_s is given for $s=0$. To obtain expressions for arbitrary s, substitute $\beta_{s-2;i}$, $\beta_{s-1;i}$, $\beta_{s;i}$, $\beta_{s+1;i}$ for $\beta_{-2;i}$, $\beta_{-1;i}$, $\beta_{0;i}$, $\beta_{1;i}$, respectively. For $k=4$, we have

$$B_0 = \begin{bmatrix} 1-\frac{\nu_{10}}{\delta_1} & -\frac{\nu_{11}}{\delta_1} & -\frac{\nu_{12}}{\delta_1} & -\frac{\nu_{13}}{\delta_1} \\ \frac{\nu_{10}}{\delta_1}-\frac{\nu_{20}}{\delta_2} & \frac{\nu_{11}}{\delta_1}-\frac{\nu_{21}}{\delta_2} & \frac{\nu_{12}}{\delta_1}-\frac{\nu_{22}}{\delta_2} & \frac{\nu_{13}}{\delta_1}-\frac{\nu_{23}}{\delta_2} \\ \frac{\nu_{20}}{\delta_2} & \frac{\nu_{21}}{\delta_2} & \frac{\nu_{22}}{\delta_2} & \frac{\nu_{23}}{\delta_2}-\frac{\nu_{33}}{\delta_3} \\ 0 & 0 & 0 & \frac{\nu_{33}}{\delta_3} \end{bmatrix},$$

where

$$\delta_i = 4\beta_{i-2;1}+8\beta_{i-3;1}\beta_{i-2;1}+4\beta^2_{i-3;1}\beta_{i-2;1}+2\beta_{i-3;2}\beta_{i-2;1}+4\beta^2_{i-2;1}+12\beta_{i-3;1}\beta^2_{i-2;1}+8\beta^2_{i-3;1}\beta^2_{i-2;1}+4\beta_{i-3;2}\beta^2_{i-2;1}+4\beta_{i-3;1}\beta^3_{i-2;1}$$
$$+4\beta^2_{i-3;1}\beta^3_{i-2;1}+2\beta_{i-3;2}\beta^3_{i-2;1}+2\beta_{i-2;2}+4\beta_{i-3;1}\beta_{i-2;2}+2\beta^2_{i-3;1}\beta_{i-2;2}+\beta_{i-3;2}\beta_{i-2;2}\,, \qquad 1\le i\le 3\,,$$

$$\nu_{10} = 4\beta_{-1;1}+8\beta_{-2;1}\beta_{-1;1}+4\beta^2_{-2;1}\beta_{-1;1}+2\beta_{-2;2}\beta_{-1;1}+4\beta^2_{-1;1}+12\beta_{-2;1}\beta^2_{-1;1}$$

$$+8\beta^2_{-2;1}\beta^2_{-1;1}+4\beta_{-2;2}\beta^2_{-1;1}+2\beta_{-1;2}+4\beta_{-2;1}\beta_{-1;2}+2\beta^2_{-2;1}\beta_{-1;2}+\beta_{-2;2}\beta_{-1;2}\,,$$

$$\nu_{11} = 6(2\beta_{-2;1}+2\beta^2_{-2;1}+\beta_{-2;2})\beta^3_{-1;1}\,,$$

$$\nu_{12} = -6(2\beta_{-2;1}+2\beta^2_{-2;1}+\beta_{-2;2})\beta^3_{-1;1}\,,$$

$$\nu_{13} = 2(2\beta_{-2;1}+2\beta^2_{-2;1}+\beta_{-2;2})\beta^3_{-1;1}\,,$$

$$\nu_{20} = 2(2\beta_{0;1}+2\beta^2_{0;1}+\beta_{0;2})\,,$$

$$v_{21} = 6\beta_{-1;1}(2\beta_{0;1} + 2\beta_{0;1}^2 + \beta_{0;2}) ,$$

$$v_{22} = 3(2\beta_{-1;1}^2 + \beta_{-1;2})(2\beta_{0;1} + 2\beta_{0;1}^2 + \beta_{0;2}) ,$$

$$v_{23} = -2(2\beta_{-1;1}\beta_{0;1} + 4\beta_{-1;1}^2\beta_{0;1} + 2\beta_{-1;2}\beta_{0;1} + 2\beta_{-1;1}^2\beta_{0;1}^2 + \beta_{-1;2}\beta_{0;1}^2 + \beta_{-1;1}\beta_{0;2} + 2\beta_{-1;1}^2\beta_{0;2} + \beta_{-1;2}\beta_{0;2}) ,$$

$$v_{33} = 2(2\beta_{1;1} + 2\beta_{1;1}^2 + \beta_{1;2}) .$$

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