Hybrid Position/Force Control: A Correct Formulation

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This paper will show conclusively that "kinematic instability" is not inherent to the hybrid position/force control scheme of robot manipulators, but is a result of an incomplete and inappropriate formulation. The inverse of the manipulator Jacobian matrix is identified as causing the kinematic instability of the hybrid position/force control scheme. Linear algebra is used to explain clearly the implications of mapping between vector spaces and to reveal why the inverse of the manipulator Jacobian matrix should not be used in hybrid position/force control. A generalized architecture for hybrid position/force control is presented that can influence both joint positions and torques. This generalized formulation also includes the control of redundant manipulators. Some sufficient conditions for kinematic stability are proposed to determine when a system may become unstable without requiring a complete system analysis. A stable hybrid position/force control scheme is given, and is demonstrated using an example that was previously shown to be unstable.

Contents

1 Introduction

Combining position and force information into one control scheme for moving the end effector in nondeterministic environments was first proposed by Craig and Raibert [2]. They called this scheme hybrid position/force control, currently referred to as hybrid control. Later, Zhang and Paul [14] modified the hybrid control scheme from a Cartesian formulation to a joint space formulation using the same method for separating the position and force constraints in the Cartesian frame of interest. In both cases, the advantage of hybrid control is that the position and force information are analyzed independently to take advantage of well-known control techniques for each, and are combined only at the final stage when both have been converted to joint torques. More recently, An and Hollerbach [1] proved that some well-accepted force control methods, including hybrid control, are unstable. Zhang [13] showed that the hybrid control scheme may become unstable in certain manipulator configurations using revolute joints.

This paper examines more closely the claim that the hybrid control scheme is kinematically unstable. We will show that the kinematic instability problem attributed to hybrid control is not fundamental, as the above published works conclude, but the result of an incorrect formulation and implementation of the hybrid control scheme. It is not clear whether the originators of the hybrid control scheme were aware of the subtleties in their approach or the ramifications it would have on different manipulator kinematics. In this paper we intend to restore the hybrid control scheme as still a very powerful, stable, and robust approach to manipulator control.

2 Background

We will use the simplified interpretation shown in Figure 1 to review the basic concepts of the original hybrid control scheme proposed by Craig and Raibert [2]. The x's in the top half of Figure

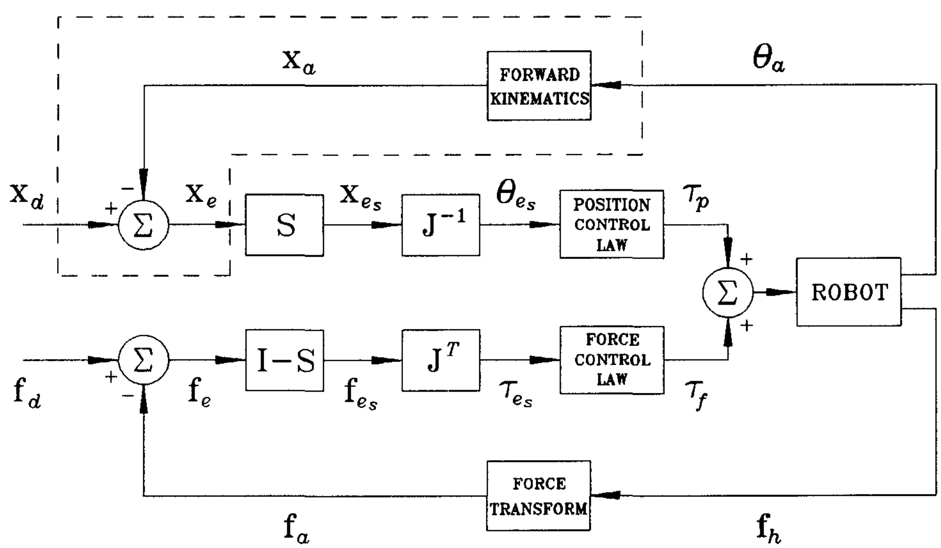

Figure 1: Original Hybrid Control Scheme

1 are 6×1 vectors representing Cartesian position and orientation,[1] and the **f**'s in the bottom half are 6×1 vectors representing the Cartesian force and moment. In this paper, the term Cartesian position includes both the position and orientation information. Likewise, the term Cartesian force includes both force and moment information. The manipulator joint position and torque values are represented by $n \times 1$ vectors $\boldsymbol{\theta}$ and τ, respectively, where n is the number of manipulator joints. All of the subscripts should be self-explanatory from the context in which they are presented. The boldface upper case letters in the boxes are matrices of the appropriate dimensions to keep the vector sizes consistent and will be explained later. The other boxes are labeled with generic terms (e.g., Position Control Law) due to their variability in any given implementation and have no direct effect on the analysis in this paper.

2.1 Position Related Equations

For any given task the position constraints are separated from the force constraints by the selection matrix **S** shown in Figure 1. **S** is a 6×6 diagonal matrix with the elements being either a one for position control or zero for no position control, which defines each degree of freedom in the Cartesian reference frame of interest. The first step is to determine the relevant or *selected* Cartesian position errors as

$$\mathbf{x}_{e_s} = \mathbf{S}\,\mathbf{x}_e \tag{1}$$

where the Cartesian error vector $\mathbf{x}_e$ is the difference between the *desired* and *actual* Cartesian locations of the manipulator. The next step is to map the Cartesian error $\mathbf{x}_{e_s}$ to a corresponding joint error $\boldsymbol{\theta}_{e_s}$ for controlling the manipulator.

The manipulator Jacobian matrix **J** is the first order approximation for transforming differential motions in joint space to differential motions in Cartesian space [8]. The following linearized relationship

$$\mathbf{x}_e = \mathbf{J}\,\boldsymbol{\theta}_e \tag{2}$$

is used to map small joint errors $\boldsymbol{\theta}_e$ to their corresponding Cartesian errors $\mathbf{x}_e$. A unique inverse mapping exists in Equation 2 when **J** is a square matrix of maximal rank. Under this condition, the joint errors are calculated from the Cartesian errors as

$$\boldsymbol{\theta}_e = \mathbf{J}^{-1}\,\mathbf{x}_e \tag{3}$$

By the Craig and Raibert approach for the hybrid control scheme, the selected joint errors are simply

$$\boldsymbol{\theta}_{e_s} = \mathbf{J}^{-1}\,\mathbf{x}_{e_s} \tag{4}$$

when using the selected Cartesian errors determined in Equation 1. We will refer to Equation 4 as the *original position solution* for $\boldsymbol{\theta}_{e_s}$.

Clearly, when **J** is singular or even near a singularity, a more general and numerically stable method is required to determine $\boldsymbol{\theta}_{e_s}$ in Equation 4. While this needs to be remembered, it is not the problem we plan to address in this paper. The kinematic instability problem was shown to occur even when the manipulator Jacobian matrix is well conditioned (An and Hollerbach 1987), meaning it is far away from a singular region in the numerical sense. Equation 4 is, however, an incorrect solution that causes the hybrid control scheme to be kinematically unstable. We will show that Equation 4 is not the only solution for $\boldsymbol{\theta}_{e_s}$, and that there is a general position solution from which a kinematically stable formulation for the hybrid control scheme may be found.

[1] Since this paper is concerned only with the kinematic behavior of the hybrid control scheme, we have intentionally omitted the Cartesian velocity and acceleration terms, while knowing their inclusion may be necessary for improved system performance.

2.2 Force Related Equations

In force control, the Cartesian force error $\mathbf{f}_e$ is calculated as the difference between the actual and desired forces. The main concept of hybrid control is to control force in the directions that are not position controlled. These are directions where the geometry is not well defined, unknown, or a certain contact force is required, making it difficult to control position. By using $\mathbf{S}^{\perp}$, the orthogonal complement of $\mathbf{S}$, the selected Cartesian force errors are

$$\mathbf{f}_{e_s} = \mathbf{S}^{\perp}\,\mathbf{f}_e \tag{5}$$

The matrix $\mathbf{S}^{\perp}$ is also a 6×6 diagonal matrix with the elements being either a one for force control or zero for no force control which defines each degree of freedom in the Cartesian reference frame of interest. Since force control is orthogonal to position control, $\mathbf{S}^{\perp} = \mathbf{I} - \mathbf{S}$ where $\mathbf{I}$ is the identity matrix. This definition for $\mathbf{S}^{\perp}$ corresponds to the transformation of the Cartesian force error $\mathbf{f}_e$ to $\mathbf{f}_{e_s}$ shown in Figure 1.

The transpose of the manipulator Jacobian matrix transforms Cartesian forces to the corresponding joint torques [8] as

$$\tau = \mathbf{J}^T\,\mathbf{f} \tag{6}$$

This relationship is correct for any $\mathbf{J}$ and $\mathbf{f}$, and not an approximation for calculating τ. In particular, using the selected Cartesian force errors determined in Equation 5, it naturally follows that the selected joint torque errors are

$$\tau_{e_s} = \mathbf{J}^T\,\mathbf{f}_{e_s} \tag{7}$$

From a kinematic standpoint, the force equations presented are sound mathematical representations, both geometrically and algebraically. They model directly the physical behavior of the environment. Hence, the force equations in Equations 5 and 7 for the hybrid control scheme are always kinematically stable and need no further discussion.

2.3 Modified Hybrid Control

Zhang and Paul [14] modified the position part of the hybrid control scheme (enclosed within the dotted line in Figure 1) using the changes shown in Figure 2. In their approach the forward kinematics was pushed out of the feedback loop to linearize the control scheme in joint space. As shown in Figure 2, the errors are now determined in joint space and mapped to the corresponding Cartesian error $\mathbf{x}_e$ using the manipulator Jacobian matrix defined in Equation 2. The relationship between the joint error θ_e and the selected joint error θ_{e_s} is obtained by combining Equations 1, 2, and 4 to get

$$\theta_{e_s} = \mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e \tag{8}$$

The claim made by Zhang and Paul was that the joint selection expression $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ in Equation 8 is the joint space equivalent form for the Cartesian selection expression $\mathbf{S}$ in Equation 1. Further justification was that $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ is a similarity transformation of $\mathbf{S}$, since both matrices have the same eigenvalues. This implies that any results derived in Cartesian space would be directly applicable in the manipulator joint space and avoid the need for any further analysis in the manipulator joint space.

This reasoning is incorrect. A similarity transformation represents the *same* transformation with respect to a different set of basis vectors, which by definition spans the same vector space [11].

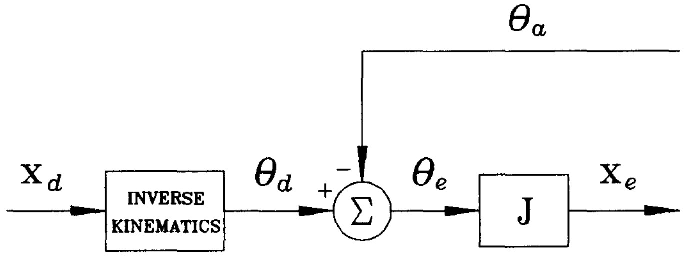

Figure 2: Modified Section of the Hybrid Control Scheme

The selection matrix $\mathbf{S}$ is a transformation in Cartesian space, while $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ is a mapping in the manipulator joint space. Therefore, $\mathbf{S}$ and $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ are not similar and any analysis done in Cartesian space must still be performed in the manipulator joint space. The derivation of Equation 8, however, is still correct.

2.4 Problem Definition

These are all the equations relevant to the hybrid control scheme regarding the kinematic stability of the two formulations. Using the simplest assumption of positive definite constant diagonal gain matrices $\mathbf{K}_p$ and $\mathbf{K}_d$ in a PD position control law, An and Hollerbach [1] showed by example how the hybrid control scheme for a two revolute joint manipulator produces an unstable system. A linearized state space approach was used by Zhang [13] to show that $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ results in an unstable system for certain manipulator configurations. These results were derived from using the equations presented above which were defined in the architectural framework of the hybrid control scheme given by Raibert and Craig [9].

The objectives now are to understand and explain why a kinematic instability problem arises in the hybrid control scheme and to show theoretically why it should never happen. The crux of the problem has to do with the formulation of Equation 4 of the Craig and Raibert approach or Equation 8 of the Zhang and Paul approach. These two equations deal only with the position part of the hybrid control scheme, which we will analyze in the next section.

3 A Correction to the Position Formulation

The kinematic instability of hybrid control has been shown to exist only in the position part of the formulation. In this section, we will show how Equation 4 in Section 2.1 is actually an incorrect derivation and that there is a general position solution for θ_{e_s}. To fully understand this solution, we first need to review some concepts from linear algebra.

3.1 Some Linear Algebra

Consider the following classical linear expression, which may be found in any text on linear algebra [7] [11]

$$\mathbf{A}\,\mathbf{x} = \mathbf{b} \tag{9}$$

In Equation 9, $\mathbf{A}$ is a constant $m \times n$ matrix with vector spaces $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{b} \in \mathbf{R}^m$. Geometrically, $\mathbf{A}$ is a mapping (or transformation) from one space to another, i.e., $\mathbf{A}: \mathbf{R}^n \rightarrow \mathbf{R}^m$. Depending on the size and rank of $\mathbf{A}$, the inverse mapping may not always exist or be unique between the two vector spaces. In general, a procedure for solving Equation 9 must distinguish among four possible conditions [6]. We will consider only the case producing an infinite number of solutions, since that is the form of our particular problem.

Equation 9 shows that any choice for $\mathbf{x}$ will produce a corresponding $\mathbf{b}$ that is always a vector in the column space of $\mathbf{A}$, denoted as $\Re(\mathbf{A})$. In other words, $\mathbf{b}$ is a linear combination of the column vectors of $\mathbf{A}$ scaled by the vector components of $\mathbf{x}$. A solution for $\mathbf{x}$ in Equation 9 will exist for a given $\mathbf{b}$ iff $\mathbf{b} \in \Re(\mathbf{A})$ [3]. When $\mathbf{A}$ contains more column vectors than necessary to span the $\mathbf{R}^m$ vector space, (i.e., $n > m$ and $\text{rank}(\mathbf{A}) = m$), an infinite number of solutions for $\mathbf{x}$ in Equation 9 will exist. The general solution to Equation 9 for this situation is

$$\mathbf{x} = \mathbf{A}^+\,\mathbf{b} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\,\mathbf{z} \tag{10}$$

where $\mathbf{A}^+$ is the pseudoinverse [11] of $\mathbf{A}$ and $\mathbf{z}$ is any arbitrary vector in $\mathbf{R}^n$. It can be shown [10] that in Equation 10 $\mathbf{A}^+\,\mathbf{b}$ is orthogonal to $(\mathbf{I} - \mathbf{A}^+\mathbf{A})\,\mathbf{z}$ and, by the Pythagorean Theorem, the unique minimum Euclidean norm solution is

$$\mathbf{x} = \mathbf{A}^+\,\mathbf{b} \tag{11}$$

3.2 Projection Matrices

There are two projection matrices associated with each vector space defined in Equation 9 that will divide a vector into two perpendicular components based on the $\mathbf{A}$ matrix transformation [4]. Hence, all vectors $\mathbf{b}$ in $\mathbf{R}^m$ can be represented as

$$\mathbf{b} = \mathbf{P}_{\mathrm{A}}\,\mathbf{b} + \mathbf{P}_{\mathrm{A}}^{\perp}\,\mathbf{b} \tag{12}$$

The vector $\mathbf{P}_{\mathrm{A}}\,\mathbf{b}$ in Equation 12 is the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$, $\Re(\mathbf{A})$. The other vector, $\mathbf{P}_{\mathrm{A}}^{\perp}\,\mathbf{b}$ is the projection of $\mathbf{b}$ onto the null space of $\mathbf{A}^T$, $\aleph(\mathbf{A}^T)$, which is the set of all independent vectors satisfying $\mathbf{A}^T\mathbf{b} = \mathbf{0}$. The two projection matrices in $\mathbf{R}^m$ for the system in Equation 9 are computed as

$$\begin{aligned} \mathbf{P}_{\mathrm{A}} &= \mathbf{A}\,\mathbf{A}^+ & &\Longrightarrow & \Re(\mathbf{P}_{\mathrm{A}}) &\equiv \Re(\mathbf{A}) \\ \mathbf{P}_{\mathrm{A}}^{\perp} &= \mathbf{I} - \mathbf{A}\,\mathbf{A}^+ & &\Longrightarrow & \Re(\mathbf{P}_{\mathrm{A}}^{\perp}) &\equiv \aleph(\mathbf{A}^T) \end{aligned} \tag{13}$$

It should be clear from Equation 13 that the direct sum of the two vector subspaces span $\mathbf{R}^m$, namely $\Re(\mathbf{A}) \oplus \aleph(\mathbf{A}^T) = \mathbf{R}^m$.

Similarly, all vectors $\mathbf{x}$ in $\mathbf{R}^n$ can be expressed as

$$\mathbf{x} = \mathbf{P}_{\mathrm{A}^T}\,\mathbf{x} + \mathbf{P}_{\mathrm{A}^T}^{\perp}\,\mathbf{x} \tag{14}$$

where the two projection matrices in $\mathbf{R}^n$ for the linear system in Equation 9 are

$$\begin{aligned} \mathbf{P}_{\mathbf{A}^T} &= \mathbf{A}^+\mathbf{A} &\Longrightarrow\quad \Re(\mathbf{P}_{\mathbf{A}^T}) &\equiv \Re(\mathbf{A}^T) \\ \mathbf{P}_{\mathbf{A}^T}^{\perp} &= \mathbf{I} - \mathbf{A}^+\mathbf{A} &\Longrightarrow\quad \Re(\mathbf{P}_{\mathbf{A}^T}^{\perp}) &\equiv \aleph(\mathbf{A}) \end{aligned} \tag{15}$$

The projection matrix $\mathbf{P}_{\mathbf{A}^T}^{\perp}$ in Equation 15 is the coefficient of the second term on the right hand side in Equation 10. This is the orthogonal complement to the column space of $\mathbf{A}$, meaning $\mathbf{A}\,\mathbf{P}_{\mathbf{A}^T}^{\perp} = \mathbf{0}$.

A projection matrix is both idempotent (i.e., $\mathbf{P}^2 = \mathbf{P}$ where each characteristic root or eigenvalue has a value of either 0 or 1), and symmetric (i.e., $\mathbf{P}^T = \mathbf{P}$). The idempotency property states that when a vector is projected onto a particular subspace, subsequent applications of the projection will not move the projected vector. The symmetry of a projection matrix is due to the fact that a projection matrix and its complement are orthogonal, namely $\mathbf{P}^T\mathbf{P}^{\perp} = (\mathbf{P}^{\perp})^T\mathbf{P} = \mathbf{0}$. One result of these two properties is that a projection matrix also satisfies the definition of a positive semidefinite matrix [11], meaning the following quadratic expression

$$\mathbf{v}^T\,\mathbf{P}\,\mathbf{v} \geq 0 \tag{16}$$

is true for all vectors $\mathbf{v} \in \mathbf{R}^n$ with $\Re(\mathbf{P}) \subseteq \mathbf{R}^n$. The norm of a projected vector is bounded by [4]

$$\|\,\mathbf{P}\,\mathbf{v}\,\| \leq \|\,\mathbf{v}\,\| \tag{17}$$

The results in Equations 16 and 17 for projection matrices will be used in a later section regarding kinematic stability.

We repeat these concepts from linear algebra to explain both the meaning and the significance of the selection matrix $\mathbf{S}$ defined in the hybrid control scheme and its effect on the manipulator Jacobian matrix. Using the selection matrix to separate position and force requirements in a Cartesian reference frame of interest is conceptually straightforward. Geometrically, the selection matrix is a projection matrix that reduces the complete Cartesian space to a desired subspace of interest. Problems arise when this selected Cartesian subspace is then mapped onto the joint space using the manipulator Jacobian matrix. We will show in the next section why this is so and how to resolve it.

3.3 The General Position Solution

The derivation of θ_{e_s} in Equation 4 is incomplete and happens to be only one solution out of an infinite number of possible solutions. Since Equation 4 has been shown to result in a kinematically unstable system, it is also considered an incorrect solution for θ_{e_s}. The problem with the original position control formulation is that a fundamental assumption of maximal rank is made implicitly when the manipulator Jacobian matrix inverse is used in Equation 3 to map *any* Cartesian errors to their corresponding joint errors. The only Cartesian error vector of interest in the hybrid control scheme is $\mathbf{x}_{e_s}$ in Equation 1, which is incorrectly used in Equation 3. When Equation 1 is combined with Equation 2 the correct result is

$$\mathbf{S}\,\mathbf{x}_e = (\mathbf{S}\,\mathbf{J})\,\theta_e \tag{18}$$

This crucial step was omitted in the original position solution of the hybrid control scheme. The significance of Equation 18 is that $\mathbf{S}$ reduces the Cartesian space on the left side of the expression, while $(\mathbf{S}\,\mathbf{J})$ maps a redundant number of manipulator joints onto this Cartesian subspace on the right. In essence, there are now more joints than necessary to satisfy the Cartesian motion constraints of the end effector. By combining Equations 1 and 18, the correct relationship between the selected Cartesian errors and the joint errors is

$$\mathbf{x}_{e_s} = (\mathbf{S}\,\mathbf{J})\,\theta_e \tag{19}$$

It should be noted that $(\mathbf{S}\,\mathbf{J})$ is a singular matrix and does not have an inverse. The form of Equation 19 is the same as Equation 9 with the terms on the left and right sides reversed. To determine the general position solution for the selected joint errors in Equation 19, the form in Equation 10 is used to get[2]

$$\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^{+}\,\mathbf{x}_{e_s} + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\mathbf{z} \tag{20}$$

The $\mathbf{z}$ in Equation 20 is an arbitrary $n \times 1$ vector in the manipulator joint space. It should be obvious that the original position solution for θ_{e_s} computed in Equation 4 will not always produce the same results as those computed in Equation 20. The original position solution for θ_{e_s} in Equation 4 is only one solution out of the infinite set of possible solutions that could be generated using Equation 20. We will refer to Equation 20 as the *general position solution* for θ_{e_s}.

3.4 A Comparison to the Original Position Solution

To fully appreciate the relationship between the original position solution for θ_{e_s} given in Equation 4 and the general position solution for θ_{e_s} in Equation 20, we use the properties given in Section 3.2 for the projection matrices of a linear system. All selected joint errors may be projected into the sum of two orthogonal vectors using the $(\mathbf{S}\,\mathbf{J})$ transformation matrix given in Equation 19 as

$$\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^{+}\,(\mathbf{S}\,\mathbf{J})\,\theta_{e_s} + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\theta_{e_s} \tag{21}$$

Substitute the solution for θ_{e_s} using Equation 4 from the original position solution into Equation 21 to get

$$\mathbf{J}^{-1}\,\mathbf{x}_{e_s} = (\mathbf{S}\,\mathbf{J})^{+}\,(\mathbf{S}\,\mathbf{J})\,\mathbf{J}^{-1}\,\mathbf{x}_{e_s} + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\mathbf{J}^{-1}\,\mathbf{x}_{e_s} \tag{22}$$

The term $(\mathbf{S}\,\mathbf{J})^{+}\,(\mathbf{S}\,\mathbf{J})\,\mathbf{J}^{-1}\,\mathbf{x}_{e_s}$ in Equation 22 may be simplified by realizing $\mathbf{J}\,\mathbf{J}^{-1} = \mathbf{I}$ and $\mathbf{S}\,\mathbf{x}_{e_s} = \mathbf{x}_{e_s}$ (see Section 3.2 for idempotency property of a projection matrix) to result in

$$\mathbf{J}^{-1}\,\mathbf{x}_{e_s} = (\mathbf{S}\,\mathbf{J})^{+}\,\mathbf{x}_{e_s} + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\mathbf{J}^{-1}\,\mathbf{x}_{e_s} \tag{23}$$

It is important to note that the first projection term in Equation 23 is the minimum norm solution part of the general form for θ_{e_s} in Equation 20. Equation 23 explicitly shows that the traditional approach of using the manipulator Jacobian matrix inverse to solve for θ_{e_s} in Equation 4 will inadvertently add an orthogonal vector $[\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\mathbf{J}^{-1}\,\mathbf{x}_{e_s}$ to the minimum norm solution. For the general position solution in Equation 20 to behave the same as the original position solution in Equation 4, compare the orthogonal projection terms in Equations 23 and 20 to immediately see that one obvious choice for the arbitrary vector $\mathbf{z}$ would be

$$\mathbf{z} = \mathbf{J}^{-1}\,\mathbf{x}_{e_s} \tag{24}$$

Both An and Hollerbach [1] and Zhang [13] have shown that, in the original hybrid control scheme, the position solution for θ_{e_s} causes a kinematically unstable system. We will show in Section 5

[2] The reasons for the change in notation of the joint error θ_e to θ_{e_s} are both to signify that these are the *selected* joint errors computed from the selected Cartesian error $\mathbf{X}_{e_s}$ and to be consistent with the notation used in the original position solution formulation.

that the minimum norm solution for θ_{e_s} is always kinematically stable. Therefore, we claim that it is the projection of this unbeknown choice for $\mathbf{z}$ in Equation 24 onto the null space of $(\mathbf{S}\,\mathbf{J})$ that causes the kinematic instability in the original hybrid control scheme.

It may be argued that the orthogonal vector $[\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})\,]\,\mathbf{z}$ in Equation 20, which is produced from any arbitrarily chosen vector $\mathbf{z}$, adds flexibility to the solution for θ_{e_s} and may be used to optimize θ_{e_s} based on some desired criterion (e.g., to minimize joint energy or to keep the joints in the middle of their operating range). We will show that not all methods chosen to influence the solution for θ_{e_s} using the orthogonal vector will guarantee system stability for all manipulator configurations. A more important issue to understand first is how the general position solution given in Equation 20 will affect the rest of the hybrid control scheme.

4 A Complete Position and Force Formulation

The general position solution for θ_{e_s} in Equation 20 of Section 3.3 assumes no force control is present and hence potentially uses the entire manipulator joint space to control position. This severely restricts the advantages of using hybrid control. In this section, a vector space approach is presented to show how the solution space of the orthogonal vector in Equation 20 is reduced to get a correct position solution for θ_{e_s} that does not preclude the use of force information in hybrid control. This results in a generalized hybrid control scheme that is no longer hindered with using the inverse of the manipulator Jacobian matrix and expands the technique to include redundant manipulators.

4.1 A Vector Space Approach

The selection matrix $\mathbf{S}$ is used as a mapping to split the Cartesian space as

$$\Re(\mathbf{S}) \oplus \Re(\mathbf{S}^{\perp}) = \mathbf{R}^{6} \tag{25}$$

The column spaces of $\mathbf{S}$ and $\mathbf{S}^{\perp}$ in Equation 25 are orthogonal complements and thus the selection matrix keeps the Cartesian position and force information independent. The Jacobian matrix $\mathbf{J}$ is used as the mapping from joint space to Cartesian space and may be used to span both spaces as

$$\begin{aligned} \Re(\mathbf{J}) \oplus \aleph(\mathbf{J}^{T}) &= \mathbf{R}^{6} \\ \Re(\mathbf{J}^{T}) \oplus \aleph(\mathbf{J}) &= \mathbf{R}^{n} \end{aligned} \tag{26}$$

The next natural step would be to combine the two mappings as $(\mathbf{S}\,\mathbf{J})$, substitute it into Equation 26, and then show all the interesting relationships among the different subspaces. We will leave that as an exercise for the reader.

It is important to understand the effect of $\mathbf{S}$ on $\mathbf{J}$ by realizing that the selection matrix projects the Jacobian matrix into two orthogonal transformations as

$$\mathbf{J} = \mathbf{S}\,\mathbf{J} + \mathbf{S}^{\perp}\,\mathbf{J} \tag{27}$$

The transpose of the matrix expression in Equation 27 is

$$\begin{aligned} \mathbf{J}^{T} &= (\mathbf{S}\,\mathbf{J} + \mathbf{S}^{\perp}\,\mathbf{J})^{T} \\ &= (\mathbf{S}\,\mathbf{J})^{T} + (\mathbf{S}^{\perp}\,\mathbf{J})^{T} \end{aligned} \tag{28}$$

The two subspaces $\Re((\mathbf{S}\,\mathbf{J})^T)$ and $\Re((\mathbf{S}^\perp\,\mathbf{J})^T)$ for the transformations in Equation 28 are complementary spaces that together span $\Re(\mathbf{J}^T)$. They are **not** orthogonal subspaces, $(\mathbf{S}\,\mathbf{J})\,(\mathbf{S}^\perp\,\mathbf{J})^T \neq \mathbf{0}$ in general, and therefore[3]

$$\Re(\mathbf{J}^T) \equiv \Re((\mathbf{S}\,\mathbf{J})^T) + \Re((\mathbf{S}^\perp\,\mathbf{J})^T) \tag{29}$$

The joint space representation of the second expression in Equation 26 may now be expressed using Equation 29 for $\Re(\mathbf{J}^T)$ as

$$\left(\Re((\mathbf{S}\,\mathbf{J})^T) + \Re((\mathbf{S}^\perp\,\mathbf{J})^T)\right) \oplus \aleph(\mathbf{J}) = \mathbf{R}^n \tag{30}$$

Since the column space of any matrix transpose is equal to the column space of its pseudoinverse [3], $\Re((\mathbf{S}\,\mathbf{J})^T) \equiv \Re((\mathbf{S}\,\mathbf{J})^+)$ and Equation 30 becomes

$$\left(\Re((\mathbf{S}\,\mathbf{J})^+) + \Re((\mathbf{S}^\perp\,\mathbf{J})^T)\right) \oplus \aleph(\mathbf{J}) = \mathbf{R}^n \tag{31}$$

Recall how Equation 25 was used to explain the separation of the Cartesian position and force vectors. Equation 31 explicitly shows how this Cartesian position and force information is interpreted in the manipulator joint space. The Cartesian positions use $\Re((\mathbf{S}\,\mathbf{J})^+)$ with the Cartesian forces using $\Re((\mathbf{S}^\perp\,\mathbf{J})^T)$. The only time the two subspaces will span the manipulator joint space is when $\aleph(\mathbf{J}) = \{\mathbf{0}\}$, which means the column vectors of $\mathbf{J}$ must all be linearly independent.

The vector space relationship in Equation 31 is a fundamental result and will be used to generalize the basic notion of hybrid control. We will expand on the meaning of each subspace in Equation 31 to show how it relates to the actual equations in hybrid control.

4.1.1 Position Space Contribution

The position mapping of the Cartesian space to the manipulator joint space in Equation 31 is spanned by $\Re((\mathbf{S}\,\mathbf{J})^+)$. This space does not include the orthogonal vector part of the general position solution for θ_{e_s} in Equation 20 of Section 3.3 and reduces the general position solution to its minimum norm solution. Substituting $\mathbf{x}_{e_s} = \mathbf{S}\,\mathbf{x}_e$ from Equation 1 into Equation 20 and solving for the minimum norm solution results in

$$\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^+\,\mathbf{S}\,\mathbf{x}_e \tag{32}$$

An interesting result that we prove in Theorem I of Appendix A for the general case regarding projection matrices and linear transformations is $(\mathbf{S}\,\mathbf{J})^+\,\mathbf{S} = (\mathbf{S}\,\mathbf{J})^+$, which simplifies Equation 32 to

$$\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^+\,\mathbf{x}_e \tag{33}$$

The solution for θ_{e_s} in Equation 33 eliminates the explicit projection of the Cartesian error $\mathbf{x}_e$ to $\mathbf{x}_{e_s}$ in Equation 1 by its implicit operation in the $(\mathbf{S}\,\mathbf{J})^+$ transformation.

[3]The two subspaces in Equation 29 may overlap, making the direct sum no longer valid, meaning that for some situations $\Re((\mathbf{S}\,\mathbf{J})^T) \cap \Re((\mathbf{S}^\perp\,\mathbf{J})^T) \neq \{0\}$.

4.1.2 Force Space Contribution

The mapping of Cartesian force to manipulator joint torques is essentially unchanged. By combining Equations 5 and 7 from Section 2.2, and using the symmetry property of projection matrices, the result may be expressed as

$$\begin{aligned} \tau_{e_s} &= \mathbf{J}^T \mathbf{S}^\perp \mathbf{f}_e \\ &= (\mathbf{S}^\perp \mathbf{J})^T \mathbf{f}_e \end{aligned} \tag{34}$$

The final matrix transformation in Equation 34 is clearly the space spanned by the $\Re((\mathbf{S}^\perp \mathbf{J})^T)$ in Equation 31.

4.1.3 Null Space Contribution

The last subspace $\aleph(\mathbf{J})$ in Equation 31 completes the coverage of the manipulator joint space. There are only two conditions when $\aleph(\mathbf{J}) \neq \mathbf{0}$: either the manipulator is in a singular configuration or it has a redundant number of joints. For redundant manipulators, this null space may be used to modify the manipulator behavior based on desired performance criteria. Since $\aleph(\mathbf{J})$ is orthogonal to both $\Re((\mathbf{S}\,\mathbf{J})^+)$ and $\Re((\mathbf{S}^\perp \mathbf{J})^T)$, the null space of $\mathbf{J}$ may be used to influence the control of both joint positions and torques, which to our knowledge has never been considered before in any formulation of hybrid control. We will show how this is done in the next section.

4.2 The Generalized Hybrid Control Scheme

From the vector space analysis, we are now able to state the correct formulations for both selected joint position and torque values to result in a generalized hybrid control scheme. The *correct position solution* for θ_{e_s} now combines Equation 33 with an additional term from the null space of $\mathbf{J}$ to get

$$\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^+ \mathbf{x}_e + [\,\mathbf{I} - \mathbf{J}^+ \mathbf{J}\,]\,\mathbf{z}_\theta \tag{35}$$

where $\mathbf{z}_\theta$ is an arbitrary position vector in the manipulator joint space and $[\,\mathbf{I} - \mathbf{J}^+ \mathbf{J}\,]$ spans $\aleph(\mathbf{J})$, since $\aleph(\mathbf{J}) \equiv \Re(\mathbf{I} - \mathbf{J}^+ \mathbf{J})$. The correct position solution in Equation 35 is a subspace of the general position solution in Equation 20 when realizing both equations have the same minimum norm component, with the null space component in Equation 35 being a reduced space of the orthogonal space of $\mathbf{S}\,\mathbf{J}$ in Equation 20, namely $\Re(\mathbf{I} - \mathbf{J}^+ \mathbf{J}) \subseteq \Re(\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J}))$. It is also interesting to note that the original position solution for θ_{e_s} in Equation 4 cannot be expressed in terms of the correct position solution for θ_{e_s} given in Equation 35 and therefore is no longer considered a feasible position solution for hybrid control.

In force control, we combine the solution for τ_{e_s} in Equation 34 with an additional term from the null space of $\mathbf{J}$ to result in

$$\tau_{e_s} = (\mathbf{S}^\perp \mathbf{J})^T \mathbf{f}_e + [\,\mathbf{I} - \mathbf{J}^+ \mathbf{J}\,]\,\mathbf{z}_\tau \tag{36}$$

where $\mathbf{z}_\tau$ is an arbitrary torque vector in the manipulator joint space. This generalized form for the selected joint torques in Equation 36 allows for a redistribution of the joint torques based on joint capabilities, which we consider a new concept for hybrid control.

Figure 3 shows the resultant generalized hybrid control scheme derived from the insights into the vector space relationship given in Equation 31. This new formulation directly takes into account the additional flexibility of redundant manipulators in the control scheme and eliminates ever having to use the inverse of the manipulator Jacobian matrix. The $(\mathbf{S}\,\mathbf{J})^+$ and $(\mathbf{S}^\perp \mathbf{J})^T$ transformations in Figure 3 always exist and are numerically stable for any manipulator.

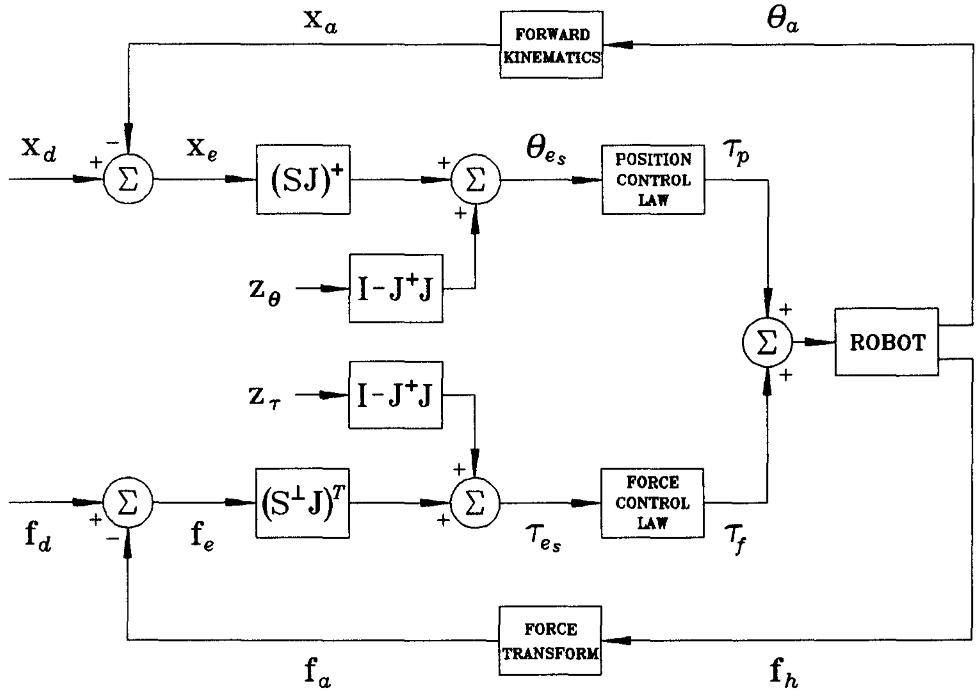

Figure 3: General Hybrid Control Scheme

5 Kinematic Stability

System stability is a result of the interaction of the kinematics, dynamics, and the control law. In hybrid control the kinematics has been shown to induce an unstable system. The notion of testing for kinematic stability has never before been defined. It is not clear that system stability and kinematic stability can be decoupled in a general sense to determine independent stability criteria that are both necessary and sufficient. In this section, only the position part of the hybrid control scheme will be analyzed to explain the conditions that we propose for determining kinematic stability.

In the original hybrid control formulation it was shown by example that the position part of the system could become unstable and that the cause was somehow related to the kinematics of the $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ transformation matrix. When a linear state space model of the position part of the system was tested for stability, this $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ term caused the system poles to migrate into the right half plane, which is the unstable region, for varying manipulator configurations. We will show that the $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ term is clearly responsible for causing an unstable system. We will do this without having to include a complete system model in the analysis by using our stated conditions for determining kinematic stability.

Before we begin the analysis, the following necessary condition for kinematic stability must be satisfied. The position control part of the system must be stable without the influence of the selection matrix S. In other words, the manipulator is always stable when using pure position control with $\mathbf{S} = \mathbf{I}$, and the θ_e's corresponding to the $\mathbf{x}_e$'s under these *normal conditions* do not produce any system instabilities. The selection matrix is not intended to stabilize an unstable system. This necessary condition should seem very natural and will always be regarded as true.

5.1 Sufficient Conditions for Kinematic Stability

Our approach is to state formally some conditions for determining how the kinematics of a system can induce instability without having to include the complete system model. It is possible these conditions may overconstrain the use of the entire stable region for the complete system and so they should be regarded only as sufficient conditions. The proofs are not rigorous and follow very basic lines of reasoning.

5.1.1 Condition I

Our first condition for kinematic stability states that the inner product of the orthogonal projections of the vectors used under normal conditions (i.e., $\mathbf{x}_e$ and θ_e) and the corresponding transformed vectors due to the selection matrix (i.e., $\mathbf{x}_{e_s}$ and θ_{e_s}) must be nonnegative. The algebraic form of this condition is, given a projection matrix $\mathbf{P}$ in the vector space $\mathbf{R}^n$ and two related vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbf{R}^n$, then for kinematic stability

$$\begin{aligned} (\mathbf{P}\,\mathbf{v}_1)^T\,(\mathbf{P}\,\mathbf{v}_2) &\geq 0 \\ (\mathbf{P}^\perp\,\mathbf{v}_1)^T\,(\mathbf{P}^\perp\,\mathbf{v}_2) &\geq 0 \end{aligned} \tag{37}$$

The purpose of the inequality constraints in Equation 37 is to insure that the respective orthogonal component vector projections of $\mathbf{v}_2$ are not in the opposite direction from the orthogonal component vector projections of $\mathbf{v}_1$. By using the properties of a projection matrix, Equation 37 simplifies to

$$\begin{aligned} \mathbf{v}_1{}^T\,\mathbf{P}\,\mathbf{v}_2 &\geq 0 \\ \mathbf{v}_1{}^T\,\mathbf{P}^\perp\,\mathbf{v}_2 &\geq 0 \end{aligned} \tag{38}$$

From a control standpoint, the inequality constraints in Equation 38 avoid the possibility of introducing positive feedback into the system to cause an unstable situation. After we have stated all of our conditions for kinematic stability, we will apply Equation 38 to the vector relationships, both in the Cartesian space (with $\mathbf{v}_1 = \mathbf{x}_e$ and $\mathbf{v}_2 = \mathbf{x}_{e_s}$) and in the manipulator joint space (with $\mathbf{v}_1 = \theta_e$ and $\mathbf{v}_2 = \theta_{e_s}$).

It should be noted that the inequality constraints in Equation 38 cover a subspace of the region allowed by the constraint $\mathbf{v}_1{}^T\,\mathbf{v}_2 \geq 0$. We originally thought $\mathbf{v}_1{}^T\,\mathbf{v}_2 \geq 0$ would be a sufficient condition for kinematic stability, but found a counterexample in the joint space analysis when using the original position solution for θ_{e_s}, which we show in Section 5.3.2.

5.1.2 Condition II

We consider the next condition a precaution. After the first condition for kinematic stability is satisfied, a second test in the analysis is to check the vector norms between the computed vectors under normal conditions (i.e., $\mathbf{x}_e$ and θ_e) to the vector norms of the vectors computed with the selection matrix (i.e., $\mathbf{x}_{e_s}$ and θ_{e_s}) for possible increases in size. This condition is related to BIBO stability [5], which states that the output vector norm of a system is bounded by a constant value k (less than infinity) times the input vector norm. We have chosen $k = 1$ for our second condition on kinematic stability to get

$$\|\mathbf{v}_2\| \leq \|\mathbf{v}_1\| \tag{39}$$

This second condition as stated in Equation 39 is applicable to any norm and not just the Euclidean norm, with $\mathbf{v}_1$ and $\mathbf{v}_2$ defined the same way as in Condition I.

5.2 Cartesian Space Test for Kinematic Stability

To test the first condition for kinematic stability in Cartesian space, the inequality constraints in Equation 38 become

$$\begin{aligned} \mathbf{x}_e{}^T \mathbf{P}_{\mathrm{S}} \mathbf{x}_{e_s} &\geq 0 \\ \mathbf{x}_e{}^T \mathbf{P}_{\mathrm{S}}^{\perp} \mathbf{x}_{e_s} &\geq 0 \end{aligned} \tag{40}$$

$\mathbf{P}_{\mathrm{S}}$ in Equation 40 is the projection matrix of the selection matrix used in Equation 1 and is just the selection matrix itself. The orthogonal projection $\mathbf{P}_{\mathrm{S}}^{\perp}$ is the orthogonal complement of $\mathbf{S}$, which is $\mathbf{S}^{\perp} = (\mathbf{I} - \mathbf{S})$. Hence, when the definition for $\mathbf{x}_{e_s}$ in Equation 1 is substituted in Equation 40, the results are

$$\begin{aligned} \mathbf{x}_e{}^T \mathbf{S}\, \mathbf{x}_e &\geq 0 \\ \mathbf{x}_e{}^T \mathbf{S}^{\perp} \mathbf{S}\, \mathbf{x}_e &= 0 \end{aligned} \tag{41}$$

Since S is a projection matrix, the first inequality constraint in Equation 41 is true for any $\mathbf{x}_e$ (see Equation 16 in Section 3.2).

For the second kinematic stability test, the norm of the transformation given in Equation 1 is

$$\| \mathbf{x}_{e_s} \| = \| \mathbf{S}\, \mathbf{x}_e \| \tag{42}$$

Using Equation 17 in Section 3.2 for the norm of the projection term in Equation 42, the upper bound on the norm for the selected Cartesian errors is

$$\| \mathbf{x}_{e_s} \| \leq \| \mathbf{x}_e \| \tag{43}$$

The conclusions in Equations 41 and 43 are not surprising. They reinforce the fact that the selected Cartesian errors will never be in the opposite direction of, nor have a norm greater than, the Cartesian errors computed from the desired and actual values. In Cartesian space, the selection matrix transformation always satisfies our sufficient conditions for kinematic stability.

5.3 Joint Space Test for Kinematic Stability

To test the first condition for kinematic stability in joint space, the inequality constraints in Equation 38 are simply[4]

$$\begin{aligned} \theta_e{}^T \mathbf{P}_{(\mathrm{SJ})^T} \theta_{e_s} &\geq 0 \\ \theta_e{}^T \mathbf{P}_{(\mathrm{SJ})^T}^{\perp} \theta_{e_s} &\geq 0 \end{aligned} \tag{44}$$

The projection matrices in Equation 44 are the joint space projections of the linear transformation $(\mathbf{S}\,\mathbf{J})$ in Equation 19. These projection matrices are

$$\begin{aligned} \mathbf{P}_{(\mathrm{SJ})^T} &= (\mathbf{S}\,\mathbf{J})^{+} (\mathbf{S}\,\mathbf{J}) \\ \mathbf{P}_{(\mathrm{SJ})^T}^{\perp} &= \mathbf{I} - (\mathbf{S}\,\mathbf{J})^{+} (\mathbf{S}\,\mathbf{J}) \end{aligned} \tag{45}$$

[4] We have found situations where the second inequality constraint in Equation 44 may be negative and still maintain a stable system, thereby reinforcing the fact that Equation 38 is only sufficient and not a necessary condition for kinematic stability. We were unable to determine the lower bound on the negative value, as it seemed to be a function of the manipulator configuration.

See Section 3.2 and compare Equation 45 to the definition in Equation 15 for the $\mathbf{P}_{\mathbf{A}^T}$ and $\mathbf{P}_{\mathbf{A}^T}^{\perp}$ projection matrices where $(\mathbf{S}\,\mathbf{J})$ is substituted for $\mathbf{A}$. The first condition as stated in Equation 44, using the projections in Equation 45, becomes

$$\begin{aligned} \theta_e{}^T (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})\,\theta_{e_s} &\geq 0 \\ \theta_e{}^T [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J})\,]\,\theta_{e_s} &\geq 0 \end{aligned} \tag{46}$$

To actually test the two conditions given for kinematic stability in the manipulator joint space, a particular solution for θ_{e_s} is required. We will present two examples for comparative purposes: (i) the correct position solution given in Equation 35 for the generalized hybrid control scheme, and (ii) the original position solution given in Equation 4 for the Craig and Raibert approach.

5.3.1 A Test of the Correct Position Solution

The conditions placed on the selected joint vectors for the manipulator in Equation 46 using the solution for θ_{e_s} in Equation 35 and as shown in Figure 3 are

$$\begin{aligned} \theta_e{}^T (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})\,\{\,(\mathbf{S}\,\mathbf{J})^+\,\mathbf{x}_{e_s} + [\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,]\,\mathbf{z}_\theta\,\} &\geq 0 \\ \theta_e{}^T [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J})\,]\,\{\,(\mathbf{S}\,\mathbf{J})^+\,\mathbf{x}_{e_s} + [\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,]\,\mathbf{z}_\theta\,\} &\geq 0 \end{aligned} \tag{47}$$

Substitute $\mathbf{x}_{e_s} = (\mathbf{S}\,\mathbf{J})\,\theta_e$ from Equation 19 into Equation 47 and simplify using the Moore-Penrose [6] properties of a pseudoinverse (i.e., $\mathbf{A} = \mathbf{A}\,\mathbf{A}^+\,\mathbf{A}$) to get

$$\begin{aligned} \theta_e{}^T (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})\,\theta_e &\geq 0 \\ \theta_e{}^T [\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,]\,\mathbf{z}_\theta &\geq 0 \end{aligned} \tag{48}$$

The matrix $(\mathbf{S}\,\mathbf{J})^+\,(\mathbf{S}\,\mathbf{J})$ in Equation 48 is the same matrix given in Equation 45 for the $\mathbf{P}_{(\mathbf{S}\mathbf{J})^T}$ projection matrix. From Equation 16 in Section 3.2, the first inequality constraint in Equation 48 is always true. Although $[\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,]$ in the second inequality constraint of Equation 48 is also a projection matrix and positive semidefinite, the inequality may not always be true for any arbitrarily chosen vector $\mathbf{z}_\theta$. However, it is always possible to find a $\mathbf{z}_\theta$ (e.g., $\mathbf{z}_\theta = \theta_e$) that will make the second inequality in Equation 48 true whenever $[\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,] \neq \mathbf{0}$.

To test the second condition for the joint solution, the vector norm of θ_{e_s} for the generalized hybrid control scheme shown in Figure 3 is

$$\|\,\theta_{e_s}\,\| \;=\; \|\,(\mathbf{S}\,\mathbf{J})^+\,(\mathbf{S}\,\mathbf{J})\,\theta_e + [\,\mathbf{I} - \mathbf{J}^+\,\mathbf{J}\,]\,\mathbf{z}_\theta\,\| \tag{49}$$

There is not enough information in Equation 49 to make any strong statements regarding the relationship between $\|\,\theta_{e_s}\,\|$ and $\|\,\theta_e\,\|$. To find the connection between the norms of the two joint error vectors, the properties given in Section 3.2 for projection matrices are used again to describe all the possible joint errors θ_e using the $(\mathbf{S}\,\mathbf{J})$ transformation matrix given in Equation 18 as

$$\theta_e = (\mathbf{S}\,\mathbf{J})^+\,(\mathbf{S}\,\mathbf{J})\,\theta_e + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J})\,]\,\theta_e \tag{50}$$

The vector norm of θ_e in Equation 50 is

$$\|\,\theta_e\,\| \;=\; \|\,(\mathbf{S}\,\mathbf{J})^+\,(\mathbf{S}\,\mathbf{J})\,\theta_e + [\,\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J})\,]\,\theta_e\,\| \tag{51}$$

Compare Equation 51 to Equation 49. The only difference between the norms of the two joint error vectors is in their orthogonal component. Since both equations have the same $(\mathbf{S}\,\mathbf{J})^+\,(\mathbf{S}\,\mathbf{J})\,\theta_e$

vector that is added to some other orthogonal vector, the difference in size between θ_{e_s} and θ_e will be determined by the difference in size of their respective orthogonal vectors. Hence, whenever the norms of the two orthogonal components satisfy the following condition

$$\| [\mathbf{I} - \mathbf{J}^+ \mathbf{J}] \mathbf{z}_\theta \| \;\leq\; \| [\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})] \theta_e \| \tag{52}$$

the relationship between the joint error norms will have $\| \theta_{e_s} \| \;\leq\; \| \theta_e \|$. The vector $\mathbf{z}_\theta$ chosen to satisfy the second inequality constraint in Equation 48 may be scaled appropriately in order to always satisfy Equation 52 as well. Hence, the correct position solution for θ_{e_s} in the generalized hybrid control scheme can always find a $\mathbf{z}_\theta$ to satisfy the sufficient conditions for kinematic stability.

5.3.2 A Test of the Original Position Solution

To test the kinematic stability conditions on the original hybrid control scheme, the relationship between θ_{e_s} and θ_e is given in Equation 8 from the modified version of the approach by Zhang and Paul where $\theta_{e_s} = \mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e$. This is an example of where our initial thought of using the constraint ${\theta_e}^T \theta_{e_s} \geq 0$ failed to be a sufficient condition for kinematic stability. We found cases where ${\theta_e}^T \mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e \geq 0$ was true when the system was actually unstable.

Substituting the solution in Equation 8 for θ_{e_s} in Equation 46, the first kinematic stability condition becomes

$$\begin{aligned} {\theta_e}^T (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})\,\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e &\geq 0 \\ {\theta_e}^T [\mathbf{I} - (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})]\,\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e &\geq 0 \end{aligned} \tag{53}$$

Simplifying the matrix expressions in Equation 53 results in

$$\begin{aligned} {\theta_e}^T (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})\,\theta_e &\geq 0 \\ {\theta_e}^T [\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J} - (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})]\,\theta_e &\geq 0 \end{aligned} \tag{54}$$

The first inequality constraint in Equation 54 is the same as the first inequality constraint in Equation 48 and is always true. In the second inequality constraint of Equation 54, $[\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J} - (\mathbf{S}\,\mathbf{J})^+ (\mathbf{S}\,\mathbf{J})]$ is not a positive semidefinite matrix. Hence, the constraint may become negative for some manipulator configurations, which could cause the system to become unstable by introducing positive feedback into the control scheme. It is interesting to note that the second inequality constraint in Equation 54 is directly related to the orthogonal vector influence on the solution for θ_{e_s} that was discussed in Section 3.4.

For the second condition of kinematic stability, the norm of θ_{e_s} in Equation 8 is

$$\| \theta_{e_s} \| \;=\; \| \mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e \| \tag{55}$$

For some manipulator configurations, $\| \mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}\,\theta_e \|$ can be greater than $\| \theta_e \|$, which means $\| \theta_{e_s} \|$ will be greater than $\| \theta_e \|$ under these conditions. Hence, the original position solution for θ_{e_s} in the Craig and Raibert hybrid control scheme does not always satisfy the sufficient conditions for kinematic stability.

These results confirm the potential kinematic instability of the original hybrid control scheme that was shown true using a linearized state space approach for a simple PD position control law of a two revolute joint manipulator by An and Hollerbach [1] (refer to Appendix B for their main result) and Zhang [13]. In both of their approaches the $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ term was responsible for the migration of the system poles into the right half plane causing instability that corresponded to the times when the sufficient conditions for kinematic stability were not satisfied.

5.4 A Stable Hybrid Control Scheme

One conclusion from the results in Section 5.3.1 for the generalized hybrid control scheme is that the arbitrary vector $\mathbf{z}_\theta$ cannot be arbitrary when constrained to satisfy the kinematic stability conditions. To ensure stability, $\mathbf{z}_\theta$ must be chosen in a manner that also satisfies both Equations 48 and 52. Even though this will ensure kinematic stability, it is not clear that continuously smooth joint motions can be maintained.

To guarantee a kinematically stable formulation of the generalized hybrid control scheme shown in Figure 3, the arbitrary joint vectors $\mathbf{z}_\theta$ and $\mathbf{z}_\tau$ are set to zero. The second inequality constraint in Equation 48 for the first kinematic stability test is now always satisfied and the second kinematic stability test follows directly from the same principles in Equation 17 of Section 3.2 to get

$$\begin{aligned} \|\theta_{e_s}\| &= \|(\mathbf{S}\,\mathbf{J})^+(\mathbf{S}\,\mathbf{J})\theta_e\| \\ &\leq \|\theta_e\| \end{aligned} \tag{56}$$

By using the minimum norm solution as the transformation from Cartesian space to the manipulator joint space, Equation 56 shows that the norm of the selected joint errors will never be larger than the norm of the joint errors produced from the Cartesian errors under normal conditions of pure position control.

The minimum norm solution for θ_{e_s} guarantees that the linear transformation in Equation 33 from the Cartesian space to the manipulator joint space will never generate a vector in the opposite direction to the projected vector θ_e and never cause an increase in the joint error vector norm, thereby always maintaining a kinematically stable system. Figure 4 shows a kinematically stable version of the generalized hybrid control scheme that was shown in Figure 3 without the null space contribution. The dotted outline shown in Figure 4 is the only difference from the original block

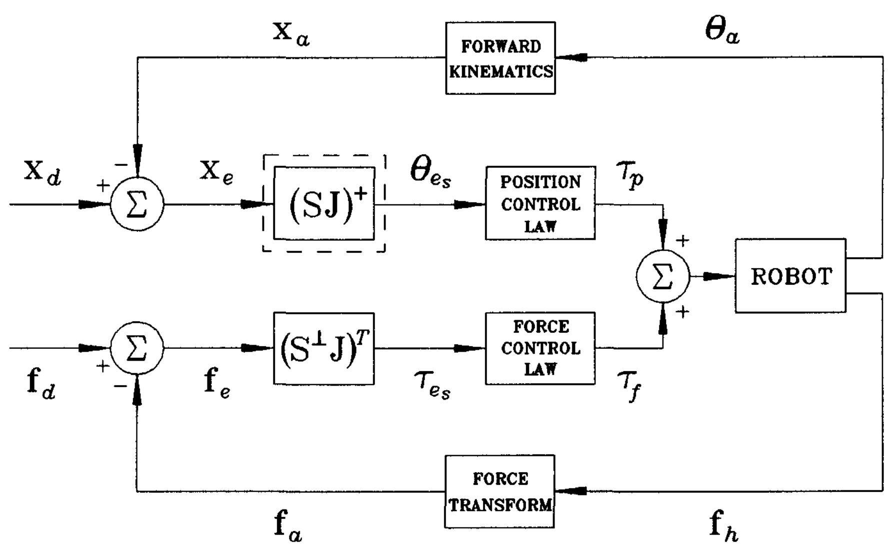

Figure 4: A Stable Hybrid Control Scheme

diagram shown in Figure 1. Although the difference may seem small, it has a tremendous impact on the robustness of hybrid control.

For example, we repeated Case 2 in An and Hollerbach [1] using Mathematica^{TM} [12], replacing the $\mathbf{J}^{-1}\,\mathbf{S}\,\mathbf{J}$ expression with our solution of $(\mathbf{S}\,\mathbf{J})^{+}(\mathbf{S}\,\mathbf{J})$. The resulting root locus plot is shown in Figure 5 with $\theta_1 = 0^o$ and θ_2 varying from -90^o to 90^o. Clearly there are no roots in the right

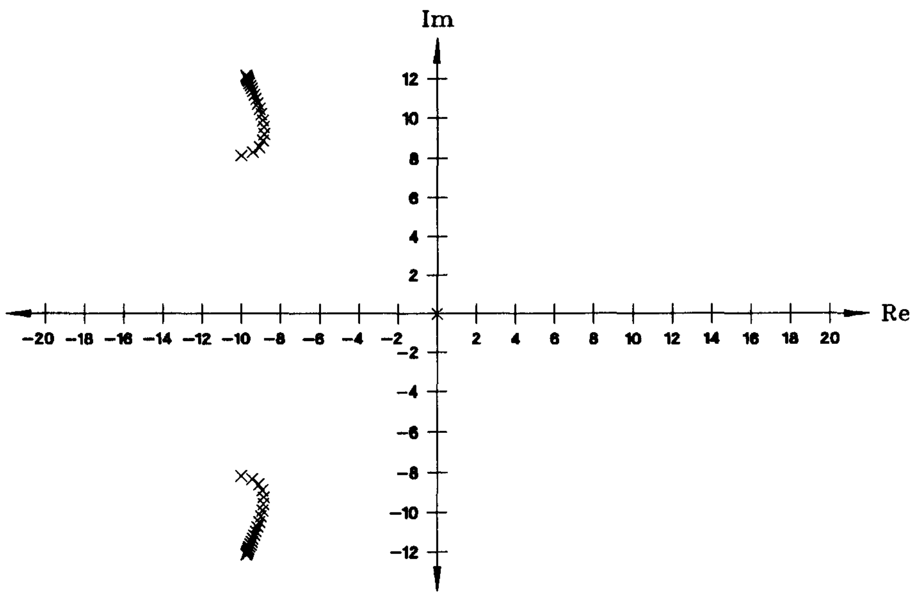

Figure 5: Stable Root Locus Plot using $\theta_{e_s} = (\mathbf{S}\,\mathbf{J})^{+}\,\mathbf{x}_e$

half plane for the given range of θ_2 values. See Figure 6 in Appendix B for a comparison to their unstable root locus plot.

6 Summary and Conclusion

We have tried to present a sound theory for hybrid position/force control of robot manipulators that will enable researchers in the field to exploit fully the power and capability of this control technique. Our efforts and results focused on three major areas considered important to hybrid control: the basic equations, a generalized architecture, and stability.

The first area reviewed the fundamental kinematic equations to show an error in the basic position formulation. The inverse of the manipulator Jacobian matrix was shown to be an incorrect and inappropriate mapping from Cartesian space to joint space. We started with the original position solution for the joints and showed how it was only one solution out of an infinite set of possible solutions. We named this formulation for the joints the general position solution. From the general position solution, we derived a subset of correct position solutions for hybrid control of which the original position solution was no longer considered a valid choice.

The second area used a vector space approach to derive a generalized architecture for the hybrid control scheme. We showed how the null space of the manipulator Jacobian matrix can influence the control of both joint positions and torques. The ability to redistribute joint torques in the force control part of hybrid control is considered a new development and result.

The third area presented some sufficient conditions for kinematic stability that do not require the complete system model in the analysis. As a result, a kinematically stable hybrid control scheme was derived. We also applied the new control scheme to demonstrate stability in an example that was previously shown to be unstable.

In addition to the standard linear algebra used throughout the paper, we showed how to make effective use of projection matrices in the analysis. We were also able to prove some interesting relationships between projection matrices and linear transformations.

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Appendix

A Some Additional Properties of Projection Matrices

The results presented in this section show what effects a projection matrix has on a linear transformation matrix and vice versa. Let $\mathbf{P}$ be an $m \times m$ projection matrix. The following theorems are true when $\mathbf{A}$ is any $m \times n$ linear transformation matrix.

Theorem I : $(\mathbf{P}\,\mathbf{A})^{+}\,\mathbf{P} = (\mathbf{P}\,\mathbf{A})^{+}$

Proof : Assume Theorem I is not true. Then

$$(\mathbf{P}\,\mathbf{A})^{+}\,\mathbf{P} \neq (\mathbf{P}\,\mathbf{A})^{+}$$

Since the row space of $(\mathbf{P}\,\mathbf{A})^{+}$ is equal to the row space of $(\mathbf{P}\,\mathbf{A})^{T}$ (which is true for any matrix [3]) and the row space of $(\mathbf{P}\,\mathbf{A})^{T}$ is equal to the column space of $(\mathbf{P}\,\mathbf{A})$ (which is also true for any matrix [7]), we can post multiply both sides of the above equation by $(\mathbf{P}\,\mathbf{A})$ without effecting the space of $(\mathbf{P}\,\mathbf{A})^{+}$ to get

$$(\mathbf{P}\,\mathbf{A})^{+}\,\mathbf{P}\,(\mathbf{P}\,\mathbf{A}) \neq (\mathbf{P}\,\mathbf{A})^{+}\,(\mathbf{P}\,\mathbf{A})$$

By the associativity law for matrices $\mathbf{P}\,(\mathbf{P}\,\mathbf{A}) \equiv (\mathbf{P}\,\mathbf{P})\,\mathbf{A}$ and by definition of a projection matrix $(\mathbf{P}\,\mathbf{P}) \equiv \mathbf{P}$ results in

$$(\mathbf{P}\,\mathbf{A})^{+}\,(\mathbf{P}\,\mathbf{A}) \neq (\mathbf{P}\,\mathbf{A})^{+}\,(\mathbf{P}\,\mathbf{A})$$

which is a contradiction.

Q.E.D.

It is important to realize that there is no condition on the rank of the linear transformation matrix $\mathbf{A}$ in the above theorem. In other words, Theorem I is true even when $\mathbf{A}$ is singular. Another interesting point to make is that the projection matrix $(\mathbf{P}\,\mathbf{A})^{+}\,(\mathbf{P}\,\mathbf{A}) = (\mathbf{P}\,\mathbf{A})^{+}\,\mathbf{A}$ using Theorem I.

Theorem II : If $\Re(\mathbf{P}\,\mathbf{A}) \equiv \Re(\mathbf{P})$, then $(\mathbf{P}\,\mathbf{A})\,(\mathbf{P}\,\mathbf{A})^{+} = \mathbf{P}$

Proof : By using one of the Moore-Penrose [6] definitions for the pseudoinverse of a matrix,

$$(\mathbf{P}\,\mathbf{A})\,(\mathbf{P}\,\mathbf{A})^{+}\,(\mathbf{P}\,\mathbf{A}) = (\mathbf{P}\,\mathbf{A})$$

From the definition of a projection matrix substitute $(\mathbf{P}\,\mathbf{A}) = \mathbf{P}\,(\mathbf{P}\,\mathbf{A})$ for the right side of the equation results in the following

$$[\,(\mathbf{P}\,\mathbf{A})\,(\mathbf{P}\,\mathbf{A})^{+} - \mathbf{P}\,]\,(\mathbf{P}\,\mathbf{A}) = \mathbf{0}$$

$(\mathbf{P}\,\mathbf{A})\,(\mathbf{P}\,\mathbf{A})^{+}$ is a projection matrix and by definition its column space is equal to $\Re(\mathbf{P\,A})$; from the condition given in the theorem this column space must be equal to $\Re(\mathbf{P})$. Hence, all of the matrices in the above equation span the same column vector space, which means the only way for the above equation to hold true is when

$$(\mathbf{P\,A})(\mathbf{P\,A})^{+} - \mathbf{P} = \mathbf{0}$$

The above equation may be rewritten to obtain the desired result of

$$(\mathbf{P\,A})(\mathbf{P\,A})^{+} = \mathbf{P}$$

Q.E.D.

Theorem III : If $\Re(\mathbf{A}) \equiv \Re(\mathbf{P})$, then $\mathbf{A}(\mathbf{P\,A})^{+} = \mathbf{P}$

Proof : Given that the column space of **A** equals the column space of **P**, then $\Re(\mathbf{P\,A}) \equiv \Re(\mathbf{P})$. Using Theorem II,

$$(\mathbf{P\,A})(\mathbf{P\,A})^{+} = \mathbf{P}$$

From the definition of a projection matrix, substitute $\mathbf{P} = \mathbf{P\,P}$ for the term on the right side of the above equation to obtain the following

$$\mathbf{P}\,[\,\mathbf{A}(\mathbf{P\,A})^{+} - \mathbf{P}\,] = \mathbf{0}$$

All three matrix terms in the above equation span the same column vector space, which means the only way for the above equation to hold true is when

$$\mathbf{A}(\mathbf{P\,A})^{+} - \mathbf{P} = \mathbf{0}$$

Or, this may be rewritten to obtain the desired result of

$$\mathbf{A}(\mathbf{P\,A})^{+} = \mathbf{P}$$

Q.E.D.

To conclude this section on projection matrices, one final result that we postulated, proved, and later found in [11] was $\mathbf{P}^{+} = \mathbf{P}$.

B The An and Hollerbach Example

We have included for reference the equations from Section 2.1 of the An and Hollerbach [1] paper regarding the stability analysis of hybrid control. It was stated that the closed-loop system described as

$$\delta\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_p\mathbf{J}^{-1}\mathbf{S}\,\mathbf{J} & -\mathbf{M}^{-1}\mathbf{K}_v\mathbf{J}^{-1}\mathbf{S}\,\mathbf{J} \end{bmatrix} \delta\mathbf{x} \tag{57}$$

must have negative real parts for the eigenvalues of the matrix to guarantee local stability at the equilibrium points.

For the two-link planar manipulator example used in the analysis,

$$\mathbf{J} = \begin{bmatrix} -l_1\, s_1 - l_2\, s_{12} & -l_2\, s_{12} \\ l_1\, c_1 + l_2\, c_{12} & l_2\, c_{12} \end{bmatrix} \tag{58}$$

where $s_i = \sin(\theta_i)$, $c_i = \cos(\theta_i)$, $s_{12} = \sin(\theta_1 + \theta_2)$, and $c_{12} = \cos(\theta_1 + \theta_2)$. The link lengths were $l_1 = 0.462\, m$ and $l_2 = 0.4445\, m$.

$$\mathbf{M} = \begin{bmatrix} \mathrm{m}_{11} & \mathrm{m}_{12} \\ \mathrm{m}_{21} & \mathrm{m}_{22} \end{bmatrix} \tag{59}$$

with,

$$\begin{aligned} \mathrm{m}_{11} &= I_1 + I_2 + m_2 l_1 l_2 c_2 + \tfrac{1}{4}(m_1 l_1^2 + m_2 l_2^2) + m_2 l_1^2 \\ \mathrm{m}_{22} &= I_2 + \tfrac{1}{4} m_2 l_2^2 \\ \mathrm{m}_{12} = \mathrm{m}_{21} &= I_2 + \tfrac{1}{4} m_2 l_2^2 + \tfrac{1}{2} m_2 l_1 l_2 c_2 \end{aligned} \tag{60}$$

The inertia values used were $I_1 = 8.095\, kg \cdot m^2$ and $I_2 = 0.253\, kg \cdot m^2$. The mass values were $m_1 = 120.1\, kg$ and $m_2 = 2.104\, kg$. The gain matrices were chosen as $\mathbf{K}_p = \mathrm{diag}[2500, 400]$ and $\mathbf{K}_v = \mathrm{diag}[300, 30]$.

In Case 2 of their paper, $\mathbf{S} = \mathrm{diag}[0, 1]$ and a root locus plot of Equation 57 was shown with $\theta_1 = 0^o$ and θ_2 varying from 90^o to 70^o. We used all of the information presented here to program in Mathematica™ [12] the above equations and definitions to verify their instability result. Our implementation produced a similar root locus plot as in Figure 4 of their paper, with our version shown in Figure 6.

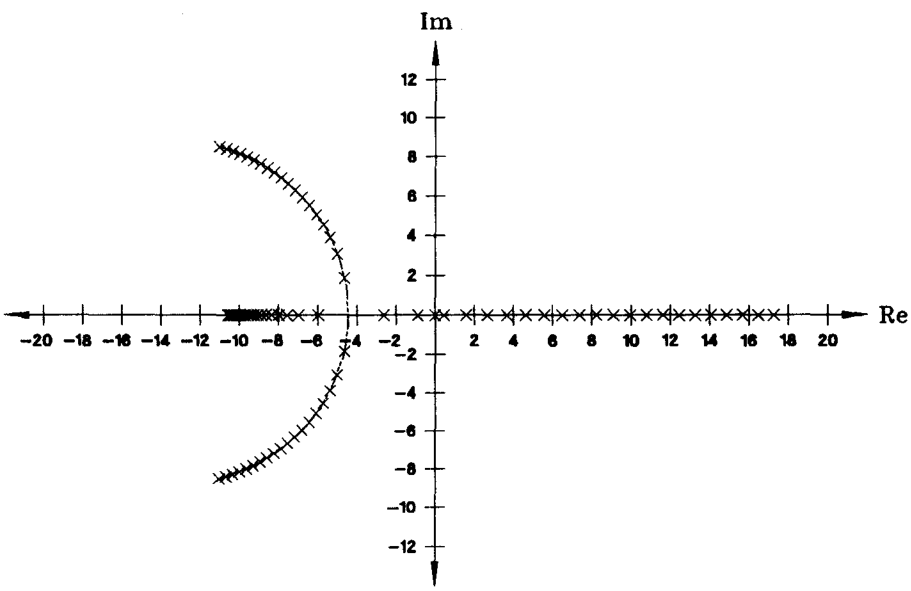

Figure 6: Unstable Root Locus Plot using $\theta_{e_s} = \mathbf{J}^{-1}\, \mathbf{S}\, \mathbf{x}_e$