The Relationship between a MMPP and a MMRP

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HPLB–NSMG–91–2
0.9
November 20, 1992

We show that the moment generating function of a Markov modulated rate process (MMRP) is related to the moment generating function of a Markov modulated Poisson process (MMPP) through a logarithmic transformation of the argument. Some implications, extensions and limitations of this relationship are presented.

1. Introduction

Markov modulated Poisson processes (MMPP) is a class of doubly stochastic point processes that is tractable and flexible enough to model fairly complex stochastic systems. Markov modulated Poisson processes have been widely used to model different aspects of communication systems [3, 6].

Markov modulated rate processes (MMRP) were analyzed by Anick *et al.*, [1]. Their analysis was based on solving a matrix eigenvalue problem which grows in complexity with the state space. In order to circumvent this problem, Stern and Elwalid [13] considered the case where the modulator is time reversible. Markov modulated rate processes have been used also to model video sources [9, 12].

In [6], Heffes and Lucantoni estimated the parameters of a Markov modulated Poisson process that models the traffic generated by the superposition of voice sources by moment matching. As far as we know, the problem of parameter estimation for Markov modulated rate processes has not been addressed in the literature.

In this paper we show that the moment generating function of a Markov modulated rate process is obtained from the moment generating function of a Markov modulated Poisson process by a logarithmic transformation of the argument. The implication of this relationship is that results obtained for Markov modulated Poisson processes have a counterpart in Markov modulated rate processes. Examples of this fact appear in the algorithm presented by Ye and Li in [14] for the analysis of queues arising in multimedia environments which is valid both for MMPP and MMRP models and in the work of Elwalid, Mitra and Stern on this topic, [5, 4].

We also generalize the relationship between the moment generating functions of counts to second order moment generating functions and extend the moment matching technique developed in [6] for Markov modulated rate processes. Finally, we show that despite the simple relationship between these two classes of processes, it is not possible to obtain the first order statistics of the buffer contents of a MMRP/D/1/∞ queue with any queue of the MMPP/GI/1/∞ family.

2. Notation

A realization of a simple point process can be described by an increasing sequence of positive random variables $(T_n)_{n\in\mathbf{N}}$, where T_n corresponds to the time of the n-th event or point of the process, see [2] for details. To each point process one can associate the counting process,

$$N_t = \sum_{n\in\mathbf{Z}} \mathbf{1}(0 < T_n \leq t),$$

which counts the number of events in the time interval $]0, t]$.

In this paper we will consider point processes modulated by a positive recurrent Markov chain X_t in equilibrium with state space $\{1, \ldots, M\}$. The equilibrium probability of X_t is denoted by $\boldsymbol{\pi} = (\pi_i)_{1 \le i \le M}$ and its intensity matrix by $\boldsymbol{Q} = (q_{ij})_{1 \le i,j \le M}$. We let,

$$q_i = -q_{ii} = \sum_{j \neq i} q_{ij}.$$

We will assume that point processes generate events at rates $r(1), \ldots, r(M)$ with, $r(1) < \ldots < r(M)$. In a communications context, these events will usually correspond to packets, bytes or bits.

In a Markov modulated rate process (MMRP), events are generated *deterministically* at a rate of $r(i)$ events per second when $X_t = i$, *i.e.*, the event rate at time t is $r(X_t)$.

In a Markov modulated Poisson process (MMPP), events are generated according to a Poisson process of rate of $r(i)$ events per second when $X_t = i$, *i.e.*, the $\mathfrak{F}_t^X$–stochastic intensity of the Markov modulated Poisson process is equal to $r(X_t)$, [2].

In the sequel, $\boldsymbol{e}$ will denote the all-ones column vector of the appropriate dimension.

3. The Moment Generating Function of the Counting Process

Consider a Markov modulated rate process with intensity matrix modulator $\boldsymbol{Q}$ and rates $r(m)$, $1 \le m \le M$. Denote by N_t the number of events in the time interval $]0, t]$ and let $0 \le t \le t'$. The analysis of Markov modulated rate processes is based on a fluid flow model [1, 13] which approximates the number of events during the time interval $]t, t']$, by the continuous random variable, $\int_t^{t'} r(X_s)ds$, *i.e.*,

$$N_{t'} - N_t \simeq \int_t^{t'} r(X_s)ds. \tag{3.1}$$

Using the approximation of equation (3.1), we derive the moment generating function of N_t. If we take in lemma A.1, $v(j) = -\log(z)r(j)$, $0 < z \le 1$,

$$z^{N_t} = \exp\{-\int_0^t v(X_s)ds\}, \tag{3.2}$$

and hence,

$$\Phi_{\mathrm{MMRP}}(z,t) = \mathsf{E}[z^{N_t}] = \boldsymbol{\pi} \exp\Big\{\Big(\boldsymbol{Q} + \log(z)\boldsymbol{R}\Big)t\Big\}\boldsymbol{e}, \tag{3.3}$$

where $\boldsymbol{R} = \mathrm{diag}\{r(1), \ldots, r(M)\}$.

Consider now a Markov modulated Poisson process with a modulator with intensity matrix $\boldsymbol{Q}$ and *Poisson* rates $r(m)$, $1 \le m \le M$. The moment generating function of counts $\Phi_{\text{MMPP}}(\cdot)$ of a Markov modulated Poisson process is given in [6], equation A.1,

$$\Phi_{\text{MMPP}}(z,t) = \boldsymbol{\pi} \exp\Big\{\Big(\boldsymbol{Q} + (z-1)\boldsymbol{R}\Big)t\Big\}\boldsymbol{e}. \tag{3.4}$$

Therefore, combining equations (3.3) and (3.4),

$$\Phi_{\text{MMRP}}\Big(z,t\Big) = \Phi_{\text{MMPP}}\Big(1+\log(z),t\Big). \tag{3.5}$$

4. Second Order Moment Generating Functions

Let t_1, $t_2 > 0$ and $t \geq 0$. In [10] it is shown that the second order generating function of a Markov modulated Poisson process is given by,

$$\begin{aligned}\Phi_{\text{MMPP}}(z_1,z_2;t_1,t_2;t) &= \mathsf{E}\Big[z_1^{N_{t_1}} z_2^{N_{t+t_1+t_2}-N_{t+t_1}}\Big] \\ &= \boldsymbol{\pi} \exp\Big\{\Big(\boldsymbol{Q} + (z_1-1)\boldsymbol{R}\Big)t_1\Big\} \exp\{\boldsymbol{Q}t\} \exp\Big\{\Big(\boldsymbol{Q} + (z_2-1)\boldsymbol{R}\Big)t_2\Big\}\boldsymbol{e}.\end{aligned} \tag{4.1}$$

The approach in [10] to prove equation (4.1) can also be applied to Markov modulated rate processes by using equation (3.1), lemma A.1 and conditioning at times t_1 and $t+t_1$. We obtain,

$$\begin{aligned}\Phi_{\text{MMRP}}\Big(z_1,z_2;t_1,t_2;t\Big) &= \mathsf{E}\Big[z_1^{N_{t_1}} z_2^{N_{t+t_1+t_2}-N_{t+t_1}}\Big] \\ &= \boldsymbol{\pi} \exp\Big\{\Big(\boldsymbol{Q} + \log(z_1)\boldsymbol{R}\Big)t_1\Big\} \exp\{\boldsymbol{Q}t\} \exp\Big\{\Big(\boldsymbol{Q} + \log(z_2)\boldsymbol{R}\Big)t_2\Big\}\boldsymbol{e}.\end{aligned}$$

Thus, we have,

$$\Phi_{\text{MMRP}}\Big(z_1,z_2;t_1,t_2;t\Big) = \Phi_{\text{MMPP}}\Big(1+\log(z_1),1+\log(z_2);t_1,t_2;t\Big) \tag{4.2}$$

Differentiating with respect to z_1 and z_2, letting $z_1 = z_2 = 1$, and using equation (36) in [10], we obtain the covariance function of a Markov modulated rate process,

$$\begin{aligned}\text{Cov}_{MMRP}(t_1,t_2;t) &= \mathsf{E}\Big[N_{t_1}\Big(N_{t+t_1+t_2} - N_{t+t_1}\Big)\Big] \\ &= \boldsymbol{\pi}\boldsymbol{R}\Big(\boldsymbol{I} - \exp\{\boldsymbol{Q}t_1\}\Big) \exp\{\boldsymbol{Q}t\}\Big(\boldsymbol{I} - \exp\{\boldsymbol{Q}t_2\}\Big)\Big(\tau\boldsymbol{e}\boldsymbol{\pi} - \boldsymbol{Q}\Big)^{-2}\boldsymbol{R}\boldsymbol{e},\end{aligned} \tag{4.3}$$

where $\tau \geq \max_i\{-q_i\}$.

Equation (4.3) does not state that a Markov modulated Poisson process and a Markov modulated rate process have the same covariance function. As the next section shows, in general, a Markov modulated Poisson process and a Markov modulated rate process with the same parameter values model different processes.

5. Parameter Estimation by Moment Matching

A moment matching methodology was presented in [6] to profile a packet stream modeled according to a two dimensional Markov modulated Poisson process.

The simple relationship derived in equation (3.5) allows us to use the methodology in [6] to estimate the parameters of Markov modulated rate processes. The moments to match are, the packet rate $M(t) = \mathsf{E}[N_t]$, the index of dispersion of counts,

$$V(t) = \frac{\mathsf{Var}(N_t)}{\mathsf{E}[N_t]},$$

and the third moment of the packet count in $]0, t]$,

$$U(t) = \mathsf{E}\Big[\big(N_t - M(t)\big)^3\Big].$$

In the sequel, $\Phi'(1,t)$, $\Phi''(1,t)$ and $\Phi'''(1,t)$ will denote the first three derivatives evaluated at $z = 1$ of the moment generating function $\Phi(z,t)$ of packet counts up to time t.

We have,

$$\begin{aligned} M(t) &= \Phi'(1,t), && (5.1)\\ V(t) &= 1 - \Phi'(1,t) + \frac{\Phi''(1,t)}{\Phi'(1,t)}, && (5.2)\\ U(t) &= \Phi'''(1,t) - 3M(t)\big(M(t)-1\big)V(t) - M(t)\big(M(t)-1\big)\big(M(t)-2\big). && (5.3) \end{aligned}$$

In the sequel, quantities referring to Markov modulated Poisson processes (respt. Markov modulated rate processes) will have the subscript MMPP (respt. MMRP). Taking derivatives in equation (3.5),

$$\begin{aligned} \Phi'_{\text{MMRP}}(z,t) &= \frac{\Phi'_{\text{MMPP}}(1+\log(z),t)}{z},\\ \Phi''_{\text{MMRP}}(z,t) &= \frac{\Phi''_{\text{MMPP}}(1+\log(z),t) - \Phi'_{\text{MMPP}}(1+\log(z),t)}{z^2},\\ \Phi'''_{\text{MMRP}}(z,t) &= \frac{\Phi'''_{\text{MMPP}}(1+\log(z),t) - 3\Phi''_{\text{MMPP}}(1+\log(z),t) + 2\Phi'_{\text{MMPP}}(1+\log(z),t)}{z^3}. \end{aligned}$$

and letting $z = 1$,

$$\Phi'_{\text{MMRP}}(1,t) = \Phi'_{\text{MMPP}}(1,t), \qquad (5.4)$$

$$\Phi''_{\mathrm{MMRP}}(1,t) = \Phi''_{\mathrm{MMPP}}(1,t) - \Phi'_{\mathrm{MMPP}}(1,t), \tag{5.5}$$
$$\Phi'''_{\mathrm{MMRP}}(1,t) = \Phi'''_{\mathrm{MMPP}}(1,t) - 3\Phi''_{\mathrm{MMPP}}(1,t) + 2\Phi'_{\mathrm{MMPP}}(1,t). \tag{5.6}$$

Therefore, combining equations (5.1), (5.2) and (5.3) with equations (5.4), (5.5) and (5.6),

$$M_{\mathrm{MMRP}}(t) = M_{\mathrm{MMPP}}(t), \tag{5.7}$$
$$V_{\mathrm{MMRP}}(t) = V_{\mathrm{MMPP}}(t) - 1, \tag{5.8}$$
$$U_{\mathrm{MMRP}}(t) = U_{\mathrm{MMPP}}(t) + 2M_{\mathrm{MMPP}}(t)\Big(1 - \frac{3}{2}V_{\mathrm{MMPP}}(t)\Big). \tag{5.9}$$

Equations (5.7), (5.8) and (5.9) allow the extension of the moment matching methodology in [6] to Markov modulated rate processes.

6. Queueing Behavior

The similarity between a Markov modulated rate process and a Markov modulated Poisson process leads us to consider how similar are queues with these processes as input streams.

Consider a MMRP/D/1/∞ queue with service rate μ and rates $0 < r(1) < \ldots < r(M)$ with $r(j) \neq \mu$ for all $j = 1, \ldots, M$. Let $W(t)$ denote its buffer content at time t, X_t be its (Markov) modulating process (with intensity matrix $\boldsymbol{Q}$ and equilibrium distribution $\boldsymbol{\pi}$) and $\boldsymbol{R} = \mathrm{diag}\{r(1), \ldots, r(M)\}$. To avoid trivialities, assume that $\boldsymbol{\pi R e} < \mu$ and $r(1) < \mu < r(M)$. We have the sample path equality,

$$\begin{aligned}\exp\{-uW(t)\}\mathbf{1}(X_t = j) = {} & \exp\{-uW(0)\}\mathbf{1}(X_0 = j) \\ & + \int_{]0,t]} \exp\{-uW(s)\}\mathbf{1}(X_s = j)J(ds) \\ & - \int_{]0,t]} \exp\{-uW(s^-)\}\mathbf{1}(X_{s^-} = j)J(ds) \\ & -u \int_0^t \exp\{-uW(s)\}\mathbf{1}(X_s = j)\mathbf{1}(W(s) > 0)\big(r(j) - \mu\big)ds,\end{aligned}$$

where $J(\cdot)$ is the point process of modulator jumps. Assuming equilibrium and taking expectations,

$$\begin{aligned}&\mathsf{E}\Big[\int_{]0,t]} \exp\{-uW(s^-)\} \sum_{i \neq j} \mathbf{1}(X_{s^-} = i)\frac{q_{ij}}{q_i} - \exp\{-uW(s^-)\}\mathbf{1}(X_{s^-} = j)J(ds)\Big] = \\ &\qquad = u\big(r(j) - \mu\big)\mathsf{E}\Big[\int_0^t \exp\{-uW(s)\}\mathbf{1}(X_s = j)\big(1 - \mathbf{1}(W(s) = 0)\big)ds\Big],\end{aligned}$$

and using standard martingale calculus,

$$\boldsymbol{W}_{\text{MMRP}}(u)\Big(\boldsymbol{Q}_{\text{MMRP}} + u(\mu\boldsymbol{I} - \boldsymbol{R}_{\text{MMRP}})\Big) = u\boldsymbol{y}(\mu\boldsymbol{I} - \boldsymbol{R}_{\text{MMRP}}), \tag{6.1}$$

where the row vector $\boldsymbol{W}_{\text{MMRP}}(u)$ has j-th component $\mathsf{E}[\exp\{-uW(0)\}\mathbf{1}(X_0 = j)]$ and the row vector $\boldsymbol{y}$ has j-th component $\mathsf{E}[\mathbf{1}(W(0) = 0)\mathbf{1}(X_0 = j)]$.

The queue length moment generating function of the MMPP/GI/1/∞ queue with service distribution $H(\cdot)$, modulator intensity matrix $\boldsymbol{Q}_{\text{MMPP}}$, and rate matrix $\boldsymbol{R}_{\text{MMPP}}$, is given by, [6, 8],

$$\boldsymbol{Y}_{\text{MMPP}}(z)\Big(z\boldsymbol{I} - \boldsymbol{A}(z)\Big) = (z-1)\boldsymbol{y}\boldsymbol{A}(z), \tag{6.2}$$

where

$$\boldsymbol{A}(z) = \int_{\mathbf{R}_+} \exp\Big\{\Big(\boldsymbol{Q}_{MMPP} + (z-1)\boldsymbol{R}_{MMPP}\Big)t\Big\}H(dt). \tag{6.3}$$

We recall that the matrix $\boldsymbol{A}(1)$ is non-negative and stochastic with left Perron-Frobenius eigenvector $\boldsymbol{\pi}_{\text{MMPP}}$, where $\boldsymbol{\pi}_{\text{MMPP}}\boldsymbol{Q}_{\text{MMPP}} = \mathbf{0}$.

Equations (6.1) and (6.2) look quite similar. We shall attempt the same approach as in sections 3 and 4, *i.e.*, find a mapping $u(z)$ from $[0,1]$ into $\mathbf{R}_+$ so that $\boldsymbol{W}_{\text{MMRP}}\big(u(z)\big) = \boldsymbol{Y}_{\text{MMPP}}(z)$. Equating equations (6.1) and (6.2) yields, after some algebra,

$$\boldsymbol{A}(z) = \frac{zu(z)}{z}\left[\frac{zu(z)}{z}\boldsymbol{I} - \boldsymbol{Q}_{\text{MMRP}}(\mu\boldsymbol{I} - \boldsymbol{R}_{\text{MMRP}})^{-1}\right]^{-1}.$$

Since $\boldsymbol{A}(1)$ is stochastic, it follows that $a = \lim_{z\uparrow 1} zu(z)/z$ exists and is positive. Then,

$$\boldsymbol{A}(1) = \left[\boldsymbol{I} - \frac{1}{a}\boldsymbol{Q}_{\text{MMRP}}(\mu\boldsymbol{I} - \boldsymbol{R}_{\text{MMRP}})^{-1}\right]^{-1},$$

and clearly, $\boldsymbol{\pi}_{\text{MMRP}}\boldsymbol{A}(1) = \boldsymbol{\pi}_{\text{MMRP}}$ so that $\boldsymbol{A}(1)$ admits the unity as an eigenvalue. However, the associated right eigenvector is $a(\mu\boldsymbol{I} - \boldsymbol{R}_{\text{MMRP}})\boldsymbol{e}$ which, because of the condition $r(1) < \mu < r(M)$, is not positive. This constradicts the fact that the matrix $\boldsymbol{A}(1)$ is stochastic.

Thus, despite the simple relationship between a Markov modulated rate process and a Markov modulated Poisson process, one cannot mimick the buffer content characteristics of the MMRP/D/1/∞ queue by the MMPP/GI/1/∞ queue.

7. Conclusion

Assume that a moment matching methodology is used to estimate the model parameters. Equations (3.5) and (4.2) show that one should expect similar modeling behaviors from a MMPP model and from a MMRP model. Thus, one could switch either way from one class of models to the other as need dictates.

For instance, if the model provided by a MMPP doesn't capture some relevant aspect of the stochastic system being modeled, there is little hope that the MMRP will do any better.

On the other hand, one may be in a situation where, for instance, a MMRP provides an acceptable model of a stochastic system but has computational algorithms that are more efficient or stable for MMPP models. Then, the results of this paper show that, in general, one can switch to a MMPP model to study the problem under consideration and gain computational efficiency and stability.

Finally, even if MMRP processes and MMPP processes exhibit similar behaviors, we have seen that the buffer content statistics of the MMRP/D/1/∞ queue cannot be reproduced by the MMPP/GI/1/∞ queue.

A. Appendix: Markov Chain Martingales

Let $X(t)$ be a Markov chain on $\{1,\ldots,M\}$ with intensity matrix $\boldsymbol{Q} = (q_{ij})_{1\leq i,j\leq M}$ such that

$$\infty > q_i = -q_{ii} \geq \sum_{j\neq i} q_{ij},$$

and let $v(\cdot)$ be a mapping from $\{1,\ldots,M\}$ into $[0,\infty[$. We have, [2, 7, 11]

Lemma A.1 *Let* $\tilde{\boldsymbol{P}}(t) = (\tilde{p}_{mn}(t))_{1\leq m,n\leq M}$, *where,*

$$\tilde{p}_{mn}(t) = \mathsf{E}\Big[\exp\Big\{-\int_0^t v(X_s)ds\Big\}\mathbf{1}\Big(X(t)=n\Big)\Big|X(0)=m\Big],\quad 1\leq m,n\leq M,$$

Then the matrix $\tilde{\boldsymbol{P}}(t)$ *is given by,*

$$\tilde{\boldsymbol{P}}(t) = \exp\{(\boldsymbol{Q}-\boldsymbol{V})t\},$$

where $\boldsymbol{V} = \mathrm{diag}\{v(1),\ldots,v(M)\}$.

Proof: (Sketch)

For $1\leq i,j\leq M$, let,

$$Y_{ij}(t) = \mathbf{1}(X_0=i)\mathbf{1}(X_t=j)\mathrm{e}^{-\int_0^t v(X_s)ds},$$

$Z_{ij}(t) = \mathsf{E}[Y_{ij}(t)]$ and $\boldsymbol{Z}_i(t) = \Big(Z_{ij}(t)\Big)_{1\leq j\leq M}$.

Let A be the point process of jumps of X. Taking expectations in the sample path equality,

$$Y_{ij}(t) = Y_{ij}(0) + \int_{]0,t]} \Big(Y_{ij}(s) - Y_{ij}(s^-)\Big) A(ds) - v(j) \int_0^t Y_{ij}(s)ds,$$

yields,

$$\begin{aligned} Z_{ij}(t) &= Z_{ij}(0) + \mathsf{E}[\mathbf{1}(X_0 = i) \int_{]0,t]} \mathrm{e}^{-\int_0^{s^-} v(X_x)dx} \Big(\sum_{k \neq j} \mathbf{1}(X_{s^-} = k)\mathbf{1}(X_s = j) - \\ &\qquad -\mathbf{1}(X_{s^-} = j)\Big) A(ds)] - v(j) \int_0^t Z_{ij}(s)ds \\ &= Z_{ij}(0) + \mathsf{E}[\mathbf{1}(X_0 = i) \int_0^t \mathrm{e}^{-\int_0^s v(X_x)dx} \Big(\sum_{k \neq j} \mathbf{1}(X_s = k)q_{kj} - \mathbf{1}(X_s = j)q_j\Big) ds] \\ &\qquad -v(j) \int_0^t Z_{ij}(s)ds \\ &= Z_{ij}(0) + \int_0^t \sum_k Z_{ik}(s)q_{kj}ds - v(j) \int_0^t Z_{ij}(s)ds. \end{aligned}$$

In vector form,

$$\boldsymbol{Z}_i(t) = \boldsymbol{Z}_i(0) + \int_0^t \boldsymbol{Z}_i(s)\Big(\boldsymbol{Q} - \boldsymbol{V}\Big)ds,$$

which implies,

$$\boldsymbol{Z}_i(t) = \boldsymbol{Z}_i(0) \exp\{(\boldsymbol{Q} - \boldsymbol{V})t\}.$$

In particular,

$$Z_{ij}(t) = \pi_i \exp\{(\boldsymbol{Q} - \boldsymbol{V})t\}\Big|_{ij},$$

and hence,

$$\mathsf{E}[\mathrm{e}^{-\int_0^t v(X_s)ds}\mathbf{1}(X_t = j)|X_0 = i] = \exp\{(\boldsymbol{Q} - \boldsymbol{V})t\}\Big|_{ij},$$

which proves the result.

■

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