1 Introduction

The description of concurrent and distributed systems in terms of partial orders is more expressive than the description based on the interleaving semantics. However it has the disadvantage that there is not an obvious algebra of partial orders. The problem is to give a nontrivial description of the operation of sequential composition. Without such an operation the description cannot be properly incremental (see [FMM 90]).

In [FMM 90], the algebra of *Concatenable Concurrent Histories* was introduced, which is an algebra of partial orders with extra information about the maximal and minimal elements which allows a nontrivial operation of sequential composition. This algebra depends on two alphabets, one of which is the alphabet of labels of elements which are maximal or minimal, and one of which is the alphabet of the labels of the other elements. In [FMM 90] a truly concurrent semantics is given to the process description language CCS using the algebra CCH; CCH is the algebra of Concatenable Concurrent Histories with the first alphabet being a singleton and the second being the set Λ of CCS actions. In this paper we will give an axiomatization for CCH, using category theory.

Our approach follows the method given in [DMM 89], where a non-commutative tensor operator is given together with special elements called symmetries. We extend this approach, introducing a biproduct structure which gives us the expressive power of the set of all bipartite histories. Bipartite histories describe the causal links between the visible actions, and take the place of the algebra of symmetries in [DMM 89].

The bulk of this paper is taken up with a proof that the axiomatization does indeed give the algebra CCH. For the reader who wishes to avoid the gory details (and they do get very gory) there is a summary, section 9, where the main results are stated. An application of the algebra CCH to Petri Nets is given in the last section.

2 Definitions: CCH and some categories

2.1 Definition - An element of CCH

Given a fixed set of labels Λ, and a label s not in Λ, an element of CCH is a triple (h, β, γ) where

- h is a labelled partial order $(V_1 \cup V_2 \cup V_3, \leq, \ell)$
- The elements of V_1 are minimal in the partial order, the elements of V_3 are maximal in the partial order and distinct from the elements of V_1, and the elements of V_2 are neither maximal nor minimal in the partial order
- $\ell(v) = s$ for $v \in V_1 \cup V_3$; $\ell(v) \in \Lambda$ for $v \in V_2$
- β is a bijection from V_1 to the set $\{1, 2, \ldots, |V_1|\}$
- γ is a bijection from V_3 to the set $\{1, 2, \ldots, |V_3|\}$

An element (h, β, γ) is defined up to isomorphisms of labelled partial orders preserving β, γ, and the selection functions for V_1, V_3.

2.2 Example

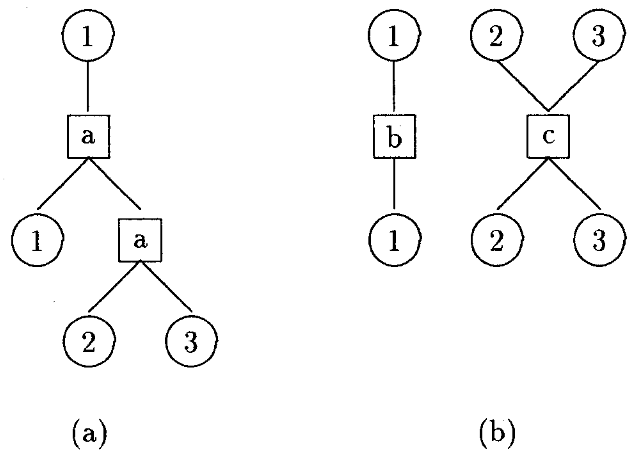


Figure 1: Two elements of CCH

The introduction of the functions β, γ allows us to discriminate between different elements of V_1, V_3 with the same label. Figure 1 illustrates two elements of CCH; the order relation is depicted through its Hasse diagram growing downwards. Elements of $V_1 \cup V_3$ are represented as circles and elements of V_2 as boxes. The functions β, γ are shown by the integers in the circles.

2.3 Parallel and sequential composition

The set CCH forms an algebra under the operations of parallel and sequential composition.

Let $cch_1 = (h_1, \beta_1, \gamma_1)$ and $cch_2 = (h_2, \beta_2, \gamma_2)$ be elements of CCH, where $h_1 = (V1_1 \cup V1_2 \cup V1_3, \leq_1, \ell_1)$ and $h_2 = (V2_1 \cup V2_2 \cup V2_3, \leq_2, \ell_2)$. Without loss of generality, $V1_1 \cup V1_2 \cup V1_3$ and $V2_1 \cup V2_2 \cup V2_3$ are disjoint.

The parallel composition $ch_1 \otimes ch_2$ is $((V_1 \cup V_2 \cup V_3, \leq, \ell), \beta, \gamma)$ where

- $V_1 = V1_1 \cup V2_1$
- $V_2 = V1_2 \cup V2_2$
- $V_3 = V1_3 \cup V2_3$
- $\leq \, = \, \leq_1 \cup \leq_2$
- $\ell(v) = s$ if $v \in V_1 \cup V_3$,
  $\ell_1(v)$ if $v \in V1_2$,
  $\ell_2(v)$ if $v \in V2_2$

- $\beta(v) = \beta_1(v)$ if $v \in V1_1$,
  $|V1_1| + \beta_2(v)$ if $v \in V2_1$
- $\gamma(v) = \gamma_1(v)$ if $v \in V1_3$,
  $|V1_3| + \gamma_2(v)$ if $v \in V2_3$

The sequential composition $ch_1; ch_2$ is defined if and only if $|V1_3| = |V2_1|$.

In this case the result of the operation is $((V_1 \cup V_2 \cup V_3, \leq, \ell), \beta, \gamma)$ where

- $V_1 = V1_1$
- $V_2 = V1_2 \cup V2_2$
- $V_3 = V2_3$
- $\leq$ is the restriction to $V_1 \cup V_2 \cup V_3$ of the transitive closure of
  $\leq_1 \cup \leq_2 \cup \{(v_1, v_2) : v_1 \in V1_3, v_2 \in V2_1, \gamma_1(v_1) = \beta_2(v_2)\}$
- $\ell(v) = s$ if $v \in V_1 \cup V_3$,
  $\ell_1(v)$ if $v \in V1_2$,
  $\ell_2(v)$ if $v \in V2_2$
- $\beta(v) = \beta_1(v)$
- $\gamma(v) = \gamma_2(v)$

2.4 Definition: Symmetric strict monoidal category

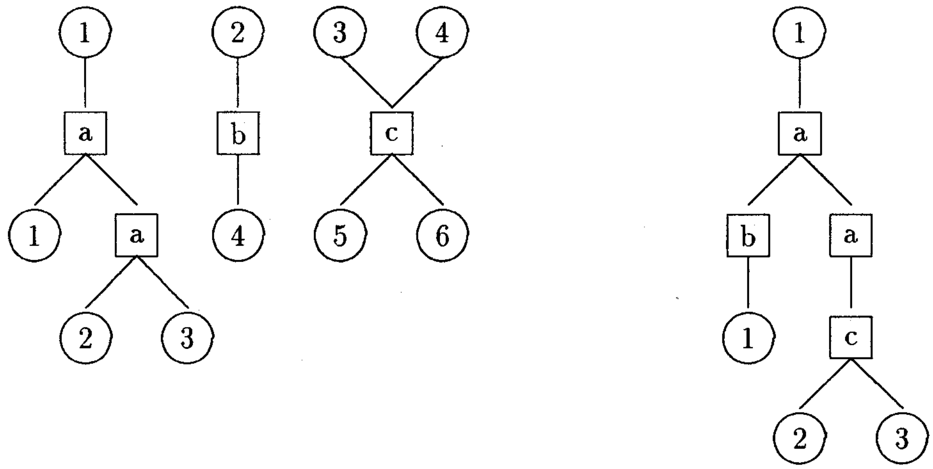
A *symmetric strict monoidal category* is a strict monoidal category (see [ML 71]) which contains a symmetry morphism $\gamma_{u,v} : u \otimes v \to v \otimes u$ for each pair of states (u, v), such that

- $\gamma_{u,v}; \gamma_{v,u} \;=\; id(u \otimes v)$
- $(\gamma_{u,v} \otimes id(w)); (id(v) \otimes \gamma_{u,w}) \;=\; \gamma_{u,v \otimes w}$
- If $a : u \to v$, $b : w \to x$, then $a \otimes b \;=\; \gamma_{u,w}; (b \otimes a); \gamma_{x,v}$

This last condition will be referred to in the rest of this paper as the coherence axiom.

3 The algebra of bipartite histories

This section is concerned with the algebra of bipartite histories, which can be considered either as an algebra of matrices with entries T,F or as the morphisms of a biproduct category. The elements of this algebra will play the role that symmetries play in [DMM 89].



(a) (b)

Figure 2: Parallel (a) and sequential composition (b) of the elements of CCH in Figure 1(a) and 1(b)

3.1 Notation for matrices

Consider the algebra of matrices with non-negative dimensions over the boolean algebra with two elements T, F. If f, g are matrices with dimensions $k \times m$, $m \times n$ respectively then write $f; g$ for the matrix obtained by matrix multiplication of f and g. If h_1, h_2 have dimensions $m1 \times n1$, $m2 \times n2$ respectively, then write $h_1 \otimes h_2$ for the $(m1 + m2) \times (n1 + n2)$ matrix satisfying $(h_1 \otimes h_2)_{i,j} =$ T if and only if $((h_1)_{i,j} =$ T or $(h_2)_{i-m1,j-m2} =$ T).

Some special matrices will be denoted as follows.

- Write $\vee$, $\wedge$ for the 2×1 and the 1×2 matrices respectively which have both entries T
- Write I_0 for the 0×0 matrix, I for the 1×1 matrix with entry T, and I_n for the tensor product of n copies of this
- Write O for the 1×1 matrix with entry F
- Write μ, ϵ for the 0×1-dimensional and the 1×0-dimensional matrices
- Write $X(i,r)$ for the $r \times r$ matrix such that $X(i,r)_{j,k} = T$ if $j = k \notin \{1, i\}$ or $j = 1, k = i$ or $j = i, k = 1$, and $X(i,r)_{j,k} = F$ otherwise

The matrix $X(i,r)$ is a "swap" matrix, which is obtained from the $r \times t$ identity matrix I_r by changing the (1,1) and (i,i) entries to F and the (1,i) and (i,1) entries to T.
If f, g are matrices with dimensions $k \times m$, $n \times m$ then write $> f, g <$ for the $(k+n) \times m$ matrix such that $> f, g <_{i,j}$ is $f_{i,j}$ if $i \leq k$ and $g_{i-k,j}$ otherwise. If f, g are matrices with dimensions $m \times k$, $m \times n$ then write $< f, g >$ for the $m \times (k+n)$ matrix such that $< f, g >_{i,j}$ is $f_{i,j}$ if $j \leq k$ and $g_{i,j-k}$ otherwise.

We will construct a biproduct category B whose morphisms are the matrices over $\{T, F\}$ whose dimensions are finite (and possibly zero). (See [ML 71] for the definition of a biproduct category; it is not essential to this paper.) The objects of B are in one-to-one correspondence with the set of non-negative integers; we will denote by $[n]$ the object of B corresponding to the integer n.

Let B be the category whose set of objects is $\{[n] : n \geq 0\}$ with a tensor product, $\otimes$, satisfying $[n_1] \otimes [n_2] = [n_1 + n_2]$, and whose morphisms from $[m]$ to $[n]$ $(m, n \geq 0)$ are just the $m \times n$ matrices over the boolean algebra with two elements. The composition of two morphisms f, g is the morphism $f; g$ and there are operations $\otimes$, $>, <$ and $<, >$ on the set of morphisms, as described in the last subsection.

It is straightforward to check that $\otimes$ is a biproduct for B, where the product and coproduct pairings of morphisms are $<, >$ and $>, <$.

3.2 Decomposition of morphisms of B

Each morphism A of B either has all entries F or is equal to exactly one morphism of the form $A1; A2; A3$ where

- $A1$ has exactly one entry T in each column, and if $A1_{i,j} = A1_{k,j+1} = \mathrm{T}$ then $k > i$
- $A2$ has at least one entry T in each row and in each column
- $A3$ has exactly one entry T in each row, and if $A1_{i,j} = A1_{i+1,k} = \mathrm{T}$ then $k > j$.

Proof

Let A be a morphism from [n] to [m] which does not have all entries F. Let the numbers of the rows of A which contain an entry T be $i_1, \ldots, i_r$ in ascending order and the numbers of the columns of A which contain an entry T be $j_1, \ldots, j_s$ in ascending order. Let $A1$ be the morphism from [n] to [r] such that $A1_{a,b}$ is T if and only if $a = i_b$. Let $A2$ be the morphism from [r] to [s] such that $A2_{a,b} = \mathrm{T}$ if and only if A contains an entry T at place i_a, j_b. Let $A3$ be the morphism from [s] to [m] such that $A3_{a,b} = \mathrm{T}$ if and only if $b = j_a$. Then A $= A1; A2; A3$, as required.

To show uniqueness, suppose that $A1; A2; A3 = B1; B2; B3$ where both expressions are of the required form. $A1$, $B1$ are determined uniquely by which of their rows have all entries F. But the i^{th} row of $A1$ has all entries F if and only if the i^{th} row of $A1; A2; A3$ has all entries F, so $A1 = B1$. Similarly $A3 = B3$. $A2$ is just the matrix obtained by deleting from $A1; A2; A3$ all rows and columns with all entries F, so is equal to $B2$.

3.3 Representation

What has the category B to do with the algebra CCH? The answer is that there is a bijection between morphisms of B and elements of CCH whose objects are all either maximal or minimal, and this bijection preserves the operations ; and $\otimes$.

The element $((V_1 \cup \{\} \cup V_3, \leq, \ell), \beta, \gamma)$ of CCH corresponds to the $|V_1| \times |V_3|$ matrix which has entry T at place (i, j) if and only if $x \leq y$ where $\beta(x) = i$ and $\gamma(y) = j$. It is straightforward to check that this bijection preserves the operators ; and $\otimes$.

An element of CCH whose objects are all either maximal or minimal is called *bipartite.*

4 An axiomatization of B

In this section we give an axiomatization for B, which does not involve the biproduct structure. This axiomatization is closely related to the axiomatization we will give for CCH.

4.1 Definition

Let B2 be the symmetric strict monoidal category generated by an object [1] and morphisms $\wedge : [1] \to [1] \otimes [1]$, $\vee : [1] \otimes [1] \to [1]$ under the following axioms.

- The identity object [0], which is the tensor product of no copies of [1], is initial and final
- $\wedge; (I \otimes \wedge) = \wedge; (\wedge \otimes I)$ where I is the identity morphism on [1]
- $(I \otimes \vee); \vee = (\vee \otimes I); \vee$
- $\wedge; \vee = \mathrm{I}$
- $\vee; \wedge = (\wedge \otimes \wedge); (I \otimes X \otimes I); (\vee \otimes \vee)$ where $X : ([1] \otimes [1]) \to ([1] \otimes [1])$ is the symmetry isomorphism
- $\wedge; (I \otimes \epsilon) = \mathrm{I}$ where $\epsilon : [1] \to [0]$
- $(I \otimes \mu); \vee = \mathrm{I}$ where $\mu : [0] \to [1]$
- $\wedge; X = \wedge$
- $X; \vee = \vee$

Note The axiom that [0] is initial and final is equivalent to the axioms
$\mu; \wedge = \mu \otimes \mu$, $\vee; \epsilon = \epsilon \otimes \epsilon$

4.2 First stage of proof that the categories B, B2 are isomorphic

It is straightforward to check that B is a symmetric, strict monoidal category and that it is generated as a symmetric strict monoidal category by the 1×2 and 2×1 dimensional matrices which have both entries T. (The symmetry isomorphism $\gamma_{n,m}$ in B is the $(n+m)\times(m+n)$ dimensional matrix whose $(i,j)^{th}$ element is T if $i = j - m$ or $j = i - n$, and F otherwise.) Each of the axioms for B2 also holds in B (with the appropriate renaming of variables) and so there is a surjective morphism of symmetric, strict monoidal categories from from B2 to B. Say that a morphism of B2 represents a given bipartite history if and only if it is mapped to the history under the morphism from B2 to B. We need to show that if two morphisms of B2 represent the same bipartite history then they are equal in B2.

If two morphisms represent the same bipartite history which has an empty set of maximal elements or an empty set of minimal elements then then are equal in B2 by the fact that [0] is initial and final.

Let I_n be the identity morphism on [n].
Let

$$\begin{aligned}\wedge_0 &= \epsilon,\ \vee_0 = \mu\\ \wedge_1 &= \vee_1 = I\\ \wedge_n &= \wedge_{n-1}; (\wedge\otimes I_{n-2}) \text{ for } n\geq 2\\ \vee_n &= (\vee\otimes I_{n-2}); \vee_{n-1} \text{ for } n\geq 2\end{aligned}$$

4.3 Lemma

Every morphism m of B2 is equal to a morphism of the form I_0, $\epsilon\otimes\ldots\otimes\epsilon$, $\mu\otimes\ldots\otimes\mu$, or

$$(\wedge_{n(1)}\otimes\ldots\otimes\wedge_{n(j)});\sigma;(\vee_{m(1)}\otimes\ldots\otimes\vee_{m(k)})$$

where $j,k\geq 1$, $n(1),\ldots,n(j),m(1),\ldots,m(k)\geq 0$, and σ is in the subalgebra of B2 generated by the symmetry isomorphisms.

Proof

The statement holds if m is a generator. If m is the tensor product of two morphisms for which the statement holds, then it also holds for m, by the functoriality of the tensor product. By induction on the minimal number of instances of generators in an expression for m, we may assume that m is of the form $m_1;m_2$ where the statement holds for m_1 and m_2. If $m:[n]\rightarrow[0]$ or $m:[0]\rightarrow[n]$ for some n then the statement holds by initiality and finality of [0]. If m_1 is a tensor product of copies of ϵ and m_2 is a tensor product of copies of μ then

$$m = (\wedge_0\otimes\ldots\otimes\wedge_0);I_0;(\vee_0\otimes\ldots\otimes\vee_0)$$

and so the statement holds.

To make the notation easier, let Π be the subalgebra of B2 generated by the symmetry isomorphisms, let W be the set of morphisms of the form

$$\wedge_{n(1)}\otimes\ldots\otimes\wedge_{n(j)}$$

for some $j \geq 1, n(1), \ldots, n(j) \geq 0$, and let V be the set of morphisms of the form

$$\vee_{n(1)} \otimes \ldots \otimes \vee_{n(j)}$$

for some $j \geq 1, n(1), \ldots, n(j) \geq 0$.

Remark 1 It follows from the axioms $(I \otimes \vee); \vee = (\vee \otimes I); \vee$, $(I \otimes \mu); \vee = \mathrm{I}$ and the functoriality of $\otimes$ that if $n(1) \geq 0$, $0 \leq a < n(2)$, then

$$(I_a \otimes \vee_{n(1)} \otimes I_{n(2)-a}); \vee_{n(2)} = \vee_{n(1)+n(2)-1}$$

Therefore, using the functoriality of $\otimes$ again, if $v_1, v_2 \in V$ and $v_1; v_2$ is defined then it is also in V. Similarly if w_1, w_2 are in W and $w_1; w_2$ is defined then it is in W.

Remark 2 It follows from the coherence axiom for the symmetry isomorphisms that if $v \in V$, $\pi \in \Pi$, and $v; \pi$ is defined, then there are some $v_1 \in V$, $\pi_1 \in \Pi$ such that $v; \pi = \pi_1; v_1$. It also follows that if $w \in W$, $\pi \in \Pi$, and $\pi; w$ is defined, then there are some $w_1 \in W$, $\pi_1 \in \Pi$ such that $w; \pi = \pi_1; w_1$.

The next stage of the proof is to prove the special case $m_1 = \vee_n, m_2 = \wedge_m$ by induction on $n + m$. If $n = 0$ or $m = 0$ the statement has already been proved. If $n = 1$ or $m = 1$ then the statement follows from the property of the identity morphism. If $n = m = 2$ then one of the axioms gives

$$m = (\wedge_2 \otimes \wedge_2); (I \otimes X \otimes I); (\vee_2 \otimes \vee_2)$$

and so the statement holds. If $n = 2 < m$ then assume (as an inductive hypothesis) that whenever $2 \leq m(1) < m$ there is some $\pi_{m(1)} \in \Pi$ such that

$$\vee_2; \wedge_{m(1)} = (\wedge_{m(1)} \otimes \wedge_{m(1)}); \pi_{m(1)}; (\vee_2 \otimes \ldots \otimes \vee_2)$$

Then $\vee_2; \wedge_m$ is equal to

$$\vee_2; \wedge_2; (\wedge_{m-1} \otimes I)$$

by the definition of $\wedge_m$ and the axiom $\wedge; (\wedge \otimes I) = \wedge; (I \otimes \wedge)$,

$$= (\wedge_2 \otimes \wedge_2); (I \otimes X \otimes I); (\vee_2 \otimes \vee_2); (\wedge_{m-1} \otimes I)$$

by one of the axioms

$$= (\wedge_2 \otimes \wedge_2); (I \otimes X \otimes I); ((\vee_2; \wedge_{m-1}) \otimes \vee_2)$$

by functoriality of $\otimes$

$$= (\wedge_2 \otimes \wedge_2); (I \otimes X \otimes I); (((\wedge_{m-1} \otimes \wedge_{m-1}); \pi_{m-1}; (\vee_2 \otimes \ldots \otimes \vee_2)) \otimes \vee_2)$$

by hypothesis,

$$= ((\wedge_2; (\wedge_{m-1} \otimes I)) \otimes (\wedge_2; (\wedge_{m-1} \otimes I))); \pi; (\vee_2 \otimes \ldots \otimes \vee_2)$$

for some $\pi \in \Pi$,

$$= (\wedge_m \otimes \wedge_m); \pi; (\vee_2 \otimes \ldots \otimes \vee_2)$$

as required.

Finally suppose $n > 2$ and that the statement holds for $\vee_{n(1)}; \wedge_m$ whenever $2 \leq n(1) \leq n$. Then

$$\vee_n; \wedge_m$$
$$= (\vee_{n-1} \otimes I); \vee_2; \wedge_m$$
$$= (\vee_{n-1} \otimes I); (\wedge_m \otimes \wedge_m); \pi_m; (\vee_2 \otimes \ldots \otimes \vee_2)$$

by the case $n = 2$

$$= ((\vee_{n-1}; \wedge_m) \otimes \wedge_m); \pi_m; (\vee_2 \otimes \ldots \otimes \vee_2)$$

by functoriality of $\otimes$

$$= ((w_1; \sigma_1; v_1) \otimes w_2); \pi_m; v_2$$

for some $v_1, v_2 \in V, w_1, w_2 \in W, \sigma \in \Pi$

$$= w_3; \sigma_2; v_1; \sigma_3; v_3$$

for some $w_3 \in W, \sigma_2, \sigma_3 \in \Pi, v_3 \in V$
which is equal to a morphism of the required form by remarks 1 and 2.

Now suppose that $m = m_1; m_2$ and $m_1 = w_1; \pi_1; v_1$, $m_2 = w_2; \pi_2; v_2$ for some $w_1, w_2 \in W, \pi_1, \pi_2 \in \Pi, v_1, v_2 \in V$. The morphism $v_1; w_2$ is a tensor product of morphisms of the form $\vee_n; \wedge_m$ for some m, n, so the statement holds for this morphism, and it must equal I_0, or a tensor product of copies of μ, or a tensor product of copies of ϵ, or a morphism $w_3; \pi_3; v_3$. If it is equal to I_0 or a tensor product of copies of μ or a tensor product of copies of ϵ then the fact that [0] is initial and final forces m to equal a morphism of the form $(\epsilon \otimes \ldots \otimes \epsilon); I_0; (\mu \otimes \ldots \otimes \mu)$, $\epsilon \otimes \ldots \otimes \epsilon$, $\mu \otimes \ldots \otimes \mu$, or I_0. If it is equal to $w_3; \pi_3; v_3$ then $m = w_1; \pi_1; w_3; \pi_3; v_3; \pi_2; v_2$ which is equal to a morphism of the required form by remarks 1 and 2. This completes the proof of the lemma. $\triangledown$

4.4 Normal form for bipartite histories

It is straightforward to show that every bipartite history with nonempty sets of minimal and maximal elements can be represented as a sequential composition $b_1; b_2; b_3$ where b_1 is the image of an element $(\wedge_{n(1)} \otimes \ldots \otimes \wedge_{n(a)})$ of W under the homomorphism from B, b_3 is the image of an element $\vee_{m(1)} \otimes \ldots \otimes \vee_{m(b)}$ of V, and b_2 is a permutation linking the i^{th} maximal element with the $p(i)^{th}$ minimal element such that

$$\text{(i) if } \Sigma_{i=1}^{\ell-1} m(i) < p(J) < p(K) \leq \Sigma_{i=1}^{\ell} m(i) \text{ and } \Sigma_{i=1}^{h-1} n(i) < J \leq \Sigma_{i=1}^{h} n(i)$$
$$\text{then } K > \Sigma_{i=1}^{h} n(i)$$

and

$$\text{(ii) if } \Sigma_{i=1}^{\ell-1} n(i) < J < K \leq \Sigma_{i=1}^{\ell} n(i) \text{ and } \Sigma_{i=1}^{h-1} m(i) < p(J) \leq \Sigma_{i=1}^{h} m(i)$$
$$\text{then } p(K) > \Sigma_{i=1}^{h} m(i)$$

Condition (i) says that if the $p(J)^{th}$ and $p(K)^{th}$ minimal elements of b_3 are linked to the same maximal element of b_3, and $p(J) < p(K)$, then $j < k$ where the j^{th} and k^{th} minimal elements of b_1 are linked to the J^{th} and K^{th} maximal elements respectively.
Condition (ii) says that if the J^{th} and K^{th} maximal elements of b_1 are linked to the same

minimal element of b_1, and $J < K$, then $j < k$ where the $p(J)^{th}$ and $p(K)^{th}$ minimal elements of b_3 are linked to the j^{th} and k^{th} maximal elements respectively.
These conditions can be ensured by means of the equations $\wedge; X = \wedge$ and $X; \vee = \vee$ in B. Moreover each bipartite history with nonempty sets of minimal and maximal elements can be represented by only one morphism of this form. The image of the algebra Π under the homomorphism from $B2$ to B is the algebra of permutations. Two elements of Π are equal if and only they are images of the same permutation. (See [DMM 89].) Therefore in order to show that any two morphisms in B2 represented by the same bipartite history are equal in B2 it is enough to show that any morphism $w; \pi; v$ in B is equal to a morphism $w_1; \pi_1; v_1$ where π_1 represents a permutation in B which satisfies the conditions (i) and (ii). Write $\gamma(p)$ for an element of Π representing a permutation p. Let

$$w = \wedge_{n(1)} \otimes \ldots \otimes \wedge_{n(n)}, \quad v = \vee_{m(1)} \otimes \ldots \otimes \vee_{m(m)}$$

Let p be any permutation.

4.5 Removing Loops

Let $\mathrm{Loop}(w, \gamma(p), v)$ be the number of pairs (J, K) such that for some h, ℓ,

$$\Sigma_{i=1}^{h-1} n(i) < J < K \leq \Sigma_{i=1}^{\ell-1} n(i)$$

and

$$\Sigma_{i=1}^{\ell-1} m(i) < p(J) < p(K) \leq \Sigma_{i=1}^{\ell} n(i)$$

This is the number of pairs (J, K) for which the J^{th} and K^{th} minimal elements of the history representing w are linked to the same maximal element, and the $p(J)^{th}$ and $p(K)^{th}$ minimal elements of the history representing v are linked to the same maximal element.
Suppose $\mathrm{Loop}(w; \gamma(p); v)$ is nonzero. The axioms $X; \vee = \vee$ and $\wedge; X = \wedge$ enable $w; \gamma(p); v$ to be rewritten as $w; \gamma(p1); v$ for some $p1$ such that $\mathrm{Loop}(w, \gamma(p1), v) = \mathrm{Loop}(w, \gamma(p), v)$ and that there are some h, ℓ for which $n(h) > 1, m(\ell) > 1$, and

$$p1((\Sigma_{i=1}^{h-1} n(i)) + 1) = (\Sigma_{i=1}^{\ell-1} m(i)) + 1,$$
$$p1((\Sigma_{i=1}^{h-1} n(i)) + 2) = (\Sigma_{i=1}^{\ell-1} m(i)) + 2.$$

Now by the coherence axiom for the symmetry isomorphisms $w; \gamma(p1); v =$

$$(\gamma_{h-1,1} \otimes I_{n-h}); (\wedge_{n(h)} \otimes \wedge_{n(1)} \otimes \ldots \otimes \wedge_{n(n)}); \gamma(p2); (\vee_{m(\ell)} \otimes \ldots \otimes \vee_{m(m)}); (\gamma_{1,\ell-1} \otimes I_{m-\ell})$$

for some $p2$ satisfying

$Loop((\wedge_{n(h)} \otimes \ldots \otimes \wedge_{n(n)}), \gamma(p2), \vee_{m(\ell)}, \ldots \vee_{m(m)}) = Loop(w, \gamma(p), v),\ p2(1) = 1,\ p2(2) = 2.$

Therefore $w; \gamma(p); v$ is equal to

$$(\gamma_{h-1,1} \otimes I_{n-h}); (\wedge_{n(h)-1} \otimes \wedge_{n(1)} \otimes \ldots \otimes \wedge_{n(n)}); (\wedge \otimes I_{n(1)+\ldots+n(n)-2}); (I_2 \otimes \gamma(p3));$$
$$(\vee \otimes I_{m(1)+\ldots+m(m)-2}); (\vee_{m(\ell)-1} \otimes \vee_{m(1)} \otimes \ldots \otimes \vee_{m(m)}); (\gamma_{1,\ell-1} \otimes I_{m-\ell})$$

where $I_2 \otimes \gamma(p3) = \gamma(p2)$.
By the axiom $\wedge; \vee = \mathrm{I}$, $w; \gamma(p); v$ is equal to

$$(\gamma_{h-1,1} \otimes I_{n-h}); (\wedge_{n(h)-1} \otimes \wedge_{n(1)} \otimes \ldots \otimes \wedge_{n(n)}); (I \otimes \gamma(p3));$$
$$(\vee_{m(\ell)-1} \otimes \vee_{m(1)} \otimes \ldots \otimes \vee_{m(m)}); (\gamma_{1,\ell-1} \otimes I_{m-\ell}),$$

$= w_1; \gamma(p4); v_1$ where $w_1 \in W, v_1 \in V$, and $Loop(w_1, \gamma(p4), v_1) < Loop(w, \gamma(p), v)$. Therefore without loss of generality $Loop(w, \gamma(p), v) = 0$.

4.6 Final stage

By using the axiom $X; \vee = \vee$ it is possible to rewrite $w; \gamma(p); v$ as $w; \gamma(p1); v$ for some $p1$ satisfying condition (i). By using the axiom $\wedge; X = \wedge$ it is possible to rewrite this as $w; \gamma(p2); v$ satisfying conditions (i) and (ii). This completes the proof that B and B2 are isomorphic categories. ∇

In the rest of the paper we will consider a larger category, which contains B as a subcategory. We will eventually be able to construct a bijection between the morphisms of this larger category and the elements of CCH, which also preserves the operators ; and $\otimes$, and whose restriction to the morphisms of B is the map described here.

5 Definitions - the category A, and layered form

This section gives the definition (in terms of generators and axioms) of the larger category containing B whose morphisms will turn out by the end of the paper to be the elements of CCH in disguise. It also gives the definition of layered form; every morphism can be written as a term in layered form, but not necessarily in a unique way.

5.1 Definition of the category A

Let Λ be a fixed set of labels. We introduce a category which has the same objects as B but more morphisms. For each label t in Λ there is a morphism (also called t, by abuse of notation) from [1] to [1]. Let A be the symmetric strict monoidal category generated by a non-identity object [1], and morphisms $t : [1] \to [1]$ $(t \in \Lambda)$, $\vee : [1] \otimes [1] \to [1]$, $\wedge : [1] \to [1] \otimes [1]$, under the following set of axioms.

- The identity object [0] is initial and final
- $\wedge; (I \otimes \wedge) = \wedge; (\wedge \otimes I)$ where I is the identity morphism on [1]
- $(I \otimes \vee); \vee = (\vee \otimes I); \vee$
- $\wedge; \vee = \mathrm{I}$
- $\vee; \wedge = (\wedge \otimes \wedge); (I \otimes X \otimes I); (\vee \otimes \vee)$ where $X : ([1] \otimes [1]) \to ([1] \otimes [1])$ is the symmetry isomorphism
- $\wedge; (I \otimes \epsilon) = \mathrm{I}$ where $\epsilon : [1] \to [0]$
- $(I \otimes \mu); \vee = \mathrm{I}$ where $\mu : [0] \to [1]$
- $\wedge; X = \wedge$
- $X; \vee = \vee$
- $t; \wedge = \wedge; (I \otimes t); (I \otimes \wedge); (\vee \otimes I)$ whenever $t \in \Lambda$

The last axiom is called the copy axiom. From now on a term will be understood to be a term of this algebra. It is clear that the subcategory of A which is the smallest symmetric strict monoidal category containing $\vee$ and $\wedge$ is B.

A term in the category A which represents a morphism from $[n]$ to $[m]$, where $n, m \geq 0$, can be given an informal representation as a picture, where

- the picture for $u1 \otimes u2$ is the picture for $u2$ to the right of the picture for $u1$
- the picture for $u1; u2$ is the picture for $u1$ above the picture for $u2$ with the upper nodes of $u2$ and the lower nodes of $u1$ identified
- the pictures for $I, X, \wedge$, and $\vee$ are just $|, \times, \wedge, \vee$, (with nodes at the ends of the lines)
- the picture for O consists of an upper and lower node
- the picture for $t \in \Lambda$ is $|$ with a label t

As an example, Figure 3 gives the pictures corresponding to the copy axiom. We found these pictures easier to work with than the formal terms, and used them to find our proofs.

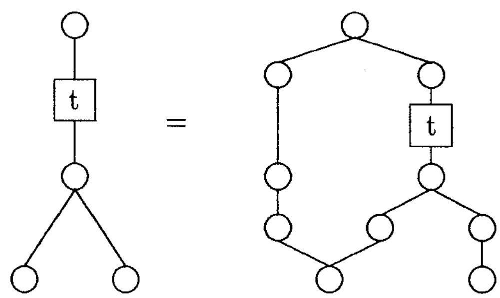


Figure 3: A pictorial representation of the copy axiom

5.2 Layered form

Using the functoriality of $\otimes$ every term can be written in the form

$$B_1; P_1; B_2; P_2; \ldots; P_{r-1}; B_r$$

for some $r \geq 1$, where $B_1, \ldots, B_r$ are morphisms of B and $P_1, \ldots, P_{r-1}$ are each of the form

$$g_1 \otimes g_2 \otimes \ldots \otimes g_s$$

where each g_i is either I or an element of Λ, and not all of $g_1, \ldots g_s$ are I. This is called Layered form.

Notice that the axioms given in the definition of the category A involve at most three layers. This was a surprising result; a first draft of this paper had axioms involving arbitrarily many layers. The price paid for having such a simple set of axioms is the length of the proof which takes up the next section.

6 Partial Ordering Condition

This section is devoted to a proof that any term can be written in a form satisfying a certain condition called the Partial Ordering Condition. This condition will be crucial when proving the representation theorem for the algebra.

Section 6.1 defines the terms used and gives the Partial Ordering Condition. Section 6.2 proves a Theorem expressing equality between certain terms, which is needed to prove the result. The proof of the Theorem is in seven stages; the first six stages prove that the theorem holds in particular cases, and the last stage uses these cases and induction to prove the theorem holds in general. Section 6.3 uses the Theorem together with a different induction to prove that any term u of the category A can be written in layered form so that any term consisting of a set of consecutive layers of u (including u itself) satisfies the partial ordering condition.

6.1 Definitions - Linkage

Suppose u is a term built from morphisms of B and elements of Λ using the operators ; and $\otimes$. Suppose that $(1 \leq i \leq n, 1 \leq j \leq m)$ where u expresses a morphism from [n] to [m]. Define Link1(u,i,j) and Link2(u,i,j) by structural induction on u as follows.

- If u is a morphism of B then Link1(u,i,j) = Link2(u,i,j) = 1 if $u_{i,j}$ = T; otherwise Link1(u,i,j) = Link2(u,i,j) = 0.
- If $u \in \Lambda$ then Link1(u,1,1) = 1 and Link2(u,1,1) = 0.
- If $u = u1 \otimes u2$ where $u1, u2$ express morphisms from [n1] to [m1] and [n2] to [m2] respectively, then Link1(u,i,j) is Link1$(u1,i,j)$ if $i \leq n1, j \leq m1$,
  Link1$(u2, i-n1, j-m1)$ if $i > n1$, $j > m1$,
  and 0 otherwise.
  Similarly Link2(u,i,j) is
  Link2$(u1,i,j)$ if $i \leq n1, j \leq m1$,
  Link2$(u2, i-n1, j-m1)$ if $i > n1$, $j > m1$,
  and 0 otherwise.
- If $u = u1; u2$ where $u1, u2$ express morphisms from [n] to [p] and [p] to [m] respectively, then
  $$Link1(u,i,j) = \Sigma_{1\leq k\leq p}(Link1(u1,i,k).Link1(u2,k,j))$$
  and
  $$Link2(u,i,j) = \Sigma_{1\leq k\leq p}(Link2(u1,i,k).Link2(u2,k,j))$$

Informally, if u is represented graphically, as described in section 5.1, Link1(u,i,j) is the number of paths between the i^{th} upper node in the picture for u and the j^{th} lower node in the picture, and Link2(u,i,j) is the number of these paths which are unlabelled.

Note that Link1(u,i,j) and Link2(u,i,j) are defined for *terms* u, not for morphisms. For instance, the terms I, $\wedge;\vee$ are equal as morphisms, but Link1(I,1,1)=1 and Link1($\wedge;\vee$,1,1)=2.

6.1.1 Definition - i linked to j by u

Say that i is linked to j by u if and only if Link1$(u, i, j) > 0$.
Informally i is linked to j by u if there is a path in the picture for u from the i^{th} upper node to the j^{th} lower node.

6.1.2 Partial ordering condition

Say that u satisfies the partial ordering condition if for each pair (i, j) either Link2(u, i, j) is zero or Link1$(u, i, j) =$ Link2$(u, i, j) = 1$.
Informally u satisfies the partial ordering condition if whenever there is an unlabelled path between two nodes there is just one of these, and no labelled path between the two nodes.
In order to prove that any morphism can be written as a term satisfying the condition we will first prove a technical result.

6.2 Theorem

If u is a term representing a morphism from [n] to [m] such that i is linked to j by u then

$$u \;=\; X(i,n);(\wedge\otimes I_{n-1});(I\otimes X(i,n));(I\otimes u);(I\otimes X(j,m));(\vee\otimes I_{m-1});X(j,m)$$

Note

X(1,n)=I_n, so that if u is a morphism from [1] to [1] the Theorem for u states that

$$u \;=\; (\wedge\otimes I_{n-1});(I\otimes u);(\vee\otimes I_{m-1})$$

The copy axiom is a special case of the Theorem, where $u = t;\wedge$ for some $t \in \Lambda$ and $i = j = 1$.

Proof

The proof goes in several stages. Let Theorem(u, i, j) be the statement of the Theorem for u, i, and j, that is the equality

$$u \;=\; X(i,n);(\wedge\otimes I_{n-1});(I\otimes X(i,n));(I\otimes u);(I\otimes X(j,m));(\vee\otimes I_{m-1});X(j,m)$$

The first stage shows that Theorem(I_n, i, i) holds whenever $1 \le i \le n$.

The second stage shows that if p is a permutation of size $n \times n$ such that $p_{i,k} =$ T and $u1$ is a morphism from [n] to [m] such that Theorem$(u1, k, j)$ holds, then Theorem$((p; u1), i, j)$ holds. A similar argument shows that if p is a permutation of size $m \times m$ such that $p_{k,j} =$ T and $u1$ is a term representing a morphism from [n] to [m] such that Theorem$(u1, i, k)$ holds, then Theorem$((u1; p), i, j)$ holds.

For stages 3, 4, 5 and 6 it is assumed that the Theorem$(u1, 1, 1)$ holds. The result of each of these stages is that Theorem$((u2; u1), 1, 1)$ holds, where $u2$ is of the form $I \otimes u3$ in stage 3, $t \otimes I_{n-1}$ in stage 4 where $t \in \Lambda$, $\vee \otimes I_{n-2}$ in stage 5, and $\wedge \otimes I_{n-1}$ in stage 6. We worked out how to prove stages 3-6 by means of the diagrams in Figure 4. (Figure 4 is on page 32.)

Stage 7 uses induction to prove the Theorem in the case $i = j = 1$, and then uses the result of stage 2 to derive the general case.

6.2.1 Stage 1

Suppose $u = I_n$. Then

$$
\begin{aligned}
& X(i,n);(\wedge \otimes I_{n-1});(I \otimes X(i,n));(I \otimes u);(I \otimes X(j,n));(\vee \otimes I_{n-1});X(j,n) \\
&= X(i,n);(\wedge \otimes I_{n-1});(I \otimes X(i,n));(I \otimes X(i,n));(\vee \otimes I_{n-1});X(i,n) \\
&= X(i,n);((\wedge;\vee) \otimes I_{n-1});X(i,n) \\
&= I_n
\end{aligned}
$$

So Theorem(I_n, i, i) holds.

6.2.2 Stage 2

Suppose that p is a permutation of size $n \times n$ and $u1$ is a morphism from [n] to [m] such that Theorem$(u1, k, j)$ holds. Since $I_a \otimes X(2,2) \otimes I_b = X(a+1, a+b+2); X(a+2, a+b+2); X(a+1, a+b+2)$ the elements $X(i,n)$ together with I generate all permutations under ;. This means that p can be written as a concatenation of matrices of the form $X(a,n)$ and we may assume by induction on the minimal length of such an expression for p that $p = X(a,n)$ for some a.

If $k = a$ then we must show that Theorem$((X(k,n);u), 1, j)$ holds. Now

$$
\begin{aligned}
& (\wedge \otimes I_{n-1});(I \otimes (X(k,n);u1));(I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\
&= (\wedge \otimes I_{n-1});(I \otimes X(k,n));(I \otimes u1);(I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\
&= X(k,n);u1
\end{aligned}
$$

since Theorem$(u1, k, j)$ holds.

If $k = 1$ then we must show that Theorem$((X(a,n);u1), a, j)$ holds. Now

$$
\begin{aligned}
& X(a,n);(\wedge \otimes I_{n-1});(I \otimes X(a,n));(I \otimes (X(a,n);u1)); \\
& \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\
&= X(a,n);(\wedge \otimes I_{n-1});(I \otimes (X(a,n);X(a,n)));(I \otimes u1); \\
& \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\
&= X(a,n);u1
\end{aligned}
$$

since Theorem$(u1, 1, j)$ holds.

Suppose that $k \neq 1, a$. Then we must show that Theorem$(X(a,n);u1, k, j)$ holds. Now

$$
\begin{aligned}
& X(k,n);(\wedge \otimes I_{n-1});(I \otimes X(k,n));(I \otimes (X(a,n);u1)); \\
& \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\
&= X(k,n);(\wedge \otimes I_{n-1});(I \otimes (X(k,n);X(a,n)));(I \otimes u1); \\
& \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m)
\end{aligned}
$$

It is straightforward to check that the matrices

$$X(k,n);(\wedge \otimes I_{n-1});(I \otimes (X(k,n);X(a,n)))$$

and

$$X(a,n);X(k,n);(\wedge \otimes I_{n-1});(I \otimes X(k,n))$$

are equal as morphisms of B (and hence as terms of the algebra). Therefore

$$\begin{aligned} & X(k,n);(\wedge \otimes I_{n-1});(I \otimes X(k,n));(I \otimes (X(a,n);u1)); \\ & \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\ = \; & X(a,n);X(k,n);(\wedge \otimes I_{n-1});(I \otimes X(k,n));(I \otimes u1); \\ & \quad (I \otimes X(j,m));(\vee \otimes I_{m-1});X(j,m) \\ = \; & X(a,n);u1 \end{aligned}$$

Since Theorem$(u1,k,j)$ holds.

The result follows.

6.2.3 Stage 3

Suppose $u = (I \otimes u3);u1$ and Theorem$(u1,1,1)$ holds. Let $u3 : [n-1] \to [r-1]$. Then

$$\begin{aligned} & (\wedge \otimes I_{n-1});(I \otimes u);(\vee \otimes I_{m-1}) \\ = \; & (\wedge \otimes I_{n-1});(I \otimes I \otimes u3);(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & (I \otimes u3);(\wedge \otimes I_{r-1});(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & (I \otimes u3);u1 \end{aligned}$$

since Theorem$(u1,1,1)$ holds,

$$= u$$

so Theorem$(u,1,1)$ holds as required.

6.2.4 Stage 4

Suppose $u = (t \otimes I_{n-1});u1$ where $t \in \Lambda$, and Theorem$(u1,1,1)$ holds. Then

$$\begin{aligned} & (\wedge \otimes I_{n-1});(I \otimes u);(\vee \otimes I_{m-1}) \\ = \; & (\wedge \otimes I_{n-1});(I \otimes t \otimes I_{n-1});(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & (\wedge \otimes I_{n-1});(I \otimes t \otimes I_{n-1});(I \otimes \wedge \otimes I_{n-1});(I \otimes I \otimes u1);(I \otimes \vee \otimes I_{m-1});(\vee \otimes I_{m-1}) \end{aligned}$$

since Theorem$(u1,1,1)$ holds,

$$\begin{aligned} = \; & (\wedge \otimes I_{n-1});(I \otimes t \otimes I_{n-1});(I \otimes \wedge \otimes I_{n-1});(I \otimes I \otimes u1);(\vee \otimes I_m);(\vee \otimes I_{m-1}) \\ = \; & (\wedge \otimes I_{n-1});(I \otimes t \otimes I_{n-1});(I \otimes \wedge \otimes I_{n-1});(\vee \otimes I_n);(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & ((\wedge;(I \otimes t);(I \otimes \wedge);(\vee \otimes I)) \otimes I_{n-1});(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & ((t;\wedge) \otimes I_{n-1});(I \otimes u1);(\vee \otimes I_{m-1}) \end{aligned}$$

by the copy axiom

$$\begin{aligned} = \; & (t \otimes I_{n-1});(\wedge \otimes I_{n-1});(I \otimes u1);(\vee \otimes I_{m-1}) \\ = \; & (t \otimes I_{n-1});u1 \\ = \; & u \end{aligned}$$

So Theorem$(u,1,1)$ holds.

6.2.5 Stage 5

Suppose $u = (\vee \otimes I_{n-2}); u1$, and Theorem$(u1, 1, 1)$ holds. Then

$$(\wedge \otimes I_{n-1}); (I \otimes u); (\vee \otimes I_{m-1})$$

$$= (\wedge \otimes I_{n-1}); (I \otimes \vee \otimes I_{n-2}); (I \otimes u1); (\vee \otimes I_{m-1})$$

$$= (\wedge \otimes I_{n-1}); (I \otimes \vee \otimes I_{n-2}); (I \otimes \wedge \otimes I_{n-2}); (I \otimes I \otimes u1); (I \otimes \vee \otimes I_{m-1}); (\vee \otimes I_{m-1})$$

since Theorem$(u1, 1, 1)$ holds,

$$= (((\wedge \otimes I); (I \otimes \vee); (I \otimes \wedge)) \otimes I_{n-2});$$

$$(I \otimes I \otimes u1); (\vee \otimes I_m); (\vee \otimes I_{m-1})$$

$$= (((\wedge \otimes I); (I \otimes \vee); (I \otimes \wedge); (\vee \otimes I)) \otimes I_{n-2}); (I \otimes u1); (\vee \otimes I_{m-1})$$

$$= ((\vee; \wedge) \otimes I_{n-2}); (I \otimes u1); (\vee \otimes I_{m-1})$$

$$= (\vee \otimes I_{n-2}); (\wedge \otimes I_{n-2}); (I \otimes u1); (\vee \otimes I_{m-1})$$

$$= (\vee \otimes I_{n-2}); u1$$

since Theorem$(u1, 1, 1)$ holds,

$$= u$$

so Theorem$(u, 1, 1)$ holds.

6.2.6 Stage 6

Suppose $u = (\wedge \otimes I_{n-1}); u1$ and Theorem$(u1, 1, 1)$ holds. Now

$$(\wedge \otimes I_{n-1}); (I \otimes u); (\vee \otimes I_{m-1})$$

$$= (\wedge \otimes I_{n-1}); (I \otimes \wedge \otimes I_{n-1}); (I \otimes u1); (\vee \otimes I_{n-1})$$

$$= (\wedge \otimes I_{n-1}); (\wedge \otimes I_n); (I \otimes u1); (\vee \otimes I_{n-1})$$

$$= (\wedge \otimes I_{n-1}); u1$$

since Theorem$(u1, 1, 1)$ holds,

$$= u$$

So Theorem$(u, 1, 1)$ holds.

6.2.7 Stage 7

Say that a term is in form F if it is of the form $I_a \otimes f \otimes I_b$ for some $a, b \geq 0$, $f \in \{O, \wedge, \vee\} \cup \Lambda$. By the coherence axiom and the functoriality of $\otimes$ each morphism linking 1 to 1 can be written as a concatenation of terms which are alternately permutations linking 1 to 1 and terms in form F. We show by induction on the number of terms of form F in such an expression that for each morphism u linking 1 to 1, Theorem$(u, 1, 1)$ holds. If there are no terms of form F in the expression then u is a permutation, so Theorem$(u, 1, 1)$ holds by stages 1 and 2. If there is at least one term of form F in the concatenation then $u = p; u2; u1$ where p is a permutation (possibly the identity permutation) linking 1 to 1, $u2$ is of form F, and $u1$ is a morphism linking 1 to 1 such that Theorem$(u1, 1, 1)$ holds, by the inductive hypothesis. If $a \geq 0$ then Theorem$(u, 1, 1)$ holds by stages 3 and 2. Suppose $a = 0$. Since 1 is linked to 1 by u we cannot have $u2 = O \otimes I_b$, so $u2 = \wedge \otimes I_b$, $\vee \otimes I_b$, or $t \otimes I_b$ for some $t \in \Lambda$. Stages 4, 5, and 6 ensure that this implies that Theorem$((u2; u1), 1, 1)$ holds, and so Theorem$(u, 1, 1)$ holds by stage 2.

Finally, suppose that i is linked to j by some morphism u from [n] to [m]. Then $u = X(i, n); v; X(j, n)$ for some morphism v linking 1 to 1. By above, Theorem$(v, 1, 1)$ holds, so by stage 2, Theorem(u, i, j) holds.

6.3 Proof of the partial ordering condition

This subsection gives two corollaries of the Theorem. The first essentially proves that any term can be written in a form satisfying the partial ordering condition. The second corollary is the stronger result that any term can be written in a layered form so that all terms consisting of consecutive layers of the layered form satisfy the partial ordering condition.

Corollary 1

Any term u in layered form which does not satisfy the partial ordering condition is equal to a term v in layered form such that Link1$(v, i, j) \leq$ Link1(u, i, j) for all pairs (i, j) and there is at least one pair (i, j) such that Link1$(v, i, j) <$ Link1(u, i, j).

Proof The proof of Corollary 1 uses the Theorem as a starting point for an argument by induction.

For ease of notation we will write $(X)^i_j$ for $Link1(X, i, j)$ throughout the proof of the corollary.

Suppose a term u in layered form does not satisfy the partial ordering condition. Without loss of generality, using the coherence axiom, we may assume that u is of the form

$$Z1; (\wedge \otimes I_n); (I \otimes Z2); Z3; (\vee \otimes I_m); Z4$$

where $Z1, Z2, Z3, Z4$ are terms in layered form,
there is some i linked to 1 by $Z1$,
there is some j linked to 1 by $Z4$, 1 is linked to 1 by $Z2$ and by each layer of $Z3$,
2 is linked to 2 by each layer of $Z3$, and
each layer of $Z3$ which is of the form $g_1 \otimes g_2 \otimes \ldots \otimes g_r$ has $g_1 = I$.

We will show that this is equal to a term in layered form

$$Z1; (\wedge \otimes I_n); (I \otimes Y2); Y3; (\vee \otimes I_m); Z4$$

where $Y2, Y3$ are terms in layered form, and

$$((I \otimes Y2); Y3)^i_j \leq ((I \otimes Z2); Z3)^i_j \text{ for all } (i,j)$$

$$((I \otimes Y2); Y3)^1_1 < ((I \otimes Z2); Z3)^1_1$$

We will prove the result by induction on the number of layers of $Z3$ of the form $g_1 \otimes \ldots \otimes g_r$.

6.3.1 Case 0

If $Z3$ has no layers of the form $g_1 \otimes \ldots \otimes g_r$ then it must be a morphism of B. Assume that this is the case. Let $Z5$ be the $s \times (m+2)$ matrix (where $Z3$ has size $s \times (m+2)$) such that $Z5_{i,j} = Z3_{i,j}$ if $(i,j) \neq (1,1)$ and $Z5_{1,1} = F$. Then $Z3 = (\wedge \otimes I_{s-1}); (I \otimes Z5); (\vee \otimes I_{m+1})$ and so

$$\begin{aligned} u &= Z1; (\wedge \otimes I_n); (I \otimes Z2); (\wedge \otimes I_{s-1}); (I \otimes Z5); (\vee \otimes I_{m+1}); (\vee \otimes I_m); Z4 \\ &= Z1; (\wedge \otimes I_n); (I \otimes \wedge \otimes I_n); (I \otimes I \otimes Z2); (I \otimes Z5); (I \otimes \vee \otimes I_m); (\vee \otimes I_m); Z4 \\ &= Z1; (\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z5; (\vee \otimes I_m))); (\vee \otimes I_m); Z4 \end{aligned}$$

and 1 is linked to 2 by $(\wedge \otimes I_n)$, 2 is linked to 2 by $(I \otimes Z2)$, 2 is linked to 2 by $Z5$, 2 is linked to 1 by $(\vee \otimes I_m)$, so 1 is linked to 1 by $(\wedge \otimes I_n); (I \otimes Z2); Z5; (\vee \otimes I_m)$. This means that we can use the Theorem, and u equals

$$Z1; (\wedge \otimes I_n); (I \otimes Z2); Z5; (\vee \otimes I_m); Z4$$

By the definition of $Z5$, $((I \otimes Z2); Z5)^i_j \leq ((I \otimes Z2); Z3)^i_j$ for all pairs (i,j) and $((I \otimes Z2); Z5)^1_1$ is less than $((I \otimes Z2); Z3)^1_1$ as required.

6.3.2 Inductive step

Now suppose that $Z3 = Z5; (I \otimes g_2 \otimes \ldots \otimes g_r); Z6$ where $Z5$ is a morphism of B and $Z6$ is I_r or is in layered form with the top layer a non-identity morphism of B. Let $Z7$ be the matrix of the same size as $Z5$ such that $Z7_{i,j} = Z5_{i,j}$ whenever $(i,j) \neq (1,1)$, $Z7_{1,1} = F$. Then u equals

$$\begin{aligned} & Z1; (\wedge \otimes I_n); (I \otimes Z2); Z5; (I \otimes g_2 \otimes \ldots \otimes g_r); Z6; (\vee \otimes I_m); Z4 \\ &= Z1; (\wedge \otimes I_n); (I \otimes Z2); (\wedge \otimes I_{s-1}); (I \otimes Z7); \\ &\quad (\vee \otimes I_{r-1}); (I \otimes g_2 \otimes \ldots \otimes g_r); Z6; (\vee \otimes I_m); Z4 \\ &= Z1; (\wedge \otimes I_n); (I \otimes \wedge \otimes I_n); (I \otimes I \otimes Z2); (I \otimes Z7); \\ &\quad (I \otimes I \otimes g_2 \otimes \ldots \otimes g_r); (\vee \otimes I_{r-1}); Z6; (\vee \otimes I_m); Z4 \\ &= Z1; (\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes g_2 \otimes \ldots \otimes g_r); X(2,r))); \end{aligned}$$

$$(I \otimes X(2,r)); (\vee \otimes I_{r-1}); Z6; (\vee \otimes I_m); Z4$$

Put

$$Z8 \;=\; (\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes g_2 \otimes \ldots \otimes g_r); X(2,r)$$

$$Z9 \;=\; \text{The term in layered form}((I \otimes X(2,r)); (\vee \otimes I_{r-1})); Z10$$

where $Z6 = Z10; Z11$ with $Z10$ the top layer of $Z6$.
Then $((I \otimes X(2,r)); (\vee \otimes I_{r-1}); Z6)$ can be rewritten in layered form as $Z9; Z11$ with fewer layers of the form $g_1 \otimes \ldots \otimes g_r$ than $Z3$,
1 is linked to 1 by $Z8$ and each layer of $Z9; Z11$, and
2 is linked to 2 by each layer of $Z9; Z11$.
Hence it is possible to use the inductive hypothesis; u is equal to a term

$$Z1; (\wedge \otimes I_n); (I \otimes Y2); Y3; (\vee \otimes I_m); Z4$$

such that

$$((\wedge \otimes I_n); (I \otimes Y2); Y3)^i_j \;\leq\; ((\wedge \otimes I_n)(I \otimes Z8); (Z9; Z11))^i_j$$

for all pairs (i,j), and

$$((\wedge \otimes I_n); (I \otimes Y2); Y3)^1_1 \;<\; ((\wedge \otimes I_n)(I \otimes Z8); (Z9; Z11))^1_1$$

It remains to prove that

$$((\wedge \otimes I_n); (I \otimes Z8); (Z9; Z11))^i_j \;\leq\; ((\wedge \otimes I_n); (I \otimes Z2); Z3)^i_j$$

for all pairs (i,j), ie.

$$((\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes \ldots \otimes g_r); X(2,r))); (Z9; Z11))^i_j$$

$$\leq\; ((\wedge \otimes I_n); (I \otimes Z2); Z5; (I \otimes g_2 \otimes \ldots \otimes g_r); Z10; Z11)^i_j$$

for all pairs (i,j).

6.3.3 Case 1 - $i = 1$

By definition of Link1,

$$((\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes \ldots \otimes g_r); X(2,r))); (Z9; Z11))^1_j$$

$$= \Sigma_k((Z9)^1_k.(Z11)^k_j)$$

$$+ \Sigma_{k1,k2,k3}((Z7)^1_{k1}.(X(2,r))^{k1}_{k2}.(Z9)^{k2+1}_{k3}.(Z11)^{k3}_j)$$

$$+ \Sigma_{k1,k2,k3,k4}((Z2)^1_{k1}.(Z7)^{k1+1}_{k2}.(X(2,r))^{k2}_{k3}.(Z9)^{k3+1}_{k4}.(Z11)^{k4}_j)$$

By definition of $Z9$, $(Z9)^{k1}_1 = (Z10)^{k1}_1$ and

$$\Sigma_{k3}((X(2,r))^{k2}_{k3}.(Z9)^{k3+1}_{k4}) \;=\; (Z10)^{k2}_{k4}$$

for any $k2, k4$, so

$$((\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes \ldots \otimes g_r); X(2,r))); (Z9; Z11))_j^1$$

$$= (Z10; Z11)_j^1$$

$$+ \Sigma_{k2}((Z7)_{k2}^1.(Z10; Z11)_j^{k2})$$

$$+ \Sigma_{k1,k2}((Z2)_{k1}^1.(Z7)_{k2}^{k1+1}.(Z10; Z11)_j^{k2})$$

By the definition of $Z7$ in terms of $Z5$, this is equal to

$$\Sigma_{k2}((Z5)_{k2}^1.(Z10; Z11)_j^{k2})$$

$$+ \Sigma_{k1,k2}((Z2)_{k1}^1.(Z5)_{k2}^{k1+1}.(Z10; Z11)_j^{k2})$$

which is equal to

$$\Sigma_{k2}(((\wedge \otimes I_n); (I \otimes Z2); Z5)_{k2}^1.(Z10; Z11)_j^{k2})$$

$$= ((\wedge \otimes I_n); (I \otimes Z2); Z5; (I \otimes g_2 \otimes \ldots \otimes g_r); Z10; Z11)_j^1$$

as required.

6.3.4 Case 2 - $i > 1$

Let $i > 1$. By definition of Link1,

$$((\wedge \otimes I_n); (I \otimes ((\wedge \otimes I_n); (I \otimes Z2); Z7; (I \otimes \ldots \otimes g_r); X(2,r))); (Z9; Z11))_j^i$$

$$= \Sigma_{k1}((I \otimes Z2); Z7; (I \otimes g_2 \otimes \ldots \otimes g_r); X(2,r))_{k1}^{i+1}.(Z9; Z11)_j^{k1+1}$$

$$= \Sigma_{k1,k2,k3,k4}((I \otimes Z2)_{k1}^{i+1}.(Z7)_{k2}^{k1}.(X(2,r))_{k3}^{k2}.(Z9)_{k4}^{k3+1}.(Z11)_j^{k4})$$

$$= \Sigma_{k1,k2,k4}((Z2)_{k1}^i.(Z7)_{k2}^{k1+1}.(Z10)_{k4}^{k2}.(Z11)_j^{k4})$$

by the definition of $Z9$ in terms of $Z10$

$$= \Sigma_{k1,k2}((Z2)_{k1}^i.(Z5)_{k2}^{k1+1}.(Z10; Z11)_j^{k2})$$

by the definition of $Z7$ in terms of $Z5$

$$= \Sigma_{k2}(((\wedge \otimes I_n); (I \otimes Z2); Z5)_{k2}^i.Link1(Z10; Z11)_j^{k2})$$

$$= ((\wedge \otimes I_n); (I \otimes Z2); Z5; (I \otimes g_2 \otimes \ldots \otimes g_r); Z10; Z11)_j^i$$

as required. This completes the proof of Corollary 1.

6.3.5 Corollary 2

Any morphism can be written as a term u in layered form such that if v is any term consisting of a series of consecutive layers of u, then v satisfies the partial ordering condition.

Proof

The proof is by induction on the minimal number of layers needed to represent the morphism as a layered term. Any term with just one layer satisfies the partial ordering condition.

Let u be a layered term $B(1); P(1); \ldots; P(r-1); B(r)$ representing the morphism such that the number

$$\Sigma_{i,j} Link1(u, i, j)$$

is minimal. By induction on the number of layers needed there is a layered term equal to $P(1); \ldots; P(r-1); B(r)$ such that any term consisting of a series of consecutive layers of the layered term satisfies the partial ordering condition. Substituting $P(1); \ldots; P(r-1); B(r)$ with this layered term in the expression for u will not increase $\Sigma_{i,j} Link1(u, i, j)$, so without loss of generality the layered term is just $P(1); \ldots; P(r-1); B(r)$.

If there is some pair (h, k) such that $k < r$, $\text{Link1}(B(1); P(1); \ldots; P(k), m, h) > 0$ for some m, and $\text{Link1}(B(k+1); P(k+1); \ldots; B(r), h, m) = 0$ for all m, then assume that k is maximal such that such a pair exists. We must have $\text{Link1}(B(k+1), h, m) = 0$ for all m. Then

$$\begin{aligned} & B(k); P(k); B(k+1) \\ = \;& B(k); (g_1 \otimes \ldots \otimes g_s); (I_{h-1} \otimes O \otimes I_{s-h}); B(k+1) \\ = \;& B(k); (g_1 \otimes g_{h-1} \otimes O \otimes g_{h+1} \ldots \otimes g_s); B(k+1) \\ = \;& B(k); (I_{h-1} \otimes O \otimes I_{s-h}); (g_1 \otimes \ldots \otimes g_s); B(k+1) \\ = \;& BB(k); P(k); B(k+1) \end{aligned}$$

where $BB(k)$ is the matrix of the same dimensions as $B(k)$ satisfying $BB(k)_{a,b} = B(k)_{a,b}$ if $b \neq h$ and F if $b = h$. Substituting $BB(k)$ for $B(k)$ in the term u leaves $\text{Link1}(v, i, j)$ and $\text{Link2}(v, i, j)$ no greater than before whenever v is a term consisting of consecutive layers of u. By performing this substitution for each h satisfying the condition given above, and using backwards induction on k, we may assume that there is no such pair (h, k).

If v is a term consisting of a series of consecutive layers of u and does not satisfy the partial ordering condition, then v must be the term $B(1); P(1); \ldots P(k)$ for some k. By Corollary 1 v can be rewritten as a term w in layered form where $Link1(w, i, j) \leq Link1(v, i, j)$ for all i, j and there is some pair (i, j) such that $Link1(w, i, j) < Link1(v, i, j)$. Since $Link1(v, i, j) > 0$ there is some m such that

$$Link1(B(k+1); P(k+1); \ldots; B(r), h, m) \; > \; 0$$

But this implies that

$$\Sigma_{i,j} Link1(w; B(k+1); P(k+1); \ldots; B(r), i, j) \; < \; \Sigma_{i,j} Link1(u, i, j)$$

which contradicts the hypothesis on u. Therefore no such v can exist. This completes the proof.

7 Normal form

This section gives a normal form for terms of the algebra, and proves that every term is equal to a term in normal form. There are six conditions for a term to be in normal form; the first two have already been introduced. It will turn out that each term is equal to a *unique* term in normal form. The definition of normal form is partly derived from the definition of normal form in [DMM 89], but in that paper a term may be equal to more than one term in normal form.

7.1 Definition of Normal form

A term u which is in *normal form* if and only if all the following conditions hold;

1. **(layering)** u is in layered form, $=B_1; P_1; \ldots; P_{r-1}; B_r$
2. **(partial ordering condition)** If v is any term consisting of a series of consecutive layers of u, v satisfies the partial ordering condition
3. **(maximal parallelism)** for each set of three consecutive layers

$$(h_1 \otimes h_2 \otimes \ldots \otimes h_n); A; (g_1 \otimes \ldots \otimes g_m)$$

   of u with A a morphism of B, if $g_i \neq I$ then there is some k such that $A_{k,i}$=T and $h_k \neq I$
4. **(independence of idles)** for each pair of consecutive layers $A; (g_1 \otimes \ldots \otimes g_m)$ with A a morphism of B, if $g_i = I$ then $|\{k : A_{k,i} = T\}| \leq 1$, and if $g_i = g_j = I$, $i \neq j$, then $|\{k : A_{k,i} = A_{k,j} = T\}| = 0$
5. **(decomposition of causal links)** Either $r = 1$, or $B_1, B_2, \ldots, B_{r-1}$ have at least one entry T in each column and $B_2, B_3, \ldots, B_r$ have at least one entry T in each row
6. **(ordering)** If $g_1 \otimes g_2 \otimes \ldots \otimes g_s = P_k$ for some k then for each $1 \leq i \leq s-1$ either $g_i = I$ or $g_i, g_{i+1} \in \Lambda$ with $g_i \prec g_{i+1}$, where $\prec$ is some fixed total ordering on Λ. Moreover, if $g_i = g_{i+1}$ for some $1 \leq i \leq s-1$, then the set $\{j : (B_k)_{j,i} = T\}$ either precedes or is equal to the set $\{j : (B_k)_{j,i+1} = T\}$ in lexicographical ordering.

7.2 Maximal parallelism

Every morphism can be expressed as a term satisfying conditions 1, 2 and 3 of normal form.

Proof

By Corollary 2, every morphism can be expressed as a term satisfying conditions 1 and 2. Suppose we have a term satisfying conditions 1 and 2, but not 3. Then there is a set of three consecutive layers

$$(h_1 \otimes h_2 \otimes \ldots \otimes h_n); A; (g_1 \otimes \ldots \otimes g_m)$$

of the term with A a morphism of B, such that for some i with $g_i \neq I$ there is no k such that $A_{k,i}$=T and $h_k \neq I$. By using permutations we may assume that $i = 1$ and $A_{k,1}$=T if and only if $1 \leq k \leq s$, for some s. We will show that the term

$$u1; (I_s \otimes h_{s+1} \otimes \ldots \otimes h_n); A; (g_1 \otimes \ldots \otimes g_m); u2$$

can be written in the form required, where the term satisfies conditions 1, and 2. Let $A1$ be the $s \times m$ matrix satisfying $\mathrm{A1}_{i,j} = A_{i,j}$ $(1 \leq i \leq s, 1 \leq j \leq m)$. Let $A2$ be the $((m-1)+(n-s)) \times (m-1)$ matrix satisfying

$$A2_{i,j} = (I_{m-1})_{i,j} \text{ if } 1 \leq i,j \leq m-1,$$

$$A2_{i,j} = A_{i+s-(m-1),j+1} \text{ if } m \leq i \leq m-1+n-s, 1 \leq j \leq m-1.$$

Now

$$\begin{aligned} &(I_s \otimes h_{s+1} \otimes \ldots \otimes h_n); A; (g_1 \otimes \ldots \otimes g_m) \\ &= (I_s \otimes h_{s+1} \otimes \ldots \otimes h_n); (A1 \otimes I_{n-s}); (I \otimes A2); (g_1 \otimes \ldots \otimes g_m) \\ &= (A1 \otimes h_{s+1} \otimes \ldots \otimes h_n); (g_1 \otimes (A2; (g_2 \otimes \ldots \otimes g_m))) \\ &= (A1 \otimes I_{n-s}); (I_m \otimes h_{s+1} \otimes \ldots \otimes h_n); (g_1 \otimes I_{n+m-s}); (I \otimes A2); (I \otimes \ldots \otimes g_m) \\ &= (A1 \otimes I_{n-s}); (g_1 \otimes I_{m-1} \otimes h_{s+1} \otimes \ldots \otimes h_n); (I \otimes A2); (I \otimes \ldots \otimes g_m) \end{aligned}$$

and by using this manipulation once for each i such that $A_{k,i} = T$ implies $h_k =$ I, the term can be written in the form required.

If the term is part of a layered expression, the rewriting may cause the maximum parallel condition to be untrue for the preceeding or following layer. However the rewriting strictly decreases the number $N(1)+N(2)+\ldots N(k)$, where there are k layers of the form $(g_1 \otimes \ldots \otimes g_n)$ and $\{j : 1 \leq j \leq n, g_i \neq I\} = N(i)$. Therefore if the rewriting is repeated for an appropriate subterm each time there is a layer which no longer satisfies the maximal parallism condition, the process must eventually terminate to leave a term satisfying conditions 1, 2, and 3.

7.3 Independence of idles

Every morphism can be expressed as a term satisfying conditions 1, 2, 3 and 4 of normal form.

Without loss of generality, using the coherence axiom, the morphism can be written in the form

$$u1; A; (I_b \otimes g_{b+1} \otimes \ldots \otimes g_n); u2$$

where A is a $m \times n$ matrix satisfying

$$\{k : A_{k,i} = T \text{ for some } 1 \leq i \leq b\} = \{1, 2, \ldots, a\},$$

$\{i : g_i = I\} = \{1, 2, \ldots, b\}$, $u1$ satisfies conditions 1, 2, 3 and 4, and the whole term satisfies conditions 1, 2 and 3.

Now

$$A; (I_b \otimes g_{b+1} \otimes \ldots \otimes g_n)$$

$$= A1; (A2 \otimes I_{n-b}); (I_b \otimes g_{b+1} \otimes \ldots \otimes g_n)$$

where $A1$ is the $m \times (a + n - b)$ matrix satisfying $A1_{i,j} = \mathrm{T}$ if and only if $i = j \leq a$ or ($j > a$ and $A_{i,j-a+b} = \mathrm{T}$), and $A2$ is the $a \times b$ matrix satisfying $A2_{i,j} = A_{i,j}$ for all $1 \leq i \leq a, 1 \leq j \leq b$

$$= A1; (I_a \otimes g_{b+1} \otimes \ldots \otimes g_n); (A2 \otimes I_{n-b})$$

and by repeating this manipulation finitely many times, and using induction on the number of layers of $u2$, the term can be rewritten in the form required.

7.4 Decomposition of causal links

Every morphism can be expressed as a term satisfying conditions 1, 2, 3, 4, and 5 of normal form.

Proof

Every morphism can be expressed as a term satisfying conditions 1, 2, 3 and 4. Suppose that

$$B(1); P(1); \ldots; P(r-1); B(r)$$

is such a term. We show by induction on r that this is equal to a term satisfying conditions 1, 2, 3, 4, and 5, with no more than $2r - 1$ layers.

Firstly, if $O_{m,n}$ is the $m \times n$ dimensional matrix with all entries F and $U1, U2$ are any morphisms of the category A from [k] to [m] and from [n] to [k] respectively then $O_{n,k}; U1 = U2; O_{k,m} = O_{n,m}$. This follows from the fact that $O_{m,n}$ is the sequential composition of a morphism from [m] to [0] with a morphism from [0] to [n], and the fact that [0] is both initial and final in A.

This result implies that if a term is in layered form and for some k all entries of B_k are F, then the morphism expressed by the term is equal to a matrix all of whose entries are F. Such a matrix is in normal form.

From now on we may assume that for each k the matrix B_k has at least one entry T. Then next stage of the proof is the case when $r = 2$. By the result on the decomposition of morphisms of B proved in an earlier section we have $B(1) = B_1(1); B_2(1)$, $B(2) = B_1(2); B_2(2)$ where

- $B_1(1)$ has at least one entry T in each column
- $B_2(1)$ has exactly one entry T in each row
- $B_1(2)$ has exactly one entry T in each column
- $B_2(2)$ has at least one entry T in each row
- if $(B_2(1))_{i,j} = (B_2(1))_{i+1,k} = T$ or $(B_1(2))_{j,i} = (B_1(2)_{j,i+1}) = T$ then $j < k$

If $B_2(1)$ and $B_1(2)$ are both square then $B(1); P(1); B(2)$ is already in the form required. If $P(1)$ has dimension 1×1 then $B_2(1)$ and $B_1(2)$ must both be square. Suppose $B_2(1)$ is not square. Then there are permutations $A1, A2$ and a morphism $B_3(1)$ of B such that $B_2(1) = A1; (\mu \otimes B_3(1)); A2$, where μ is the 0×1 matrix. Therefore

$$B(1); P(1); B(2) = B_1(1); A1; (\mu \otimes B_3(1)); A2; P(1); B(2)$$
$$= B_1(1); A1; (\mu \otimes B_3(1)); (g_1 \otimes \ldots \otimes g_k); A3; B(2)$$

for some permutation $A3$ and $g_1, \ldots, g_k \in \Lambda \cup \{I\}$, where $P(1)$ has dimension $k \times k$

$$= B_1(1); A1; ((\mu; g_1) \otimes (B_3(1); (g_2 \otimes g_3 \otimes \ldots \otimes g_k))); A3; B(2)$$
$$= B_1(1); A1; (\mu \otimes (B_3(1); (g_2 \otimes \ldots \otimes g_k))); A3; B(2)$$

since $[0]$ is an initial object,

$$= B_1(1); A1; B_3(1); (g_2 \otimes \ldots \otimes g_k); (\mu \otimes I_{k-1}); A3; B(2)$$
$$= C; (g_2 \otimes \ldots \otimes g_k); D$$

for some morphisms C, D of B. By induction on k this is equal to a term of the form required. If $B_1(2)$ is not square there is a similar proof.

Now suppose that $r > 2$ and use induction on r. Suppose we have a term

$$B(1); P(1); \ldots; B(r-1); P(r-1); B(r)$$

Without loss of generality

$$B(1); P(1); \ldots; P(r-2); B(r-1)$$

satisfies condition 5. In particular there is an entry T in each row of $B(r-1)$, and so in each row of $B(r-1); P(r-1); B(r)$. By the case $r = 2$, $B(r-1); P(r-1); B(r)$ is equal to a term $B1; P; B2$ satisfying conditions 1, 2, and 3, and since there is an entry T in each row of $B1, P, B2$ there is an entry T in each row and each column of $B1$. Therefore

$$B(1); P(1); \ldots P(r-2); B1; P; B2$$

satisfies conditions 1 and 5, and it is straightforward to check that the substitution of $B1; P; B2$ for $B(r-1); P(r-1); B(r)$ leaves the term still satisfying conditions 2, 3, and 4.

7.5 Ordering

Every morphism can be expressed as a term in normal form.

Proof

Given a term satisfying conditions 1, 2, 3, 4 and 5, a judicious choice of permutations to pre- and post-multiply the matrices B_i will transform it into an equal term in normal form. Therefore every morphism can be represented by a term in normal form.

8 Representation Theorem

We are at last ready to describe the bijection between the set of morphisms of the category A and the set CCH. First we will give a map U from the set of terms to the set CCH, such that any two terms which represent the same morphism have the same image under U. We prove that U is surjective, and finally that the restriction of U to the set of terms in normal form is injective. Since every morphism of the category A is equal to a term in normal form this proves that there is a bijection between the set of morphisms and CCH.

8.1 Definition - underlying partial order

If u is a term of the algebra, the *underlying partial order* $U(u)$ of u is the element of CCH defined structurally as follows;

- If u is a morphism of B from [n] to [m] then $U(u)$ is the element $((V_1 \cup \{\} \cup V_3, \leq, \ell), \beta, \gamma)$ for which $|V_1| = n$, $|V_3| = m$, and if $x, y \in V$ then $x \leq y$ if and only if $x \in V_1, y \in V_3$ and $u_{\beta(x),\gamma(y)} = T$
- If $t \in \Lambda$ then $U(t)$ has three elements $v1, v2, v3$ labelled s,t,s respectively such that $vi \leq vj$ if and only if $i \leq j$
- $U(u1; u2)$ is the element $U(u1); U(u2)$
- $U(u1 \otimes u2)$ is the element $U(u1) \otimes U(u2)$

By the representation of the morphisms of B, $U(u)$ is well defined whenever u is a morphism of B. It is straightforward to check that the left hand and right hand side of the axioms introduced in the definition of the category A have the same underlying partial orders. Therefore two terms which are equal in the algebra have the same underlying partial order.

8.2 Lemma - every element of CCH is an underlying partial order

Given any element of CCH whose internal elements are labelled with the set Λ there is some term u of the algebra whose underlying partial order is the element.

Proof

Use induction on the number of objects in the element of CCH which are neither minimal nor maximal. If all the objects are either minimal or maximal then the element is $U(u)$ for the matrix u which has an entry T at place (i, j) if the minimal object labelled i is comparable with the maximal object labelled j, and has an entry F at place (i, j) otherwise.

Suppose C is an element $((V1 \cup V2 \cup V3, \leq, \ell), \beta, \gamma)$ of CCH with V2 nonempty, and that any element of CCH with fewer than $|V2|$ objects which are neither maximal nor minimal is $U(u)$ for some term u. Pick an object x which is minimal in V2. Let D be the element

$$(((V1 \cup \{\overline{x}\}) \cup (V2 \backslash \{x\}) \cup V3, \prec, \ell_D), \beta_D, \gamma)$$

of CCH where

- $\overline{x} \notin V1 \cup V2 \cup V3$,
- $y \prec z$ if and only if $y \leq z$ or $y = \overline{x}$ and $x \leq z$,
- $\ell_D(y) = \ell(y)$ if $y \in V2 \backslash \{x\}$,
- $\beta_D(y) = \beta(y)$ if $y \neq \overline{x}$, $\beta_D(\overline{x}) = |V1|+1$.

D has fewer objects which are neither maximal nor minimal than C, so D $= U(u)$ for some term u. Let E be the $n \times (n+1)$ matrix which has entry T at place (i, j) if $i = j \leq n$ or $j = n+1, y \leq x$ where $y \in V1$ and $\ell(y) = i$, and entry F at place (i, j) otherwise. Then C $= U(E; (I_n \otimes \ell(x)); u)$, which is the underlying partial order of a term of the algebra, as required.

8.3 If $U(u1) = U(u2)$ then $u1 = u2$

This is the second part of the Representation Theorem. The main work in proving this has already been done, in the proof of normal form. Without loss of generality $u1$ and $u2$ are in normal form. We will show that they are the same term in normal form, by induction on the number of layers of $u1$. The proof follows [DMM 89]. If $u1$ is a morphism of B then all elements of $U(u1)$ are either maximal or minimal, and so $u2$ is a morphism of B, and $u1 = u2$ by the representation theorem for morphisms of B.

Suppose that $u1$ is $B(1); P(1); \ldots; P(r-1); B(r)$ in layered form. Let L(1) be the set of elements of $U(u1)$ which are not minimal but which are not greater than any non-minimal element of $U(u1)$. Let $P(1) = g_1 \otimes \ldots \otimes g_n$. It is easy to see (using the condition of maximal parallelism) that there is a bijection between the set $\{j : g_j = t \in \Lambda\}$ and the set of elements of $L(1)$ labelled t, such that an element e in $L(1)$ labelled t is greater than e_i, the minimal element of $U(u1)$ for which $\beta(e_i) = i$, if and only if $B(1)_{i,j} = \mathrm{T}$ where j is the corresponding element of $\{j : g_j = t\}$.

By the partial ordering condition, $Link2(B(1); P(1), i, j) > 0$ for some j if and only if there is some element e of $U(u1)$ which is not in the set $L(1)$, which is greater than e_i, and there is no element f of $U(u1)$ with $e > f > e_i$. Hence the set

$$\{i : \text{ there is some } j \text{ such that } B(1)_{i,j} = T, g_j = I\}$$

is determined by $U(u1)$. By the condition on independence of idles, the number of elements of this set is equal to $|\{i : g_i = I\}|$. Therefore the multiset $\{g_i : 1 \leq i \leq n\}$ is determined by $U(u1)$, and hence the first part of the ordering condition ensures that $P(1)$ is determined by $U(u1)$.

Now for each $t \in \Lambda$ the set $\{\{i : B(1)_{i,j} = T\} : g_j = t\}$ is determined by $U(u1)$, so the second part of the ordering condition ensures that for each j with $g_j \in \Lambda$ the set $\{i : B(1)_{i,j} = T\}$ is determined by $U(u1)$. Suppose now that $g_j = I$. The condition on independence of idles and the second part of the ordering condition force the set $\{i : B(1)_{i,j} = T\}$ to be the singleton whose element is the j^{th} smallest element of

$$\{i : \text{ there is some } j \text{ such that } B(1)_{i,j} = T, g_j = I\}$$

Hence $B(1)$ is determined by $U(u1)$.

Suppose that $U(u1)$ is $((V_1 \cup V_2 \cup V_3, \leq, \ell), \beta, \gamma)$. There is only one element C of CCH satisfying $U(B(1); P(1)); C = U(u1)$. This is the element $C =$

$$((\{f_1, \ldots, f_n\} \cup V_2 \backslash L(1) \cup V_3 \backslash L(1), \prec, \ell), \beta, \gamma 1)$$

where $f_1, \ldots, f_n$ are not in $V_1 \cup V_2 \cup V_3$,

$$f \prec e, e, f \in S \text{ if and only if}$$

$$(e, f \in V_2 \cup V_3, f < e \text{ or } f = f_j \text{ for some } j, B(1)_{i,j} = T, e \in V_2 \cup V_3, e_i < e)$$

$$\beta 1(f_j) = j \text{ for all } j$$

The term $u2$ is the term $B(1); P(1); u3$ where $U(u3)$=$U(B(2); P(2); \ldots; B(r))$=$C$, so by induction on r, $u3$ is the term $B(2); P(2); \ldots; B(r)$ and $u1$, $u2$ are the same term, as required.

9 Summary

The algebra CCH is the algebra of morphisms of the category A. A is the symmetric strict monoidal category generated by a non-identity object [1], and morphisms $t : [1] \rightarrow [1]$ $(t \in \Lambda)$, $\vee : [1] \otimes [1] \rightarrow [1]$, $\wedge : [1] \rightarrow [1] \otimes [1]$, under the following set of axioms.

- The identity object [0] is initial and final
- $\wedge; (I \otimes \wedge) = \wedge; (\wedge \otimes I)$ where I is the identity morphism on [1]
- $(I \otimes \vee); \vee = (\vee \otimes I); \vee$
- $\wedge; \vee = \mathrm{I}$
- $\vee; \wedge = (\wedge \otimes \wedge); (I \otimes X \otimes I); (\vee \otimes \vee)$ where $X : ([1] \otimes [1]) \rightarrow ([1] \otimes [1])$ is the symmetry isomorphism
- $\wedge; (I \otimes \epsilon) = \mathrm{I}$ where $\epsilon : [1] \rightarrow [0]$
- $(I \otimes \mu); \vee = \mathrm{I}$ where $\mu : [0] \rightarrow [1]$
- $\wedge; X = \wedge$
- $X; \vee = \vee$
- $t; \wedge = \wedge; (I \otimes t); (I \otimes \wedge); (\vee \otimes I)$ whenever $t \in \Lambda$

The smallset subcategory of A which is a symmetric strict monoidal category and contains $\vee$, $\wedge$ is B. The algebra of morphisms of B is the algebra of bipartite histories, and is isomorphic to the algebra of non-negative-dimensional matrices over the boolean algebra with two elements.

10 Applications to Petri Nets and CCS

10.1 A connection with Petri Nets

Suppose there is a Petri Net with set of places P, and set of events E, such that there are no events whose multisets of preplaces or postplaces are empty.

Form the algebra freely generated under $\_;\_$ and $\otimes$ by the set

$$\{gen(e) : e \in E\} \cup \{gen(p) : p \in P\} \cup \{gen(p,q) : p,q \in P \text{ (possibly } p = q)\}$$

where
$g1 \otimes g2$ is always defined;
$g1; g2$ is defined if and only if the string $maxlabels(g1)$ is equal to the string $minlabels(g2)$;
$minlabels(gen(e))$ is some ordering of the multiset of preplaces of e;
$maxlabels(gen(e))$ is some ordering of the multiset of postplaces of e;
$minlabels(gen(p)) = maxlabels(gen(p)) = p$;
$minlabels(gen(p,q)) = p.q$;
$maxlabels(gen(p,q)) = q.p$;
$minlables(g1 \otimes g2) = minlabels(g1).minlabels(g2)$;
$maxlables(g1 \otimes g2) = maxlabels(g1).maxlabels(g2)$;
$minlables(g1; g2) = minlables(g1)$, $maxlables(g1; g2) = maxlables(g2)$.

There is a partial algebra homomorphism from this algebra to CCH (with E as the alphabet Λ) sending $gen(p)$ to I, $gen(p,q)$ to $X(2,2)$, and $gen(e)$ to $\vee_{r(e)}; t_e; \wedge_{s(e)}$, where $r(e) =$ |multiset of preplaces of e| and $s(e) =$ |multiset of postplaces of e|. Take the set of labelled partial orders obtained by removing the maximal and minimal elements of the underlying partial order of elements in the image of this homomorphism. This is exactly the set of all partial orders of events which can be obtained by computations in the Petri Net. Moreover, there is a bijection between the set of all partial orders of events obtained by computations in the Petri net and the expressions

$$I_{n1}; P_1; B_2; \ldots; P_{r-1}; I_{n2}$$

where

$$B_1; P_1; B_2; \ldots; P_{r-1}; B_r$$

is an expression in normal form equal to an element in the image, and $n1, n2$ are the number of generators in $\{t_e : e \in E\} \cup \{I\}$ in P_1, P_{r-1} respectively.

10.2 A model for CCS

In [FMM 90] the category A whose set of morphisms is CCH is given the structure of a CCS model. The operations of CCH are defined on objects of the category, and the morphisms represent proofs of computations from one object to another. See the reference for details.

11 References

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