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The BMAP/GI/1 Queue with Server Set-Up Times and Server Vacations

Josep M. Ferrandiz

Hewlett-Packard Laboratories
Filton Road, Stoke Gifford
Bristol, BS12 6QZ
United Kingdom

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Using Palm-martingale calculus, we derive the workload characteristic function and queue length moment generating function for the BMAP/GI/1 queue with server vacations. In the queueing system under study, the server may start a vacation at the completion of a service or at the arrival of a customer finding an empty system. In the latter case we will talk of a server set-up time. The distribution of a set-up time or of a vacation period after a departure leaving a non-empty system behind is conditionally independent of the queue length and workload. Furthermore, the distribution of the server set-up times may be different from the distribution of vacations at service completion times. The results are particularized to the M/GI/1 queue and to the BMAP/GI/1 queue (without vacations).

1. Introduction

A class of versatile point processes was introduced in [10] by M. Neuts and further extended by Lucantoni in [8]. The first class of point processes has been referred to as N-processes in the queueing literature and the second one to batch markovian arrival processes (BMAP). They are a wide generalization of Poisson processes and they encompass a large class of numerically tractable point processes as special cases. For instance, in [6], Markov modulated Poisson processes, a subclass of batch markovian arrival processes, were used to model the superposition of voice sources. The N/GI/1/∞ queue was studied in [13] and the results therein were extended to the case of batch markovian arrival processes in [8]. In all cases, the analysis was based on the matrix geometric methodology introduced by M. Neuts in [11, 12].

Queueing systems with vacations have been widely studied. A complete survey article can be found in [5]. In [2], a pseudo-conservation law for a multiclass queueing system with vacations is studied and [7] contains the analysis of the M/G/1 queueing system with finite buffer under more sophisticated vacation schedules.

The contributions of this paper can be found in the approach used and in the results obtained in the analysis of a queueing system under a very general vacation schedule. In fact, the analysis technique used allows the derivation of these new results with very little extra effort.

The approach used to analyze the queueing system with vacations under consideration relies on the so-called Palm-martingale calculus [1, 4] instead of the usual matrix approach [11, 12]. The analysis of a stochastic system via Palm-martingale calculus can be divided into three parts. First, one establishes sample path relations between quantities of interest. Then, one takes expectations in the sample path equalities derived. Palm probabilities provide the natural link between the time and event averages typically involved in expectations of sample path equalities [4]. The Palm probability framework avoids the manipulation of empirical averages and limiting distributions, thus reducing this step to mere computations. One typically obtains a relationship between Palm probabilities and the stationary probability. Finally, with the help of martingale calculus, one relates the Palm probabilities obtained in the previous step to the stationary probability. Hence, the basic element needed to analyze a stochastic system in equilibrium via Palm-martingale calculus is an underlying stationary point process admitting a stochastic intensity. In contrast with the classical matrix approach, this is the only mathematical structure required. (Of course, one still has to find the appropriate point processes to carry out the analysis). Therefore, one should expect that some currently open problems in queueing theory can be successfully solved via Palm-martingale calculus. In order to illustrate the methodology, we rederive some results obtained in [13] and [9] via Palm-martingale calculus.

The second contribution of this paper is the extension of results contained in [13] and [9]. The vacation schedule considered here is more general than the one in [9] since we allow server vacations to start when a customer joins an empty system or at departures that leave behind a non-empty system. In the former case the server vacation will be called a server set-up time. The vacation schedule considered here does not allow service interruption. In the model under study, the vacation length distribution can be different in both cases and is independent of the queue length and system workload. Our problem formulation allows also the server to take consecutive vacations. Furthermore, the input process considered here is more general than the one in [9] since it allows batch arrivals. However, it is hard to compare specific results since the vacations analyzed in [9] can start only at the end of a busy period whereas in this paper, they start at departures that do not start a busy period or at arrivals that start a busy period (set-up times). In that sense, the residual vacation time seen by an arrival finding an empty system in the vacation schedule analyzed in [9] corresponds to the server set-up times of this paper. On the other hand, this paper does not address computational issues and factorization issues as respectively done in [9, 8].

This paper is divided as follows. The notation and results used related to Palm-martingale calculus can be found in [1] and [3]. Section 2 contains a brief summary of results needed in Palm probability and martingales. In section 3, BMAP-processes are introduced emphasizing their properties from a Palm-martingale calculus viewpoint. Section 4 contains the description of the BMAP/GI/1 queueing system with server vacations analyzed. In section 5 we obtain formulas of relevant distributions at event times, *i.e.*, arrival times and departure times. Some of these results appear elsewhere [13, 9] and are derived here to illustrate Palm-martinagle calculus. In section 6 we compute the moment generating function of the queue length vector. In section 7 we derive the characteristic function of the the workload vector. The results obtained are particularized to the BMAP/GI/1 queue without vacations in section 8. In section 9, the results obtained are also particularized for the M/GI/1 queueing system with server vacations. The conclusions are contained in section 10.

2. Overview of Palm Probability and Martingales

In this section, we present a brief summary of results in Palm probability and martingales used in this paper.

We introduce the notation related to Palm probability. A formal presentation on this subject and further details can be found in [1].

On a probability space $(\Omega, \mathfrak{F})$ we consider a stationary stochastic process $Z(t)$, $t \in \mathbb{R}$, a measurable set B, and a simple point process $N = (T_n)_{n \in \mathbb{Z}}$, *i.e.*, such that $T_n < T_{n+1}$, with the standard labelling convention, $T_n \leq 0$, $n \leq 0$ and $T_n > 0$, $n > 0$.

For each $n \in \mathbb{Z}$, define the operator θ_{T_n} which associates to a trajectory $\omega \in \Omega$, a new trajectory $\theta_{T_n}\omega$ obtained by shifting ω by n points of N, *e.g.*, $Z(T_m, \theta_{T_n}\omega) = Z(T_{m+n}, \omega)$. It is customary to use the notations $Z(T_n) = Z(0) \circ \theta_{T_n}$ and $\mathbf{1}_{\theta_{T_n}\omega \in B} = \mathbf{1}_B \circ \theta_{T_n}$. In order to simplify the notation, we let,

$$\begin{aligned} \int_{(0,t]} Z(s)N(ds) &= \sum_{n \in \mathbb{Z}} Z(T_n)\mathbf{1}_{\{0 < T_n \leq t\}}, \\ \int_{(0,t]} \mathbf{1}_B \circ \theta_s N(ds) &= \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{\theta_{T_n}\omega \in B\}}\mathbf{1}_{\{0 < T_n \leq t\}}, \end{aligned}$$

and in particular, $N_{(0,t]} = \int_{(0,t]} N(ds)$.

If $\lambda_N = \mathsf{E}[N_{(0,1]}] < \infty$, the Palm probability of B with respect to N is defined as,

$$\mathsf{P}_N^0(B) = \frac{\mathsf{E}\left[\int_{(0,1]} \mathbf{1}_B \circ \theta_s N(ds)\right]}{\mathsf{E}\left[\int_{(0,1]} N(ds)\right]}.$$

Similarly, the Palm expectation of $Z(\cdot)$ is given by,

$$\mathsf{E}_N^0[Z(0)] = \frac{\mathsf{E}\left[\int_{(0,1]} Z(s)N(ds)\right]}{\mathsf{E}\left[\int_{(0,1]} N(ds)\right]}.$$

The Palm probability is invariant under θ_{T_n}, *i.e.*, $\mathsf{P}_N^0(B) = \mathsf{P}_N^0(\theta_{T_n}B)$.

If $N' = (T'_n)_{n \in \mathbb{Z}}$ is another simple point process (with the standard labelling convention), the so-called exchange formula relates the Palm probabilities associated to two point processes,

$$\lambda_N \mathsf{E}_N^0[Z(0)] = \lambda_{N'}\mathsf{E}_{N'}^0\left[\int_{(0,T'_1]} Z(s)N(ds)\right],$$

where $\lambda_{N'} = \mathsf{E}[N'_{(0,1]}]$.

For the notation and results related to martingales the reader is referred to [3]. The main result needed in martingale theory is presented in the sequel. Let $\mathfrak{F}_t$ be a filtration of the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and N a stationary point process adapted to $\mathfrak{F}_t$ with $(\mathsf{P}, \mathfrak{F}_t)$–stochastic intensity λ_t and rate $\lambda = \mathsf{E}[N_1]$. Then, if $Z(t)$ is a non-negative

$\mathfrak{F}_t$–predictable process, we have,

$$\lambda t \mathsf{E}_N^0[Z(0)] = \mathsf{E}\Big[\int_0^t \lambda_s Z(s)ds\Big].$$

In fact, this last equation is the link between time and event averages mentionned earlier [4] and is nothing but the definition of stochastic intensity of a point process combined with the definition of Palm probability associated to that point process.

3. Palm-martingale Analysis of BMAP

Batch markovian arrival processes (BMAP) were introduced by M. Neuts in [10] and further extended and refined by Lucantoni in [8]. In this section we derive the main properties of BMAP via Palm-martingale calculus.

Let $(X_t, \mathfrak{F}_t^X)$ be an irreducible positive recurrent continuous time Markov chain on a finite state space $E = \{1, \ldots, L\}$ with equilibrium distribution $\boldsymbol{\pi}$. The process X_t will be referred to as the phase of the batch markovian arrival process. The jump times of X_t (the jumps can be from a state into itself) form a point process $(\tilde{T}_n)_{n\in\mathbf{Z}}$. We associate to this point process a sequence of marks $(\tilde{U}_n)_{n\in\mathbf{Z}}$ with values on $\mathbf{N}$.

In this paper, the marked point process $\tilde{A} = (\tilde{T}_n, \tilde{U}_n)_{n\in\mathbf{Z}}$ will be the used to generate the input process to a single server queue with server vacations and set-up times. If $\tilde{U}_n > 0$, the mark $\tilde{U}_n$ will represent the size of the batch of customers arriving at time $\tilde{T}_n$. If $\tilde{U}_n = 0$, no customer will join the queue at time $\tilde{T}_n$ and thus, $\tilde{T}_n$ will not be an arrival time to the queueing system. The marked point process $\tilde{A} = (\tilde{T}_n, \tilde{U}_n)_{n\in\mathbf{Z}}$ is such that,

1. $\tilde{A}$ admits a $(\mathsf{P}, \mathfrak{F}^X)$– stochastic intensity λ_{X_t}.
2. The marks of $\tilde{A}$ and the phase are conditionally independent of the arrival time given the phase just before the arrival. Their conditionnal moment generating function is given by,

$$\mathsf{E}_{\tilde{A}}^0[z^{\tilde{U}_0}\mathbf{1}(X_0 = j)|X_{0^-} = i] = p_{ij}(z),$$

   with $\sum_j p_{ij}(1) = 1$. Note that, as opposed to [8], we allow transitions from a state into itself regardless of the value of $\tilde{U}_n$.

We briefly examine some properties of $\tilde{A} = (\tilde{T}_n, \tilde{U}_n)_{n\in\mathbf{Z}}$ needed throughout the paper using the Palm-martingale methodology. Let,

$$N_t = \sum_{n\in\mathbf{Z}} \tilde{U}_n \mathbf{1}(0 < \tilde{T}_n \le t),$$

be the aggregated batch count up to time t, *i.e.*, the total number of arrivals in $]0,t]$. Consider the sample path equality,

$$\begin{aligned} z^{N_t}\mathbf{1}(X_t = j) &= \mathbf{1}(X_0 = j) + \sum_{n\in\mathbf{Z}} \mathbf{1}(0 < \tilde{T}_n \le t)\Big(z^{N_{\tilde{T}_n}}\mathbf{1}(X_{\tilde{T}_n} = j) - z^{N_{\tilde{T}_n^-}}\mathbf{1}(X_{\tilde{T}_n^-} = j)\Big) \\ &= \mathbf{1}(X_0 = j) + \int_{]0,t]} \Big(z^{N_s}\mathbf{1}(X_s = j) - z^{N_{s^-}}\mathbf{1}(X_{s^-} = j)\Big)\tilde{A}(ds), \end{aligned}$$

where, in the above equations, if $Z(t)$, $t \in \mathbf{R}$, is a stochastic process, $Z(\tilde{T}_n^-)$ and $Z(s^-)$ denote the value of $Z(\cdot)$ just before a point of $\tilde{A}$. Let $Y_{ij}(t) = \mathsf{E}[\mathbf{1}(X_0 = i)\mathbf{1}(X_t = j)z^{N_t}]$. Multiplying the above sample path equality by $\mathbf{1}(X_0 = i)$ and taking expectations, yields after standard martingale manipulations,

$$\begin{aligned} Y_{ij}(t) &= \mathbf{1}(i=j)\pi_j + \mathsf{E}\Big[\mathbf{1}(X_0 = i)\int_{]0,t]}\Big(z^{N_s}\mathbf{1}(X_s = j) - z^{N_{s^-}}\mathbf{1}(X_{s^-} = j)\Big)\tilde{A}(ds)\Big] \\ &= \mathbf{1}(i=j)\pi_j + \mathsf{E}\Big[\mathbf{1}(X_0 = i)\int_{]0,t]} z^{N_{s^-}}\Big(z^{\tilde{U}_s}\sum_l \mathbf{1}(X_{s^-} = l)\mathbf{1}(X_s = j) - \\ &\qquad -\mathbf{1}(X_{s^-} = j)\Big)\tilde{A}(ds)\Big] \\ &= \mathbf{1}(i=j)\pi_j + \mathsf{E}\Big[\mathbf{1}(X_0 = i)\int_0^t \Big(\sum_l z^{N_s}\mathbf{1}(X_s = l)\lambda_l p_{lj}(z) - z^{N_s}\mathbf{1}(X_s = j)\lambda_j\Big)ds\Big]. \end{aligned}$$

Defining the vector $Y_i(t) = (Y_{ij}(t))_{j\in E}$, the previous equation can be rewritten in matrix form as,

$$Y_i(t) = Y_i(0) + \int_0^t Y_i(s)\boldsymbol{D}(z)ds, \tag{3.1}$$

where the matrix $\boldsymbol{D}(z) = (D_{ij}(z))_{i,j\in E}$ has entries,

$$D_{ij}(z) = \begin{cases} \lambda_i p_{ij}(z), & \text{if } i \ne j, \\ \lambda_i\big(p_{ii}(z) - 1\big) & \text{if } i = j, \end{cases}$$

and for $j \in E$, $Y_{ij}(0) = \mathbf{1}(i=j)\pi_i$. Solving the integral equation (3.1),

$$Y_i(t) = Y_i(0)\exp\{\boldsymbol{D}(z)t\},$$

i.e.,

$$Y_{ij}(t) = \mathsf{E}[\mathbf{1}(X_0 = i)\mathbf{1}(X_t = j)z^{N_t}] = \pi_i \exp\{\boldsymbol{D}(z)t\}\Big|_{ij},$$

where here and in the sequel, the $(i,j)-$th entry of a matrix $\boldsymbol{B}$ will be denoted by $\boldsymbol{B}|_{ij}$.

A similar notation will be used for vectors. Therefore after conditioning, [10, 8],

$$\mathsf{E}[\mathbf{1}(X_t = j)z^{N_t}|X_0 = i] = \exp\{\boldsymbol{D}(z)t\}\Big|_{ij}, \tag{3.2}$$

and adding with respect to i and j,

$$\mathsf{E}[z^{N_t}] = \boldsymbol{\pi}\exp\{\boldsymbol{D}(z)t\}\boldsymbol{e}, \tag{3.3}$$

where here and in the sequel, $\boldsymbol{e}$ will denote the all-ones column vector.

Letting $z = 1$ in equation (3.2) implies,

$$\mathsf{P}(X_t = j|X_0 = i) = \exp\{\boldsymbol{D}(1)t\},$$

for all $t \geq 0$, *i.e.*, $\boldsymbol{D}(1)$ is the intensity matrix of the Markov chain X_t so that $\boldsymbol{D}(1)\boldsymbol{e} = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{D}(1) = \mathbf{0}$.

The customer arrival rate is, [8],

$$\lambda_N = \mathsf{E}[N_1] = \boldsymbol{\pi}\frac{d}{dz}\exp\{\boldsymbol{D}(z)\}\Big|_{z=1}\boldsymbol{e} = \boldsymbol{\pi}\boldsymbol{D}'(1)\boldsymbol{e}.$$

In the sequel we will let $\boldsymbol{D} = \boldsymbol{D}(1)$, $\boldsymbol{D}' = \boldsymbol{D}'(1)$, $\boldsymbol{D}_0 = \boldsymbol{D}(0)$ and to avoid trivialities, we will assume that $\boldsymbol{D}_0 \neq \boldsymbol{D}$.

We define the simple marked point process $A = (T_n, U_n)_{n\in\mathbf{Z}}$ by deleting the points of $\tilde{A}$ with zero batch size. Thus, $A = (T_n, U_n)_{n\in\mathbf{Z}}$ is the actual input process to the queue and the number of arrival times during $]0, t]$ is,

$$A_t = \sum_{n\in\mathbf{Z}} \mathbf{1}(0 < T_n \leq t) = \sum_{n\in\mathbf{Z}} \mathbf{1}(\tilde{U}_n > 0)\mathbf{1}(0 < \tilde{T}_n \leq t).$$

We evaluate λ_A, the rate of A, and $\mathsf{E}_A^0[U_0]$. Since [1],

$$\mathsf{P}(T_1 > t) = \lambda_A \int_t^\infty \mathsf{P}_A^0(T_1 > s)ds,$$

and,

$$\mathsf{P}(T_1 > t) = \mathsf{E}[\mathbf{1}(N_t = 0)] = \boldsymbol{\pi}\exp\{\boldsymbol{D}_0 t\}\boldsymbol{e},$$

Combining the last two equations, taking derivatives and letting $t = 0$,

$$\lambda_A = \mathsf{E}[A_1] = -\boldsymbol{\pi}\boldsymbol{D}_0\boldsymbol{e}.$$

Finally, from the definition of the Palm probability $\mathsf{P}_A^0(\cdot)$,

$$\mathsf{E}_A^0[U_0] = \frac{\lambda_N}{\lambda_A} = -\frac{\boldsymbol{\pi D' e}}{\boldsymbol{\pi D_0 e}}. \tag{3.4}$$

Note that, since $U_0 \geq 1$, P_A^0 a.s, we have $\mathsf{E}_A^0[U_0] \geq 1$.

4. The BMAP/G/1 Queue with Vacations

In the sequel, we assume that the queueing system under study is in equilibrium. The queue length and system workload at time t are will be denoted by Q_t and $W(t)$.

4.1 Notation

As mentioned in section 3, the sequence of arrival times to the queue is denoted by $T_n, n \in \mathbb{Z}$, and its associated counting process by A. The sequence of departure times will be denoted by $(T'_n)_{,n\in\mathbb{Z}}$ and its associated counting process is denoted by D. Thus, D_t is the number of departures in $]0,t]$, *i.e.*,

$$D_t = \sum_{n\in\mathbb{Z}} \mathbf{1}(0 < T'_n \leq t),$$

and the departure rate is $\lambda_N = \mathsf{E}[D_1]$.

Services start at times S_n, $n \in \mathbb{Z}$. The sequence of service demands $(\sigma_n)_{n\in\mathbb{Z}}$ is i.i.d. with distribution $H(\cdot)$ such that $H(0) = 0$. We also let $\phi(u) = \mathsf{E}[\exp\{-u\sigma_0\}]$.

Let $B(t)$ be the "busy server indicator", *i.e.*, $B(t) = 1$ if the server is busy at time t and $B(t) = 0$ if it is idle. The server is idle either when it is on vacation or when the system is empty .

Let $\mathcal{R}(t)$ be the residual service time at time t. If the system is empty at time t, then $\mathcal{R}(t) = 0$. If $B(t) = 0$ and the system is not empty, $\mathcal{R}(t)$ is equal to a full service. If $B(t) = 1$, it is equal to $t - T'_1(\theta_t)$, where θ_t is the shift operator, [1]. Thus,

$$\mathcal{R}(t) = \begin{cases} 0 & \text{if } Q_t = 0 \\ \sigma & \text{if } B(t) = 0 \text{ and } Q_t > 0 \\ t - T'_1(\theta_t) & \text{if } B(t) = 1 \end{cases}$$

where σ denotes an independent random variable distributed according to $H(\cdot)$. The sequence of jump times discontinuities of $\mathcal{R}(t)$ is denoted by $(T''_n)_{n\in\mathbb{Z}}$ and its associated counting process by Δ. Thus, the number of jump discontinuities of $\mathcal{R}(\cdot)$ in the time

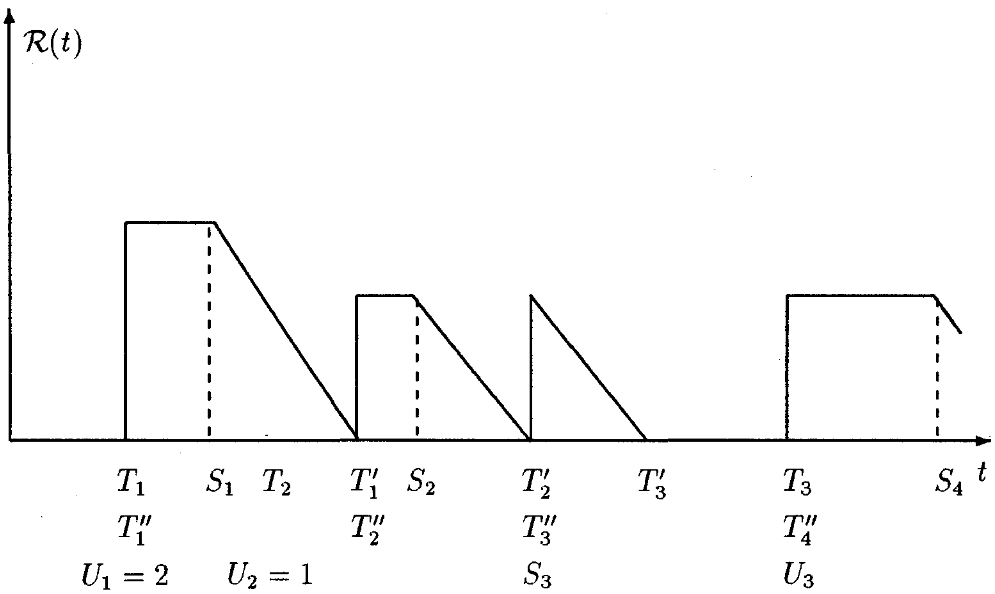

Figure 4.1: The residual service time

interval $]0, t]$ is,

$$\Delta_t = \sum_{n \in \mathbb{Z}} \mathbf{1}(0 < T''_n \leq t).$$

Note that the rate of Δ is equal to the customer input rate, *i.e.*, $\lambda_N = \mathsf{E}[N_1] = \mathsf{E}[\Delta_1] = \mathsf{E}[D_1]$. Since Δ has jump discontinuities either at the arrival times with non-zero group size finding an empty system or at the departure times leaving a non-empty system behind, we have, [1],

$$\lambda_N \mathsf{E}^0_\Delta[f] = \lambda_A \mathsf{E}^0_A[\mathbf{1}(Q_{0^-} = 0)f] + \lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)f], \tag{4.1}$$

where f is a random variable. The notation introduced in this section is shown in figure 4.1.

4.2 The Server Vacations

We assume that services cannot be interrupted by a server vacation. We define,

$$\gamma_{V_A}(u) = \mathsf{E}^0_A[\exp\{-uS_1\}|Q_{0^-} = 0] \tag{4.2}$$
$$\gamma_{V_D}(u) = \mathsf{E}^0_D[\exp\{-uS_1\}|Q_{0^-} > 0]. \tag{4.3}$$

Thus, $\gamma_{V_A}(\cdot)$ is the characteristic function of the server set-up time. Similarly, $\gamma_{V_D}(\cdot)$ is the characteristic function of the vacation duration given that a departure leaves a non-empty system behind. Moreover, we assume that if a batch of customers arrives to an empty system, only one customer in the batch generates a server set-up time. The independence assumptions on $\gamma_{V_D}(\cdot)$ show that the server takes a vacation after a departure leaving behind a non-empty system with probability $\gamma_{V_D}(0)$ and starts a new service with probability $1-\gamma_{V_D}(0)$. A similar statement holds for set-up times.

The definitions of $\gamma_{V_A}(\cdot)$ and $\gamma_{V_D}(\cdot)$ are quite general. In either case, we do not exclude the possibility that $S_1 = 0$ with positive probability. For instance, if $S_1 = 0$ with probability one, the system under study becomes the BMAP/GI/1 queue analyzed in [8]. Another important case is obtained if just after a departure that leaves behind a non-empty system, we have $S_1 = 0$ with probability one, and when a customer joins an empty system, we have $S_1 > 0$ with probability one, then we are considering a system that only requires server set-up times, *i.e.*, it only starts server vacations when a customer joins an empty queue. This type of schedule is similar to the one analyzed in [9]. The server set-up times of this paper correspond to the residual vacation time at the beginning of a busy period for the vacations analyzed in [9]. Consecutive vacations, are also covered by the present formulation if the number of consecutive vacations or set-up times taken by the server is an independent random variable with moment generating function $P_i(\cdot)$, $i = A, D$. It is easy to show then that if $v_i(\cdot)$, $i = A, D$, is the moment generating function of a single vacation period, then it suffices to take,

$$\gamma_{V_i}(u) = P_i\big(v_i(u)\big), \quad i = A, D,$$

to analyze the case of consecutive vacations.

In order to simplify future equations we also define,

$$\gamma_A(u) = \mathsf{E}_A^0[\exp\{-uT_1'\}|Q_{0^-} = 0] = \gamma_{V_A}(u)\phi(u) \tag{4.4}$$

$$\gamma_D(u) = \mathsf{E}_D^0[\exp\{-uT_1'\}|Q_{0^-} > 0] = \gamma_{V_D}(u)\phi(u) \tag{4.5}$$

4.3 The System Utilization

We evaluate now the system utilization, $\rho = \mathsf{E}[\mathbf{1}(Q_0 > 0)]$. From a utilization viewpoint, the n-th customer brings a load equal to $\sigma_n + V_n$ where V_n is the duration of the server vacation before the service of the n-th customer starts. Hence, ρ satisfies the equation,

$$\begin{aligned}\rho &= \mathsf{E}[\mathbf{1}(Q_0 > 0)] \\ &= \lambda_N \mathsf{E}_\Delta^0[\int_0^{T"_1} \mathbf{1}(Q_s > 0)ds]\end{aligned}$$

$$= \lambda_N \mathsf{E}^0_\Delta[\sigma_0 + V_0].$$

Using equation (4.1) and letting $d_i = -\gamma_i'(0)$, $i = A, D$,

$$\begin{aligned}\rho &= \lambda_N \mathsf{E}^0_\Delta[\sigma_0 + V_0] \\ &= \lambda_A \mathsf{E}^0_A[(\sigma_0 + V_0)\mathbf{1}_{\{Q_{0-}=0\}}] + \lambda_N \mathsf{E}^0_D[(\sigma_0 + V_0)\mathbf{1}_{\{Q_0=0\}}] \\ &= \lambda_A\Big(d_A \mathsf{E}^0_A[\mathbf{1}_{\{Q_{0-}=0\}}] + d_D(\mathsf{E}^0_A[U_0] - \mathsf{E}^0_A[\mathbf{1}_{\{Q_{0-}=0\}}])\Big). \qquad (4.6)\end{aligned}$$

5. Distributions at Event Times

5.1 Distributions at Arrival Times

We begin with a general result.

Proposition 5.1 *Let A be the point process of (non-empty) arrival times of a BMAP process with phase process X_t. For $0 \le z, w \le 1$, the following equality holds,*

$$\lambda_A \mathsf{E}^0_A[z^{U_0} w^{Q_{0-}} \mathbf{1}(X_0 = j)] = \boldsymbol{q}(w)\Big(\boldsymbol{D}(z) - \boldsymbol{D}_0\Big)\Big|_j \qquad (5.1)$$

where $\boldsymbol{q}(w) = (q_1(w), \ldots, q_L(w))$ with $q_j(w) = \mathsf{E}[w^{Q_0}\mathbf{1}(X_0 = j)]$.

Proof: From the definition of Palm probability,

$$\begin{aligned}\lambda_A \mathsf{E}^0_A[z^{U_0} w^{Q_{0-}} \mathbf{1}(X_0 = j)] &= \mathsf{E}\Big[\int_{]0,1]} z^{U_s} w^{Q_{s-}} \mathbf{1}(X_s = j) A(ds)\Big] \\ &= \mathsf{E}\Big[\int_{]0,1]} z^{\tilde U_s} w^{Q_{s-}} \mathbf{1}(X_s = j) \tilde A(ds)\Big] \\ &\quad - \mathsf{E}\Big[\int_{]0,1]} w^{Q_{s-}} \mathbf{1}(X_s = j)\mathbf{1}(\tilde U_s = 0) \tilde A(ds)\Big].\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathsf{E}\Big[\int_{]0,1]} z^{\tilde U_s} w^{Q_{s-}} \mathbf{1}(X_s = j) \tilde A(ds)\Big] &= \mathsf{E}\Big[\int_{]0,1]} z^{\tilde U_s} w^{Q_{s-}} \sum_i \mathbf{1}(X_{s-} = i)\mathbf{1}(X_s = j) \tilde A(ds)\Big] \\ &= \mathsf{E}\Big[\int_{]0,1]} w^{Q_{s-}} \sum_i \mathbf{1}(X_{s-} = i) p_{ij}(z) \tilde A(ds)\Big] \\ &= \mathsf{E}\Big[\int_0^1 w^{Q_s} \sum_i \mathbf{1}(X_s = i) p_{ij}(z)\lambda_i ds\Big].\end{aligned}$$

Letting $z = 0$ in the previous equation,

$$\mathsf{E}\Big[\int_{]0,1]} w^{Q_{s-}} \mathbf{1}(X_s = j)\mathbf{1}(\tilde{U}_s = 0)\tilde{A}(ds)\Big] = \mathsf{E}\Big[\int_0^1 w^{Q_s} \sum_i \mathbf{1}(X_s = i)p_{ij}(0)\lambda_i ds\Big].$$

The result follows after combining the last two equations with the definition of $\boldsymbol{D}(z)$.

■

Letting $w = 0$ in proposition 5.1,

$$\lambda_A \mathsf{E}_A^0[z^{U_0}\mathbf{1}(Q_{0-} = 0)\mathbf{1}(X_0 = j)] = \boldsymbol{y}\Big(\boldsymbol{D}(z) - \boldsymbol{D}_0\Big)\Big|_j, \tag{5.2}$$

where $\boldsymbol{y} = \boldsymbol{q}(0)$, *i.e.*, $y_j = \mathsf{E}[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = j)]$.

Letting $w = 1$ in proposition 5.1,

$$\lambda_A \mathsf{E}_A^0[z^{U_0}\mathbf{1}(X_0 = j)] = \boldsymbol{\pi}\Big(\boldsymbol{D}(z) - \boldsymbol{D}_0\Big)\Big|_j. \tag{5.3}$$

In particular, letting $z = 1$ in equation (5.3), we obtain the phase distribution at arrival times,

$$\mathsf{E}_A^0[\mathbf{1}(X_0 = j)] = \frac{\boldsymbol{\pi}\boldsymbol{D}_0\big|_j}{\boldsymbol{\pi}\boldsymbol{D}_0\boldsymbol{e}},$$

Note that taking derivatives with respect to z and letting $z = 1$ in equation (5.3) yields equation (3.4).

5.2 Distributions at Departure Times

The result in this section was also obtained in [13, 8]. We present here a derivation based on Palm-martingale calculus. Let $Y_t = z^{Q_t}\mathbf{1}(X_t = m)$. The process Y_t satisfies the evolution equation,

$$Y_t = Y_0 + \int_{]0,1]}(Y_s - Y_{s-})\tilde{A}(ds) + \int_{]0,1]}(Y_s - Y_{s-})D(ds).$$

Letting $t = 1$, taking expectations, assuming equilibrium and using the definition of Palm probability $\mathsf{P}_D^0(\cdot)$ we obtain,

$$\begin{aligned}\lambda_N(z-1)\mathsf{E}_D^0[\mathbf{1}(X_0 = j)z^{Q_0}] &= \mathsf{E}\Big[\int_{]0,1]} z^{Q_{s-}}\Big(z^{U_s}\mathbf{1}(X_s = j) - \mathbf{1}(X_{s-} = j)\Big)\tilde{A}(ds)\Big] \\ &= \mathsf{E}\Big[\int_{]0,1]} z^{Q_{s-}}\Big(z^{U_s}\sum_i \mathbf{1}(X_{s-} = i)\mathbf{1}(X_s = j) - \end{aligned}$$

$$
\begin{aligned}
& \qquad -\mathbf{1}(X_{s^-}=j)\big)\tilde{A}(ds)\Big] \\
=& \mathsf{E}\Big[\int_0^1 z^{Q_s}\Big(\sum_i \mathbf{1}(X_s=i)p_{ij}(z)\lambda_i-\mathbf{1}(X_s=j)\lambda_j\Big)ds\Big] \\
=& \int_0^1 \sum_i q_i(z)D_{ij}(z)ds.
\end{aligned}
$$

Therefore, letting $\boldsymbol{q}_D(z)$ be the row vector with j-th component $\mathsf{E}_D^0[\mathbf{1}(X_0=j)z^{Q_0}]$, we get,

$$
\lambda_N \boldsymbol{q}_D(z)=\frac{\boldsymbol{q}(z)\boldsymbol{D}(z)}{z-1}. \tag{5.4}
$$

However, equation (5.4) is not suitable for the computation of the phase distribution at departures, $\pi_{D,l}=\mathsf{E}_D^0[\mathbf{1}(X_0=l)]$, $1\le l\le L$.

5.3 Phase Distribution at Departure Times

For $0\le z\le 1$ we define,

$$
\begin{aligned}
A_{ij}(z) &= \mathsf{E}\Big[z^{N_{\sigma_0}}\mathbf{1}(X_{\sigma_0}=j)|X_0=i\Big] \\
&= \mathsf{E}[\exp\{\boldsymbol{D}(z)\sigma_0\}\Big|_{ij}] \\
&= \int_{\mathbf{R}_+}\exp\{\boldsymbol{D}(z)t\}\Big|_{ij}H(dt).
\end{aligned}
$$

where the service independence was used, and let $\boldsymbol{A}(z)=(A_{ij}(z))$. Similarly, we let

$$
\begin{aligned}
\Gamma_{A,ij}(z) &= \mathsf{E}_A^0\Big[z^{N_{T_1'}}\mathbf{1}(X_{T_1'}=j)|X_0=i,Q_{0^-}=0\Big], \\
\Gamma_{D,ij}(z) &= \mathsf{E}_D^0\Big[z^{N_{T_1'}}\mathbf{1}(X_{T_1'}=j)|X_0=i,Q_0>0\Big],
\end{aligned}
$$

and $\boldsymbol{\Gamma}_A(z)=(\Gamma_{A,ij}(z))$ and $\boldsymbol{\Gamma}_D(z)=(\Gamma_{D,ij}(z))$. Hence, to summarize,

$$
\boldsymbol{A}(z) = \mathsf{E}\Big[\exp\{\boldsymbol{D}(z)\sigma_0\}\Big], \tag{5.5}
$$

$$
\boldsymbol{\Gamma}_{V_A}(z) = \mathsf{E}_A^0\Big[\exp\{\boldsymbol{D}(z)S_1\}|Q_{0^-}=0\Big], \tag{5.6}
$$

$$
\boldsymbol{\Gamma}_{V_D}(z) = \mathsf{E}_D^0\Big[\exp\{\boldsymbol{D}(z)S_1\}|Q_0>0\Big]. \tag{5.7}
$$

Furthermore,

$$
\boldsymbol{\Gamma}_A(z)=\mathsf{E}_A^0\Big[\exp\{\boldsymbol{D}(z)T_1'\}|Q_{0^-}=0\Big]=\boldsymbol{\Gamma}_{V_A}(z)\boldsymbol{A}(z), \tag{5.8}
$$

$$
\boldsymbol{\Gamma}_D(z)=\mathsf{E}_D^0\Big[\exp\{\boldsymbol{D}(z)T_1'\}|Q_0>0,\Big]=\boldsymbol{\Gamma}_{V_D}(z)\boldsymbol{A}(z). \tag{5.9}
$$

In the sequel we let $\boldsymbol{A}=\boldsymbol{A}(1)$ and $\boldsymbol{\Gamma}_i(1)=\boldsymbol{\Gamma}_i$, $i=A,D$.

In this section we compute $\boldsymbol{\pi}_D$ as a function of $\boldsymbol{y} = (y_1, \dots, y_L)$ where $y_j = \mathsf{E}[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = j)]$. From the exchange formula [1], taking into account that under $\mathsf{P}^0_\Delta(\cdot)$ we have a.s., $0 < T_1' \le T_1''$ and using equation (4.1),

$$\begin{aligned}\lambda_N \mathsf{E}^0_D[\mathbf{1}(X_0 = j)] &= \lambda_N \mathsf{E}^0_\Delta\Big[\sum_{k=1}^{D_{T_1''}} \mathbf{1}(X_{T_k'} = j)\Big] \\ &= \lambda_N \mathsf{E}^0_\Delta\Big[\mathbf{1}(X_{T_1'} = j)\Big] \\ &= \lambda_A \mathsf{E}^0_A[\mathbf{1}(Q_{0^-} = 0)\mathbf{1}(X_{T_1'} = j)] + \\ &\qquad + \lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)\mathbf{1}(X_{T_1'} = j)]. \end{aligned} \tag{5.10}$$

But,

$$\lambda_A \mathsf{E}^0_A[\mathbf{1}(Q_{0^-} = 0)\mathbf{1}(X_{T_1'} = j)] = \sum_i \lambda_A \mathsf{E}^0_A[\mathbf{1}(Q_{0^-} = 0)\mathbf{1}(X_0 = i)]\boldsymbol{\Gamma}_A\Big|_{ij}, \tag{5.11}$$

and from proposition 5.1 with $z = 1$ and $w = 0$,

$$\lambda_A \mathsf{E}^0_A[\mathbf{1}(Q_{0^-} = 0)\mathbf{1}(X_{T_1'} = j)] = \boldsymbol{y}\big(\boldsymbol{D} - \boldsymbol{D}_0\big)\boldsymbol{\Gamma}_A\Big|_j. \tag{5.12}$$

Similarly,

$$\lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)\mathbf{1}(X_{T_1'} = j)] = \sum_i \lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)\mathbf{1}(X_0 = i)]\boldsymbol{\Gamma}_D\Big|_{ij}. \tag{5.13}$$

From,

$$\mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)\mathbf{1}(X_0 = i)] = \pi_{D,i} - \mathsf{E}^0_D[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = i)],$$

and equation (5.4) with $z = 0$,

$$\lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = i)] = -\boldsymbol{y}\boldsymbol{D}_0\Big|_i,$$

we get,

$$\lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_{T_1'} = j)] = \lambda_N \boldsymbol{\pi}_D \boldsymbol{\Gamma}_D\Big|_j + \boldsymbol{y}\boldsymbol{D}_0\boldsymbol{\Gamma}_D\Big|_j. \tag{5.14}$$

Thus, combining equations (5.10), (5.11), (5.12), (5.13) and (5.14) in vector form,

$$\lambda_N \boldsymbol{\pi}_D\big(\boldsymbol{I} - \boldsymbol{\Gamma}_D\big) = \boldsymbol{y}\Big(\boldsymbol{D}\boldsymbol{\Gamma}_A + \boldsymbol{D}_0\big(\boldsymbol{\Gamma}_D - \boldsymbol{\Gamma}_A\big)\Big). \tag{5.15}$$

The matrix $\boldsymbol{\Gamma}_D$ is stochastic and $\boldsymbol{\pi}\boldsymbol{\Gamma}_D = \boldsymbol{\pi}$. Therefore, $\boldsymbol{\pi}_D$ is the unique positive solution of the linear system of equation (5.15) such that $\boldsymbol{\pi}_D \boldsymbol{e} = 1$.

A closed form can be derived by noting that the matrix $\boldsymbol{\Gamma}_D - \boldsymbol{I} - \boldsymbol{e}\boldsymbol{\pi}$ is invertible,

$$\lambda_N \boldsymbol{\pi}_D = \boldsymbol{y}\Big(\boldsymbol{D}\boldsymbol{\Gamma}_A + \boldsymbol{D}_0\big(\boldsymbol{\Gamma}_D - \boldsymbol{\Gamma}_A\big)\Big)\Big(\boldsymbol{I} - \boldsymbol{\Gamma}_D + \boldsymbol{e}\boldsymbol{\pi}\Big)^{-1} + \lambda_N \boldsymbol{\pi} \tag{5.16}$$

In order to simplify future equations we let,

$$\boldsymbol{F} = \Big(\boldsymbol{D}\boldsymbol{\Gamma}_A + \boldsymbol{D}_0\big(\boldsymbol{\Gamma}_D - \boldsymbol{\Gamma}_A\big)\Big)\Big(\boldsymbol{I} - \boldsymbol{\Gamma}_D + \boldsymbol{e}\boldsymbol{\pi}\Big)^{-1}, \tag{5.17}$$

i.e.,

$$\lambda_N \boldsymbol{\pi}_D = \boldsymbol{y}\boldsymbol{F} + \lambda_N \boldsymbol{\pi}. \tag{5.18}$$

From the previous derivation we also obtain the following two lemmas.

Lemma 5.2 *The following equality holds,*

$$\lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 > 0)\mathbf{1}(X_0 = j)] = \boldsymbol{y}\Big(\boldsymbol{F} + \boldsymbol{D}_0\Big)\Big|_j + \lambda_N \pi_j. \tag{5.19}$$

where $\boldsymbol{y} = (y_1, \ldots, y_L)$ *with* $y_j = \mathsf{E}[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = j)]$.

Lemma 5.3 *The following equality holds,*

$$\lambda_N \mathsf{E}^0_D[z^{Q_0}\mathbf{1}(Q_0 > 0)\mathbf{1}(X_0 = j)] = \frac{\boldsymbol{q}(z)\boldsymbol{D}(z)}{z-1}\Big|_j + \boldsymbol{y}\boldsymbol{D}_0\Big|_j, \tag{5.20}$$

where $\boldsymbol{y} = (y_1, \ldots, y_L)$ *with* $y_j = \mathsf{E}[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = j)]$.

Proof: We have, from equation (5.4),

$$\begin{aligned} \lambda_N \mathsf{E}^0_D[z^{Q_0}\mathbf{1}(Q_0 > 0)\mathbf{1}(X_0 = j)] &= \lambda_N \mathsf{E}^0_D[z^{Q_0}\mathbf{1}(X_0 = j)] - \lambda_N \mathsf{E}^0_D[\mathbf{1}(Q_0 = 0)\mathbf{1}(X_0 = j)] \\ &= \frac{\boldsymbol{q}(z)\boldsymbol{D}(z)\Big|_j}{z-1} + \boldsymbol{y}\boldsymbol{D}_0\Big|_j. \end{aligned}$$

■

6. The System Queue Length

6.1 The Queue Length Moment Generating Function

In this section we compute $q_j(z) = \mathsf{E}[z^{Q_0}\mathbf{1}(X_0 = j)]$ where $0 \le z < 1$ and $j \in E$.

From the inversion formula, using the fact that on $\{Q_0 > 0\}$, $T_1' = T_1''$ and the definition of $\boldsymbol{D}(z)$, [10],

$$\begin{aligned}\mathsf{E}[\mathbf{1}(Q_0 > 0)z^{Q_0}\mathbf{1}(X_0 = j)] &= \lambda_N \mathsf{E}^0_\Delta[\int_0^{T_1'} z^{Q_t}\mathbf{1}(X_t = j)dt] \\ &= \lambda_N \mathsf{E}^0_\Delta[z^{Q_0}\int_0^{T_1'} z^{N_t}\mathbf{1}(X_t = j)dt] \\ &= \lambda_N \mathsf{E}^0_\Delta[z^{Q_0}\sum_i \mathbf{1}(X_0 = i)\int_0^{T_1'} \exp\{\boldsymbol{D}(z)t\}\Big|_{ij}dt].\end{aligned}$$

Hence, for $0 \le z < 1$,

$$\begin{aligned}&\mathsf{E}[\mathbf{1}(Q_0 > 0)z^{Q_0}\mathbf{1}(X_0 = j)] = \\ &= \lambda_A \mathsf{E}^0_A\Big[z^{U_0}\mathbf{1}(Q_{0-} = 0)\sum_i \mathbf{1}(X_0 = i)\boldsymbol{D}^{-1}(z)\int_0^{T_1'}\boldsymbol{D}(z)\exp\{\boldsymbol{D}(z)t\}dt\Big|_{ij}\Big] + \\ &\qquad + \lambda_N \mathsf{E}^0_D\Big[z^{Q_0}\mathbf{1}(Q_0 > 0)\sum_i \mathbf{1}(X_0 = i)\boldsymbol{D}^{-1}(z)\int_0^{T_1'}\boldsymbol{D}(z)\exp\{\boldsymbol{D}(z)t\}dt\Big|_{ij}\Big] \\ &= \lambda_A \mathsf{E}^0_A\Big[z^{U_0}\mathbf{1}(Q_{0-} = 0)\sum_i \mathbf{1}(X_0 = i)\Big]\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_A(z) - \boldsymbol{I}\Big)\Big|_{ij} + \\ &\qquad + \lambda_N \mathsf{E}^0_D\Big[z^{Q_0}\mathbf{1}(Q_0 > 0)\sum_i \mathbf{1}(X_0 = i)\Big]\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_D(z) - \boldsymbol{I}\Big)\Big|_{ij} \\ &= \boldsymbol{y}\Big(\boldsymbol{D}(z) - \boldsymbol{D}_0\Big)\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_A(z) - \boldsymbol{I}\Big)\Big|_j + \\ &\qquad + \Big(\frac{\boldsymbol{q}(z)\boldsymbol{D}(z)}{z-1} + \boldsymbol{y}\boldsymbol{D}_0\Big)\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_D(z) - \boldsymbol{I}\Big)\Big|_j.\end{aligned} \tag{6.1}$$

Since $q_j(z) = y_j + \mathsf{E}[\mathbf{1}(Q_0 > 0)z^{Q_0}\mathbf{1}(X_0 = j)]$ and since for $0 \le z < 1$, the matrix $z\boldsymbol{I} - \boldsymbol{\Gamma}_D(z)$ is invertible, it follows, after some algebra, that $\boldsymbol{q}(z)$ is given by,

$$\boldsymbol{q}(z) = (z-1)\boldsymbol{y}\Big(\boldsymbol{\Gamma}_A(z) + \boldsymbol{D}_0\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_D(z) - \boldsymbol{\Gamma}_A(z)\Big)\Big)\Big(z\boldsymbol{I} - \boldsymbol{\Gamma}_D(z)\Big)^{-1}, \tag{6.2}$$

for $0 \le z < 1$ with the boundary condition,

$$\lim_{z\uparrow 1}\boldsymbol{q}(z) = \boldsymbol{\pi}. \tag{6.3}$$

6.2 Average Queue Length

As an application of the results obtained in this section we obtain the joint distribution of average queue length and phase.

We consider the matrices,

$$\boldsymbol{M}(z) = (z-1)\boldsymbol{\Gamma}_A(z)\Big(z\boldsymbol{I} - \boldsymbol{\Gamma}_D(z)\Big)^{-1}, \tag{6.4}$$

and

$$\boldsymbol{N}(z) = \begin{cases} (z-1)\boldsymbol{D}^{-1}(z)\Big(\boldsymbol{\Gamma}_D(z) - \boldsymbol{\Gamma}_A(z)\Big)\Big(z\boldsymbol{I} - \boldsymbol{\Gamma}_D(z)\Big)^{-1} & \text{if } \boldsymbol{\Gamma}_A(z) \neq \boldsymbol{\Gamma}_D(z), \\ \boldsymbol{0} & \text{if } \boldsymbol{\Gamma}_A(z) = \boldsymbol{\Gamma}_D(z), \end{cases} \tag{6.5}$$

so that,

$$\boldsymbol{q}(z) = \boldsymbol{y}\Big(\boldsymbol{M}(z) + \boldsymbol{D}_0\boldsymbol{N}(z)\Big). \tag{6.6}$$

Let $\xi(z)$ be the Perron-Frobenius eigenvalue of $\boldsymbol{D}(z)$ and $\boldsymbol{u}(z)$ (resp. $\boldsymbol{v}(z)$) be the associated left (resp. right) eigenvector. We assume that these eigenvectors are normalized as follows,

$$\begin{aligned} \boldsymbol{u}(z)\boldsymbol{v}(z) &= 1, \\ \boldsymbol{u}(z)\boldsymbol{e} &= 1. \end{aligned}$$

Since $\lim_{z\uparrow 1} \boldsymbol{D}(z) = \boldsymbol{D}$, we have,

$$\begin{aligned} \lim_{z\uparrow 1} \xi(z) &= 0, \\ \lim_{z\uparrow 1} \boldsymbol{u}(z) &= \boldsymbol{\pi}, \\ \lim_{z\uparrow 1} \boldsymbol{v}(z) &= \boldsymbol{e}. \end{aligned}$$

Furthermore, the matrix $\boldsymbol{\Gamma}_i(z), i = A, D$, admits $\gamma_i\Big(-\xi(z)\Big)$ as an eigenvalue and $\boldsymbol{v}(z)$ (respt. $\boldsymbol{u}(z)$) as its associated left (respt. right) eigenvector [13].

An elementary computation shows that,

$$\boldsymbol{u}(z)\boldsymbol{M}(z) = \mu(z)\boldsymbol{u}(z), \tag{6.7}$$

$$\boldsymbol{u}(z)\boldsymbol{N}(z) = \nu(z)\boldsymbol{u}(z), \tag{6.8}$$

where,

$$\mu(z) = \frac{(z-1)\gamma_A\Big(-\xi(z)\Big)}{z - \gamma_D\Big(-\xi(z)\Big)}, \tag{6.9}$$

$$\nu(z) = \frac{(z-1)\gamma_A\big(-\xi(z)\big)\Big(\gamma_D\big(-\xi(z)\big) - \gamma_A\big(-\xi(z)\big)\Big)}{\xi(z)\Big(z - \gamma_D\big(-\xi(z)\big)\Big)}, \tag{6.10}$$

for $0 \leq z < 1$. A similar result holds for the right eigenvector $\boldsymbol{v}(z)$. Thus, $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ are eigenvectors of $\boldsymbol{M}(z)$ with associated eigenvalue $\mu(z)$ and of $\boldsymbol{N}(z)$ with eigenvalue $\nu(z)$.

Let

$$\begin{aligned} \boldsymbol{M} &= \lim_{z \uparrow 1} \boldsymbol{M}(z), \\ \boldsymbol{N} &= \lim_{z \uparrow 1} \boldsymbol{N}(z). \end{aligned}$$

Taking limits as as $z \uparrow 1$ in equations (6.7) and (6.8), it follows that $\boldsymbol{\pi}$ and $\boldsymbol{e}$ are also eigenvectors of $\boldsymbol{M}$ and $\boldsymbol{N}$. The eigenvalue associated with $\boldsymbol{\pi}$ and $\boldsymbol{e}$ of the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ is obtained by taking limits as $z \uparrow 1$ in equations (6.9) and (6.10) and taking into account that [12],

$$\xi'(1) = \boldsymbol{\pi}\boldsymbol{D}'\boldsymbol{e} = \lambda_N. \tag{6.11}$$

We obtain,

$$\mu = \lim_{z \uparrow 1} \mu(z) = \frac{1}{1 - \lambda_N d_D}, \tag{6.12}$$

$$\nu = \lim_{z \uparrow 1} \nu(z) = \frac{d_D - d_A}{1 - \lambda_N d_1}, \tag{6.13}$$

where $d_i = -\gamma_i'(0)$, $i = A, D$.

Therefore, from section 5.3,

$$\lim_{z \uparrow 1} \frac{\boldsymbol{q}(z)\boldsymbol{D}(z)}{z - 1} = \boldsymbol{y}\boldsymbol{F} + \lambda_N \boldsymbol{\pi}, \tag{6.14}$$

which implies, applying L'Hopital's rule,

$$\boldsymbol{q}'(1)\boldsymbol{D} + \boldsymbol{\pi}\boldsymbol{D}' = \boldsymbol{y}\boldsymbol{F} + \lambda_N \boldsymbol{\pi}. \tag{6.15}$$

The derivation of q proceeds as follows. Taking derivatives and letting $z = 1$ in,

$$\Big(\boldsymbol{M}(z) + \boldsymbol{D}_0\boldsymbol{N}(z)\Big)\boldsymbol{v}(z) = \Big(\mu(z) + \nu(z)\boldsymbol{D}_0\Big)\boldsymbol{v}(z),$$

yields,

$$\left(\boldsymbol{M}' + \boldsymbol{D}_0\boldsymbol{N}'\right)\boldsymbol{e} + \left(\boldsymbol{M} + \boldsymbol{D}_0\boldsymbol{N}\right)\boldsymbol{v}' = \left(\mu' + \nu'\boldsymbol{D}_0\right)\boldsymbol{e} + \left(\mu + \nu\boldsymbol{D}_0\right)\boldsymbol{v}',$$

where $\boldsymbol{M}' = \boldsymbol{M}'(1)$, $\mu' = \mu'(1)$, $\boldsymbol{v}' = \boldsymbol{v}'(1)$, $\boldsymbol{N}' = \boldsymbol{N}'(1)$ and $\nu' = \nu'(1)$. On the other hand, from equation (6.6),

$$q = \boldsymbol{q}'(1)\boldsymbol{e} = \boldsymbol{y}\left(\boldsymbol{M}' + \boldsymbol{D}_0\boldsymbol{N}'\right)\boldsymbol{e},$$

therefore combining these two last equations and taking into account that,

$$\boldsymbol{\pi} = \lim_{z\uparrow 1} \boldsymbol{y}\left(\boldsymbol{M}(z) + \boldsymbol{D}_0\boldsymbol{N}(z)\right)$$

and that $\boldsymbol{\pi}\boldsymbol{v}' = 0$,

$$q = \mu'(1-\rho) + \nu'\boldsymbol{y}\boldsymbol{D}_0\boldsymbol{e} + \boldsymbol{y}\left(\mu\boldsymbol{I} + \nu\boldsymbol{D}_0\right)\boldsymbol{v}'. \tag{6.16}$$

The values of μ' and ν' are obtained from equations (6.9) and (6.10),

$$\begin{aligned} \mu' &= \frac{2d_A\lambda_N - 2d_Ad_D\lambda_N^2 + \lambda_N^2\gamma_D''(0) + d_D\xi''(1)}{2(1-\lambda_N d_D)^2}, \\ \nu' &= \mu'(d_D - d_A) - \frac{\lambda_N\mu}{2}\left(\gamma_D''(0) - \gamma_A''(0)\right). \end{aligned}$$

The values of $\boldsymbol{v}'$ and $\xi''(1)$ are obtained by applying Theorem A.2.3 of [12] to $\boldsymbol{D}(z)$. We obtain,

$$\boldsymbol{v}' = \left(\boldsymbol{D} - \boldsymbol{e}\boldsymbol{\pi}\right)^{-1}\left(\lambda_N\boldsymbol{I} - \boldsymbol{D}'\right)\boldsymbol{e} \tag{6.17}$$

and

$$\xi''(1) = -\boldsymbol{\pi}\boldsymbol{D}''(1)\boldsymbol{e} - 2\boldsymbol{\pi}\boldsymbol{D}'\boldsymbol{v}'. \tag{6.18}$$

Higher moments of the queue length can be obtained by successive derivation of equations (6.2), (6.9) and (6.10) and repeated application of Theorem A.2.3 in [12] to $\boldsymbol{D}(z)$.

7. The System Workload

For $1 \le j < L$, we define $\psi_j(u) = \mathsf{E}[\mathbf{1}(X_0 = j)\exp\{-uW(0)\}]$ and let $\boldsymbol{\psi}(u) = (\psi_1(u), \ldots, \psi_L(u))$. The process,

$$Y_j(t) = \exp\{-uW(t)\}\mathbf{1}(X_t = j),$$

satisfies the sample path equality,

$$Y_j(t) = Y_j(0) + \int_{]0,t]} \Big(Y_j(s) - Y_j(s^-)\Big)\tilde{A}(ds) + u \int_0^t \mathbf{1}(B(s) = 1)Y_j(s)ds,$$

where $B(\cdot)$ is the busy server indicator. Taking expectations and following an approach identical to that of section 3 and proposition 5.1,

$$\sum_i \psi_i(u)D_{ij}\big(\phi(u)\big) = -u\mathsf{E}\Big[\mathbf{1}(B(0) = 1)\mathbf{1}(X_0 = j)\exp\{-uW(0)\}\Big], \qquad (7.1)$$

where the service independence was used.

We compute next the right hand side of equation (7.1). Note first that for $0 \le t \le T_1'$,

$$W(t) = W(0) + \sum_{k=1}^{N_t} \sigma_k - (t - S_1)^+.$$

Then, letting σ_0 denote the duration of the service starting at S_1 and applying the inversion formula [1],

$$\begin{aligned}
&\mathsf{E}\Big[\mathbf{1}(B(0) = 1)\mathbf{1}(X_0 = j)\exp\{-uW(0)\}\Big] = \\
&= \lambda_N \mathsf{E}^0_\Delta\Big[\int_{S_1}^{T_1'} \mathbf{1}(X_t = j)\exp\{-uW(t)\}dt\Big] \\
&= \lambda_N \mathsf{E}^0_\Delta\Big[\sum_i \mathbf{1}(X_0 = i)\exp\{-uW(0)\}\int_0^{\sigma_0} \mathbf{1}(X_{t+S_1} = j)\phi(u)^{N_{t+S_1}} e^{ut}dt\Big] \\
&= \lambda_N \mathsf{E}^0_\Delta\Big[\sum_i \mathbf{1}(X_0 = i)\exp\{-uW(0)\}\Big(e^{u\sigma_0}\exp\{\boldsymbol{D}\big(\phi(u)\big)\sigma_0\} - \boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1} \\
&\qquad\qquad \exp\{\boldsymbol{D}\big(\phi(u)\big)S_1\}\Big|_{ij}\Big].
\end{aligned} \qquad (7.2)$$

To simplify the notation, let,

$$\begin{aligned}
Z_j = \sum_i \mathbf{1}(X_0 = i)\exp\{-uW(0)\}\Big(e^{u\sigma_0}\exp\{\boldsymbol{D}\big(\phi(u)\big)\sigma_0\} - \boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1} \\
\exp\{\boldsymbol{D}\big(\phi(u)\big)S_1\}\Big|_{ij}.
\end{aligned}$$

Then, from equation (4.1),

$$\mathsf{E}\Big[\mathbf{1}(B(0) = 1)\mathbf{1}(X_0 = j)\exp\{-uW(0)\}\Big] = \lambda_A \mathsf{E}^0_A\Big[\mathbf{1}(Q_{0^-} = 0)Z_j\Big] + \lambda_N \mathsf{E}^0_D\Big[\mathbf{1}(Q_0 > 0)Z_j\Big]. \qquad (7.3)$$

Let $\sigma_{n,k}$, $1 \le k \le U_n$, be the service demand for the k-th customer arriving at time T_n.

Since on $\{Q_{0^-} = 0\}$ we have P_A^0 a.s.,

$$W(0) = \sum_{k=1}^{U_0} \sigma_{0,k}, \qquad \mathsf{P}_A^0 - \text{a.s.},$$

with $\sigma_{0,k} = \sigma_0$ for some $1 \le k \le U_0$, the first term in the right hand side of equation (7.3) becomes using the service independence and proposition 5.1,

$$\begin{aligned}
&\lambda_A \mathsf{E}_A^0\Big[\mathbf{1}(Q_{0^-} = 0)Z_j\Big] = \\
&\quad = \lambda_A \mathsf{E}_A^0\Big[\mathbf{1}(Q_{0^-} = 0)\sum_i \mathbf{1}(X_0 = i)\phi(u)^{U_0-1}\Big(\boldsymbol{A}\big(\phi(u)\big) - \phi(u)\boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1} \\
&\qquad\qquad \boldsymbol{\Gamma}_{V_A}\big(\phi(u)\big)\Big|_{ij}\Big] \\
&\quad = \frac{1}{\phi(u)}\boldsymbol{y}\Big(\boldsymbol{D}\big(\phi(u)\big) - \boldsymbol{D}_0\Big)\Big(\boldsymbol{A}\big(\phi(u)\big) - \phi(u)\boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1}\boldsymbol{\Gamma}_{V_A}\big(\phi(u)\big)\Big|_j.
\end{aligned} \tag{7.4}$$

Since on $\{Q_0 > 0\}$, we have P_D^0 a.s.,

$$W(0) = \sum_{k=1}^{Q_0} \sigma_k, \qquad \mathsf{P}_D^0 - \text{a.s.},$$

where σ_k is the service demand of the k-th customer and $\sigma_k = \sigma_0$ for some $1 \le k \le Q_0$, the second term in the right hand side of equation (7.3) becomes, using the service independence and equation (5.4),

$$\begin{aligned}
&\lambda_N \mathsf{E}_D^0\Big[\mathbf{1}(Q_0 > 0)Z_j\Big] = \\
&\quad = \lambda_N \mathsf{E}_D^0\Big[\mathbf{1}(Q_0 > 0)\sum_i \mathbf{1}(X_0 = i)\phi(u)^{Q_0-1}\Big(\boldsymbol{A}\big(\phi(u)\big) - \phi(u)\boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1} \\
&\qquad\qquad \boldsymbol{\Gamma}_{V_D}\big(\phi(u)\big)\Big|_{ij}\Big] \\
&\quad = \frac{1}{\phi(u)}\Big(\frac{\boldsymbol{q}\big(\phi(u)\big)\boldsymbol{D}\big(\phi(u)\big)}{\phi(u) - 1} + \boldsymbol{y}\boldsymbol{D}_0\Big)\Big(\boldsymbol{A}\big(\phi(u)\big) - \phi(u)\boldsymbol{I}\Big)\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1} \\
&\qquad\qquad \boldsymbol{\Gamma}_{V_D}\big(\phi(u)\big)\Big|_j.
\end{aligned} \tag{7.5}$$

Combining equations (7.4), (7.5), (5.8), (5.9) and (6.2) in matrix form,

$$\boldsymbol{\psi}(u) \;=\; -\frac{u}{\phi(u)}\boldsymbol{y}\Big[\boldsymbol{\Gamma}_{V_A}\big(\phi(u)\big) + \boldsymbol{D}_0\boldsymbol{D}^{-1}\big(\phi(u)\big)\Big(\boldsymbol{\Gamma}_{V_D}\big(\phi(u)\big) - \boldsymbol{\Gamma}_{V_A}\big(\phi(u)\big)\Big)\Big]$$

$$\left[\boldsymbol{I} + \boldsymbol{\Gamma}_D\big(\phi(u)\big)\Big(\phi(u)\boldsymbol{I} - \boldsymbol{\Gamma}_D\big(\phi(u)\big)\Big)^{-1}\right]$$
$$\left[\boldsymbol{A}\big(\phi(u)\big) - \phi(u)\boldsymbol{I}\right]\left[u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\right]^{-1},$$

where we used the fact that the matrices $\boldsymbol{\Gamma}_i(z), \boldsymbol{A}(z), \boldsymbol{\Gamma}_{V_i}(z)$ and $\boldsymbol{D}(z)$ commute. Finally, observing that,

$$\boldsymbol{I} + \boldsymbol{\Gamma}_D\big(\phi(u)\big)\Big(\phi(u)\boldsymbol{I} - \boldsymbol{\Gamma}_D\big(\phi(u)\big)\Big)^{-1} = \phi(u)\Big(\phi(u)\boldsymbol{I} - \boldsymbol{\Gamma}_D\big(\phi(u)\big)\Big)^{-1},$$

and using equations (6.2), (5.8) and (5.9),

$$\boldsymbol{\psi}(u) = \frac{u}{1-\phi(u)}\boldsymbol{q}\big(\phi(u)\big)\left[\boldsymbol{I} - \phi(u)\boldsymbol{A}^{-1}\big(\phi(u)\big)\right]\left[u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\right]^{-1}. \qquad (7.6)$$

8. The BMAP/GI/1/∞ Queue

For completeness, we particularize the results obtained to the BMAP/GI/1/∞ queueing system analyzed in [8]. This queueing system is a generalization of the N/GI/1/∞ queue originaly studied by Ramaswami in [13]. The server vacations are removed by letting $\gamma_i(u) = \phi(u)$, $\gamma_{V_i}(u) = 1$, $\boldsymbol{\Gamma}_i(z) = \boldsymbol{A}(z)$ and $\boldsymbol{\Gamma}_{V_i}(z) = 1$, $i = A, D$.

For the BMAP/GI/1/∞ queueing system, $\rho = -\lambda_N\phi'(0)$ with $\lambda_N = \boldsymbol{\pi}\boldsymbol{D}'\boldsymbol{e}$. Proposition 5.1 and equation (5.4) hold unchanged. Equation (5.17) becomes,

$$\boldsymbol{F} = \boldsymbol{D}\boldsymbol{A}\big(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi}\big)^{-1}. \qquad (8.1)$$

Therefore, the phase distribution at departure times is,

$$\lambda_N\boldsymbol{\pi}_D = \boldsymbol{y}\boldsymbol{D}\boldsymbol{A}\big(\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi}\big)^{-1} + \lambda_N\boldsymbol{\pi},$$

and propositions 5.2 and 5.3 hold with $\boldsymbol{F}$ given by equation (8.1). Equation (6.2) gives the characteristic function of the queue length vector of the BMAP/GI/1/∞ queue in closed form,

$$\boldsymbol{q}(z) = (z-1)\boldsymbol{y}\boldsymbol{A}(z)\big(z\boldsymbol{I} - \boldsymbol{A}(z)\big)^{-1}, \quad 0 \le z < 1. \qquad (8.2)$$

For the workload, equation (7.6) becomes,

$$\boldsymbol{\psi}(u) = u\boldsymbol{y}\Big(u\boldsymbol{I} + \boldsymbol{D}\big(\phi(u)\big)\Big)^{-1}, \quad u > 0. \qquad (8.3)$$

$$\boldsymbol{\psi}(0) = \boldsymbol{\pi}. \qquad (8.4)$$

These results were obtained in [13, 8].

The joint distribution of the average queue length and phase is given by,

$$\boldsymbol{q}'(1) = \left[y\boldsymbol{A}\boldsymbol{D} - \boldsymbol{\pi}\Big(q\boldsymbol{I} + \boldsymbol{D}'\big(\boldsymbol{I} - \boldsymbol{A}\big)\Big)\right]\Big(\boldsymbol{D}\big(\boldsymbol{I} - \boldsymbol{A}\big) - \boldsymbol{e}\boldsymbol{\pi}\Big)^{-1}, \tag{8.5}$$

and q is equal to,

$$q = \mu'(1-\rho) + \mu \boldsymbol{y}\boldsymbol{v}'. \tag{8.6}$$

The corresponding values of μ and μ' are given by,

$$\begin{aligned} \mu &= \frac{1}{1-\rho}, \\ \mu' &= \frac{2\rho(1-\rho) + \lambda_N^2 \phi''(0) - \phi'(0)\xi''(1)}{2(1-\rho)^2}. \end{aligned}$$

The values of $\boldsymbol{v}'$ and $\xi''(1)$ remain unchanged and are given by equations (6.17) and (6.18).

9. The M/GI/1 Queue with Server Vacations

The results obtained in this paper considerably simplify when the input to the queue is a Poisson process, *i.e.*, in the M/GI/1 case. For an M/GI/1 queue,

$$\begin{aligned} \boldsymbol{D}(z) &= \lambda(z-1), \\ \boldsymbol{D} &= 0, \end{aligned}$$

and thus,

$$\begin{aligned} \boldsymbol{\Gamma}_i(z) &= \gamma_i\big(\lambda(1-z)\big), \quad i = A, D, \\ \boldsymbol{A}(z) &= \phi\big(\lambda(1-z)\big). \end{aligned}$$

Furthermore, equation (5.18) reduces to $\boldsymbol{q}_D(z) = \boldsymbol{q}(z)$ (the so-called departure theorem) and the PASTA property holds [4]. For consistency, we keep the notation, $y = \mathsf{E}[\mathbf{1}(Q_0 = 0)]$. From equation (4.6), since for the M/G/1 queue, $y = 1 - \rho$ we have,

$$y = \frac{1 - \lambda d_D}{1 + \lambda(d_A - d_D)}, \tag{9.1}$$

where $d_i = -\gamma_i'(0)$, $i = A, D$. The system is stable if $\lambda d_D < 1$ and $d_A < \infty$.

The queue length is obtained directly from equation (6.2),

$$q(z) = \frac{1-\lambda d_D}{1-\lambda(d_D-d_A)} \frac{\gamma_D\big(\lambda(1-z)\big) - z\gamma_D\big(\lambda(1-z)\big)}{\gamma_D\big(\lambda(1-z)\big)-z},$$

where $q(z) = \mathsf{E}[z^{Q_0}]$. Furthermore, applying twice L' Hopital's rule we obtain the Pollaczek-Khintchine formula for the M/GI/1 queue with the vacation schedule considered,

$$\mathsf{E}[Q_0] = \lambda \frac{\big(1-\lambda d_D\big)\big(\lambda\gamma_A''(0)+2d_A\big)+\lambda^2 d_A\gamma_D''(0)}{2\big(1+\lambda(d_D-d_A)\big)\big(1-\lambda d_D\big)}.$$

Finally for the workload,

$$\psi(u) = \frac{q\big(\phi(u)\big)}{\phi\big(\lambda(1-\phi(u))\big)} \frac{u}{1-\phi(u)} \frac{\phi\big(\lambda(1-\phi(u))\big)-\phi(u)}{u+\lambda\big(\phi(u)-1\big)}. \tag{9.2}$$

10. Conclusions

In this paper we have studied the BMAP/GI/1/∞ queue with server vacations and server set-up times. The analysis, based on Palm-martingale calculus, studied the system behavior between discontinuity jumps of the residual service times. We have computed distributions of interest at arrival and departure times. The queue length moment generating function vector has been given in closed form. The results obtained have been particularized to the BMAP/GI/1 queue without vacations and the M/GI/1 queue with the same server vacation schedule.

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