A Game Theoretic Framework for SLA Negotiation

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game theory, SLA, automation, contract, price, penalty

An SLA/Contract is an agreement between a client and a service provider. It specifies desired levels of service and penalties in case of default. The objective of this work is to develop a mathematical model for the negotiation process before agreeing to an SLA. The model will be based on Game Theory models of signaling games. The idea is to try to capture the bargaining process that occurs when clients are offered not just a take-it-or-leave-it contract but also the opportunity for them to express their preferences via a counteroffer. Of course, many times this counteroffer does not meet the service provider preferences and that's how the bargaining begins. Then having the game defined we'll try to find equilibria of the game.

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Abstract

An SLA/Contract is an agreement between a client and a service provider. It specifies desired levels of service and penalties in case of violations. The objective of this work is to develop a mathematical model for the negotiation process before an agreement is reached. The model will be based on Game Theory models of signaling games. The idea is to try to capture the bargaining process that occurs when clients are offered not just a *take-it-or-leave-it* contract but also the opportunity for them to express their preferences via a counteroffer. Of course, many times this counteroffer does not meet the service provider objectivess and that's how the bargaining begins. Then having the game defined we try to find equilibria of the game.

[0]Work developed during stay at HP Labs Palo-Alto

1. Introduction

A Service Level Agreement is (SLA/Contract) is an agreement between a provider and a consumer which is comprised of Service Level Objectives that guarantee quality of service (such as availability, performance and reliability), a promise of payment and penalties to impose in case the objectives are not met. The study of such contracts has become increasingly important with the increasing use of IT outsourcing procedures, which had reached 56 billion in 2000 and was expected to reach 100 by 2005 (Dermikan et al. 2005). While the original practice of IT outsourcing contracts involved complicated measures to safeguard the clients interest against the many potential mishaps, a more modern approach has focused on a system of penalties and rewards based on observed quality of service, serving as a monetary compensation that insures the client in case the service is suboptimal (Dermikan et al. 2005).

Of course the design of a SLA requires interaction between the provider and the consumer, since even though the provider may know the level of service that can be arranged only the consumer knows the level of service that he requires. Therefore it's only natural for the design of a SLA that the provider and the consumer negotiate in order to achieve a SLA that is good enough for both parties. Of course the develpment of this negotiation is very complex and may eventually result in a SLA that doesn't work for any of the parties.

The objective of this work is to develop a mathematical model for the negotiation process before agreeing to an SLA. The model will be based on Game Theory models of signaling games[4]. The idea is to try to capture the bargaining process that occurs when clients are offered not just a *take-it-or-leave-it* contract but also the opportunity for them to express their preferences via a counteroffer. Of course, many times this counteroffer does not meet the service provider preferences and that's how the bargaining begins.

The idea is to find equilibria of this game, that is to say situations where no one wants to change their strategy. Since there is some signaling involved, these equilibria must be consequent with the Bayes' rule[1]. This means that in an equilibrium given the strategy of the players I won't update my beliefs and therefore I will have no incentive to change my strategy.

First in section 2. we will describe the problem we want to model. In section 3. we will present the game that will work as a base for the diferent models, and in section B we will describe several models, make examples of application of those models and characterize the equilibria if possible. Later, in section 6., we will describe other works develped in this area. Finally in section 7. we will recapitulate the results that the model establishes and also comment on the contrast of the model with reality.

[1]The reason the equilibria are required to comply with the Bayes' rule is that if they do not follow the Bayes' rule then it means that the player is not updating her beliefs in a rational way.

2. The Problem

In this article, we are considering a negotiation scenario between a service provider and a client. The service provider provides services (which could be n-tier services e.g. cluster of appservers/databases, or three-tiered services) that the client intends to use. The service provider furnishes the customer SLA templates to begin with. For example, if the client is seeking to use an HPC service, the service provider may provide different templates which provide choices between the price and time to completion. Since different clients have varied needs, clients would like to modify some of the contracts in order to adjust it more closely to their needs, for example if a contract template states a price of 200 USD for 10 hrs in time to completion, the client may need to readjust it to obtain the results earlier in 8 hrs. The client may be willing to pay a premium and a total amount of 400 USD. The client in this case will negotiate and provide a counter-offer. The service provider will weigh-in on the counter offer based on its overall objectives. The service provider by looking at the counteroffer gets an idea of the client's preferences and decides to make a final offer. The client may then accept the new offer or decide to abandon the negotiation.

This is the problem we are looking forward to model. This is a simplification of a scenario where the service provider may not only have to solve the negotiation problem with the client, but also think on how to efficiently allocate resources across clients. Likewise the client may be negotiating with one or more enterprises for the same service. Of course every client is different, even the same client at different stages in time might behave completely different, although as a service provider gets to know different kinds of clients, it may be able to classify its clients based on common properties that the client have into certain classes (e.g. gold, silver bronze). Also different clients have varying degree of quality sensistivity, large enterprises are highly sensitive to quality requirements while small and medim business may not be as sensitive.

3. The Model

We will model the problem as a dynamic game between the service provider and the client. Before describing the game development, let's describe the elements of the game. There are initial contracts $\mathcal{M} = \{(p_j, q_j) | j = 1, \ldots, m\}$, the client may have different utility functions based on its type, this type is denoted as $\lambda \in \Lambda = \{\lambda_i | i = 1, \ldots, n\}$, there are probabilities known to each of the players that refer to the client's type, these probabilities will be denoted as $\mathbb{P}(\lambda = \lambda_i) = r_i$ and finally there will be utility functions that describe the utility of commiting to a contract: for the service provider the utility will be denoted as $u_2(p, q)$ and for the client of type λ will be denoted as $u_1((p, q), \lambda)$.

Next we describe the game and how it develops:

- The client approaches the service provider, chooses one of the contracts $(p, q) \in \mathcal{M}$ and modifies it to his advantage. We'll model this modification as choosing a contract (p', q') such that $\|(p', q') - (p, q)\| \leq \varepsilon p$ for a given parameter ε which will be refered to as *negotiation percentage*.
- The service provider receives this counteroffer and extracts some information about the client's type. He uses this information to update the initial beliefs about the client's type and give a final counteroffer. Following the previous idea if (p', q') is the client's counteroffer, the service provider chooses a contract (p^*, q^*) such that $\|(p^*, q^*) - (p', q')\| \leq \eta p'$, the parameter η will be refered to as the service provider *negotiation percentage*.

Going into details, player 2 will be the service provider with a utility function $u_2 : \mathbb{R}^2 \to \mathbb{R}$ and $u_2((p, q))$ describes service provider's utility when the contract (p, q) is established. Player 1 would be the client with a utility function $u_1 : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ and $u_1((p, q), \lambda)$ represents the client's type $\lambda \in \Lambda$ utility when the contract (p, q) is established. Here the usual hypothesis are assumed, that is, $u_{1p} > 0, u_{2p} < 0, u_{1q} < 0, u_{2q} > 0, u_{1pp} \leq 0, u_{1qq} \leq 0, u_{1pq} = 0, u_{2pp} \leq 0, u_{2qq} \leq 0, u_{2pq} = 0$. We will also assume that the client accepts any contract such that $u_1((p, q), \lambda) \geq 0$, the service provider will accept any contract such that $u_2((p, q)) \geq 0$. The game played will be $G = (A_1, A_2, F)$ where A_1 and A_2 represent action space for player 1 and player 2 respectively and F is the payoff function of the game. The action space for the client will be

$$A_1 = \{(p, q) \in \mathbb{R}^2 | \|(p, q) - (p_j, q_j)\| \leq p_j \varepsilon \text{ for some } (p_j, q_j) \in \mathcal{M}\}. \tag{1}$$

For the service provider or player 2, the action space will depend on the action taken by player 1:

$$A_2(p', q') = \{(p, q) \in \mathbb{R}^2 | \|(p, q) - (p', q')\| \leq p' \eta\}. \tag{2}$$

The payoff function $F : A_1 \times A_2 \times \Lambda \to \mathbb{R}^2$ is as follows:

$$F^1((p', q'), (p, q), \lambda_k) = (u_1((p, q), \lambda_k))^+ \tag{3}$$

$$F^2((p', q'), (p, q), \lambda_k) = \begin{cases} 0 & \text{Si } u_1((p, q), \lambda_k) < 0 \\ u_2(p, q) & \text{Si } u_1((p, q), \lambda_k) \geq 0 \end{cases} \tag{4}$$

Here we will add an additional assumption in order to simplfy the subject:

Assumption 1 *Given the assumptions in section* 3. *then the nondominated strategies for the client are finite. Moreover there are at most nm possible counteroffers (one for each contract in the menu) and are solution to the problem:*

$$\max \quad u_1((p,q),\lambda) \tag{5}$$

$$s.t. \quad \|(p,q)-(p_j,q_j)\| \le p_j\varepsilon \tag{6}$$

Here we consider the case when a client try to pretend to have another type. A proposition follows:

Proposition 2 *Given the assumptions in section* 3. *then the nondominated strategies for the service provider given a counteroffer from the client* (p', q') *are finite. Moreover there are at most* m *possible final offers (one for each client type) and are solution to the problem:*

$$\max \quad u_2((p,q)) \tag{7}$$

$$s.t. \quad \|(p,q)-(p',q')\| \le p'\eta \tag{8}$$

$$u_1((p,q),\lambda) \ge 0 \tag{9}$$

The assumption and the proposition that followed make it much easier to study the game and therefore its equilibria. Now we extend the payoff function to the set of strategies:

$$\mathcal{A}_1 = \{\sigma_1 = (\sigma_1(\cdot|\lambda_i))_{i=1}^m | \sum_{(p,q)\in A_1} \sigma_1((p,q)|\lambda_i) = 1 \quad \forall i = 1,\dots,m.\}. \tag{10}$$

$$\mathcal{A}_2 = \{\sigma_2 = (\sigma_2(\cdot|(p,q)))_{(p,q)\in A_1} | \sum_{(r,s)\in A_2} \sigma_2((r,s)|(p,q)) = 1 \quad \forall (p,q) \in A_1\}. \tag{11}$$

In the following way,$F : \mathcal{A}_1 \times \mathcal{A}_2 \times \Lambda \to \mathbb{R}^2$:

$$F^1(\sigma_1,\sigma_2,\lambda_k) = \sum_{(p,q)\in A_1} \sum_{(r,s)\in A_2(p,q)} \sigma_1((p,q)|\lambda_k)\sigma_2((r,s)|(p,q))F^1((p,q),(r,s),\lambda_k). \tag{12}$$

$$F^2(\sigma_1,\sigma_2,\lambda_k) = \sum_{(p,q)\in A_1} \sum_{(r,s)\in A_2(p,q)} \sigma_1((p,q)|\lambda_k)\sigma_2((r,s)|(p,q))F^2((p,q),(r,s),\lambda_k). \tag{13}$$

The idea is to find an equilibrium, that is a situation where neither the client nor the service provider would like to change their behaviour. The concept of equilibrium in this framework is the *perfect bayesian equilibrium* or PBE. A PBE is a trio (beliefs, client's strategy, service provider strategy) $=(\mu, \sigma_1^*, \sigma_2^*)$ that satisfies[2]:

$$(P1) \quad \forall i, \sigma_1^* \in \arg\max_{\alpha_1\in\mathcal{A}_1} F^1(\alpha_1,\sigma_2^*,\lambda_i),$$

$$(P2) \quad \forall a_1, \sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2\in\mathcal{A}_2} \sum_{i=1}^m \mu(\lambda_i|a_1)F^2(a_1,\alpha_2,\lambda_i),$$

$$(B) \quad \text{If } \sum_{i=1}^m r_i\sigma_1^*(a_1|r_i) > 0, \mu(\lambda_j|a_1) = \frac{r_j\sigma_1^*(a_1|\lambda_j)}{\sum_{i=1}^m r_i\sigma_1^*(a_1|\lambda_i)}.$$

[2] Here we considered the natural extension of the utility functions to mixed strategies

(P1) and (P2) mean neither the service provider nor the client have incentive to change their strategies since they are maximizing their utility. (B) means that given the actions of the client, the service provider doesn't change her beliefs about the client's type.

Some results have been deduced for the model, such as:

Proposition 3 *Given the model described, any strategy that satisfies (P2) is such that:*

$$(\forall (p', q') \in A_1)(\exists (p, q) \in A_2(p', q')) \quad \sigma_2^*((p, q)|(p', q')) = 1$$

4. Non Linear Numeric Example: 3 Clients 3 Initial Contracts

The model considered for this example consists on a client's utility function $u_1(p, q, \lambda_i) = a_i - c_i e^{-\lambda_i q} - p$ where the constants $\{a_i\}_{i=1}^{m}$ and $\{c_i\}_{i=1}^{m}$ are given, and a service provider utility function:

$$u_2(p, q, \lambda) = \begin{cases} p - d - ce^{zq} & \text{If client type } \lambda \text{ accepts} \\ 0 & \text{Otherwise} \end{cases}$$

where the constants d, c and z are given. It was also considered $m = n = 3$, that is 3 types of client and 3 initial contracts.

Initially we have the clients and the service provider, with their respective utility functions. In figure 1 we show the contour of the utility function asociated to the reservation utility, that is to say any contract with utility lower than the described is not acceptable, any contract with utility greater than the described is acceptable. The contracts with greater utility are in the direction of the arrow next to the utility function. For example for the service provider the contracts with greater utility are found by increasing the price or lowering the quality.

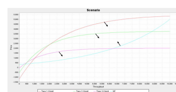

Figure 1: Utility functions of the players.

This allows to define negotiation areas, that is contracts that are rationally acceptable for both the service provider and the client. For example in figure 2 we see the negotiation area between the service provider and the client type 2.

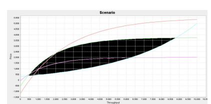

Figure 2: Negotiation area between the type 2 client and the service provider.

The same can be done for the type 3 client and the type 1 client.

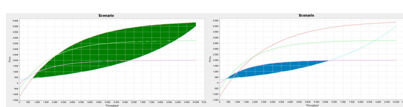

Figure 3: Negotiation areas between the type 1 and 3 client and the service provider.

Initially the service provider offers 3 contracts that can be located in the graph as seen in figure 4.

The client has a certain area in which he can modify the contract, as seen in figure 5:

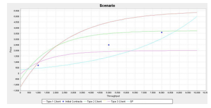

Figure 4: Location of the initial contracts.

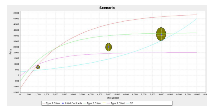

Figure 5: Area of posible replyes from the client.

From within these contracts the client may choose his reply, of course this reply will have less price and more quality if possible. For example possible client replyes might be like the ones described in figure 6.

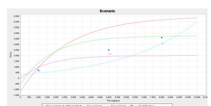

Figure 6: Possible client replyes.

Now seeing this counter offer the service provider tries to infer information about the client and designs an optimal counteroffer within a certain negotiation range. In figure 7 we present a negotiation range.

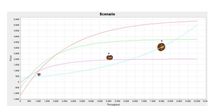

Figure 7: Service provider's negotation range.

In figure 8 we present the final schema of the negotiation. This is the actual output for the application developed.

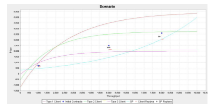

Figure 8: Service provider's negotation range.

5. An Example Of Application

The application was developed as a Java Swing frame based application using Eclipse's Visual Editor. The applications first screen consist of a dialog where you input all the parameters for the problem: Negotiation parameters, utility function parameters (This may be estimated also) and the initial contracts.

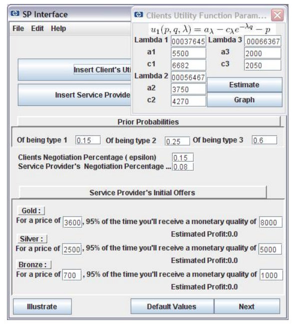

Figure 9: Application Screen 1.

The next screen requires as an input a maximum number of iterations, with default value as 100. These iterations are for computing the action space of the players. As we saw in proposition 1 the client's actions can be found by solving the problem (5), therefore according to Karush Kuhn Tucker conditions the solution to (5) is such that:

$$p = p_j - \frac{\varepsilon}{(e^{2\lambda_i q} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} e^{\lambda_i q} \tag{14}$$

$$q = q_j + \frac{c_i \lambda_i \varepsilon}{(e^{2\lambda_i q} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{15}$$

Therefore we used the following fixed point iteration to solve (15):

$$q_0 = q_j + \frac{c_i \lambda_i \varepsilon p_j}{(1 + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{16}$$

$$q_{n+1} = q_j + \frac{c_i \lambda_i \varepsilon p_j}{(e^{2\lambda_i q_n} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{17}$$

Here q_0 which corresponds to the solution of the linealized problem[3] and the number of iterations made are defined by the input parameter. To obtain the value of p we just replace the obtained value of q in (14).

In the second screen of the application the program prints the optimal replies for the client as well as a maximum error, this error is the difference:

$$Err = q_N - (q_j + \frac{c_i \lambda_i \varepsilon}{(e^{2\lambda_i q_N} + c_i^2 \lambda_i^2)^{\frac{1}{2}}})$$

[3] Just linealizing the utility function.

where N is the maximum number of iterations, then the error is the difference between the last iteration and what would be the next iteration. The number printed is the maximum error among all the computations.

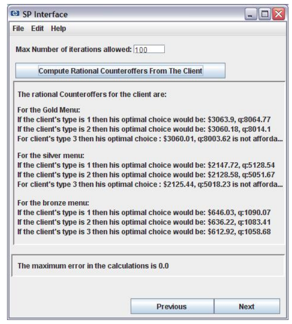

Figure 10: Application Screen 2.

After having the solution for the client's replyes the next step in the program is to find the service provider replyes. In order to do this the problem to solve is described by (7), to find the optimal solution to this problem we first found the solution to the following problems:

$$
\begin{aligned}
(Pb1) \quad & \max_{(p,q)} \quad u_2(p,q,\lambda) && (18)\\
& \text{s.t.} \quad \|(p,q)-(p',q')\| \leq p'\eta && (19)\\
(Pb2) \quad & u_1(p,q,\lambda) = 0 && (20)\\
& \|(p,q)-(p',q')\| = p'\eta && (21)
\end{aligned}
$$

To solve (Pb1) we used an analogous method, obtained the Karush Kuhn Tucker Conditions and the followed the following fixed point schema in order to solve them:

$$q_0 = q' - \frac{cz\eta p'}{(1+c^2z^2)^{\frac{1}{2}}} \tag{22}$$

$$q_{n+1} = q' - \frac{cz\eta p'}{(e^{-2zq_n}+c^2z^2)^{\frac{1}{2}}} \tag{23}$$

To solve (Pb2) we used (20) to obtain a relationship $p = f(q) + p'$ for some function f that does not depend on p. and replaced on (21). The following equation was obtained after replacing:

$$(q-q')^2 = \eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q} - p')^2$$

Here $f(q) = a_i - c_i e^{-\lambda_i q}$. This problem has two solutions, both may be feasible. Then we followed two fixed point schemas:

$$q_0^0 = q' \tag{24}$$

$$q_{n+1}^0 = q' - \sqrt{|\eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q_n^0} - p')^2|} \tag{25}$$

$$q_0^1 = q' \tag{26}$$

$$q_{n+1}^1 = q' + \sqrt{|\eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q_n^1} - p')^2|} \tag{27}$$

To illustrate the fixed points we are computing let's see figure 11

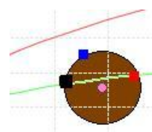

Figure 11: Fixed points solution.

In figure 11 we represent the negotiation area of the service provider by the brown circle, the client's utility function by the green curve and the client's reply by the pink center of the circle. The solution to (23) is the blue square which would be the optimal without considering if the client accepts or not, the solution to (25) would be the black square, which would be a feasible solution for both the client and the service provider, and the other feasible solution would be (27). To obtain the optimal reply given the type we consider out of these 3 problems (if applicable) which one gives the most utility.

Screens 3 and 4 of the application show the optimal reply of the service provider given each type.

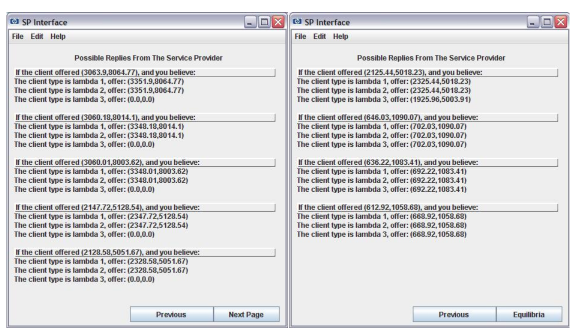

Figure 12: Application's screen 3 and 4.

Now that we the information regarding the actions of the player what's left is to find a perfect Bayesian equilibrium. Before explaining the methodology followed, first let's show the *natural* order of the problem:

- The client follows a strategy $\sigma_1(\cdot|\lambda_i)$

→ That strategy induces some beliefs $\mu(\lambda_j|a_1) = \frac{r_j \sigma_1^*(a_1|\lambda_j)}{\sum_{i=1}^m r_i \sigma_1^*(a_1|\lambda_i)}$

→ The service provider decides to play $\sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2 \in \mathcal{A}_2} \sum_{i=1}^m \mu(\lambda_i|a_1) F^2(a_1, \alpha_2, \lambda_i)$

→ The client obtains a utility $F^1(\sigma_1(\cdot|\lambda_i), \sigma_2^*, \lambda_i)$

Following this idea we decided to take an initial strategy for the client as an input, and use it to compute a utility $F^1(\sigma_1(\cdot|\lambda_i), \sigma_2^*, \lambda_i)$ following the schema mentioned above. Then by using as motivation replicator dynamics we updated the strategy of the client as follows:

$$\sigma_1^{n+1}((p,q)|\lambda) = \sigma_1^n((p,q)|\lambda) \frac{F^1((p,q), \sigma_2^*, \lambda)}{\sum_{(p'q')} \sigma_1^n((p',q')|\lambda) F^1((p',q'), \sigma_2^*, \lambda)}$$

This means that whatever action that gives greater utility is used more often that those that give less utility. Since the utility depends on the service provider response it's not clear that this dynamics converge to some value. We have the following proposition:

Proposition 4 *Consider the algorithm referenced above, when used over σ^0 such that $\sigma^0((p_j,q_j)|(p',q')) > 0 \forall (p',q') \in A_1 \quad (p_j,q_j) \in A_2$, if $\sigma_1^n \rightarrow_n \hat{\sigma_1}$ then $(\hat{\sigma_1}, sigma_2^*, \mu)$ with σ_2^*, μ obtained by the algorithm form a PBE.*

The application screen 5 is where you input an initial strategy and compute the clients utilities, while application screen 6 shows the beliefs induced and the optimal reply by the service provider. In screen 6 there is a RD button that iterates with replicator dynamics, there is a graphic button which shows the final situation and there is a Txt button that creates and prints the results on the text file: results.txt.

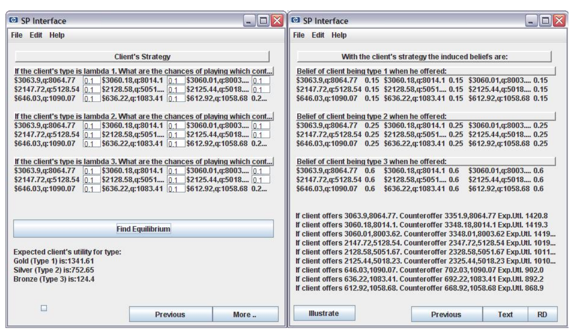

Figure 13: Application's screen 5 and 6.

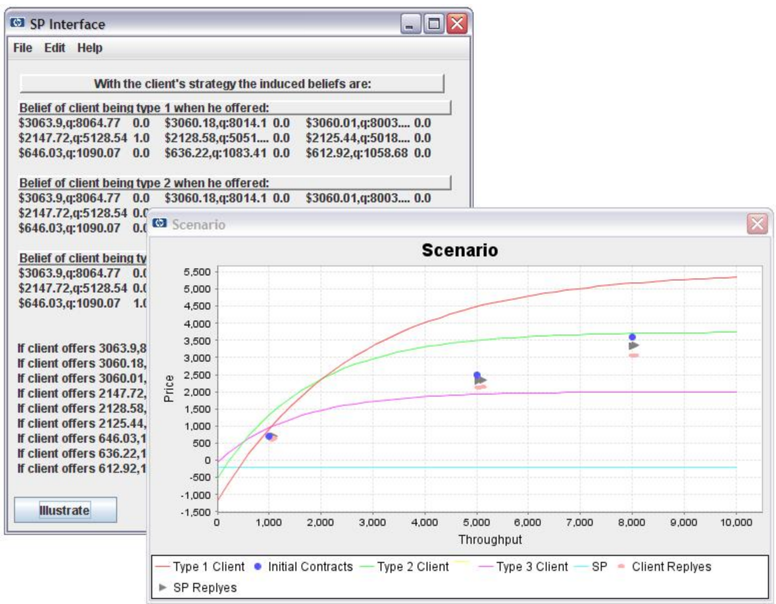

Figure 14: Application's final screen.

6. Related Work

Bargaining has attracted a lot of interest from economists for a long time. The emphasis done here is based on signaling games as presented on [4], there is also a great development of this topic in [2] and [6]; The signaling game that is somewhat similar to the one described is the education game presented, where the degree adquired acted as a signal for the employer. Different approaches to bargaining in multiple stages have been made by different authors, for example [5] used fictitious play to replicate the bargaining process, which is an interesting dynamic that has many appreciable properties. [7] uses an evolutionary approach to bargainnig such as the one used in the numerical implementation designed to find the PBE.

This work is also incomplete as it is of great importance the resource allocation problem while bargaining with two or more clients, since if both clients accept and the resources aren't enough the lack of service may lead to great money loses on penalizations and reputation.

Also the design of the initial contracts is of great importance since if they are optimal for the clients initially there may be no need for bargaining. A great deal about designing optimal contracts can be seen in [3].

Experiments may be used to validate the model, although as seen in [1] designing an experiment is a very delicate issue, where cleaning the test tubes in order to isolate the effects is fundamental.

7. Conclusions

The objective of this work was to develop a game theoretic framework for SLA negotiation in order to study the dynamics of the negotiation process. There were several models developed, starting from really simple models which could be studied completely to more complex models where only partial results could be stated, the idea of making the models more complex was to be able to reproduce as many aspects of real life bargaining as possible. The last model developd was the one presented on this article, this model considered the fact that counteroffers usually don't go over a certain range, and that negotiation proceses on IT markets are not usually long which is the main reason to consider a 2 stage bargaining model. The models had very similar dynamics, which is good in the sense that it means that there is some sort of robustness of the model. It is important to notice that the results have economic intuition, and are able to replicate in some level the negotiation when the players have enough information about each other and also agree in negotiation ranges. This hypotheses even though sound as too binding, are common sense in real negotiation, the problem is that the parameters such as the negotiation range differ from each player, and are usually hidden. The problem in developing a model where all of the parameters are hidden are that the results will be so variable that it would not be possible to conclude anything.

After defining the model the next step was the development of an application that in some sense solved the model, this application is to assist decisions more than a decision making application, this is due to the fact that people usually are not completely aware of their utility functions and make decisions base on an idea of their utility function.

There are some small adjustments that may be able to help produce more accurate results, such as consider that if the contract is not accepted the utility may be $u(0, 0, \lambda)$ rather than 0. This is useful to represent situations where the reserve value for all the different client varies. The best way to decide which adjustments strenghten the model, is to compare results from the model with actual negotiations, and in order to do this there is first a calibration stage where the parameters such as utility functions and negotiation precentages are defined, for example, using previous negotation data or assigned by an expert. Considering the idea as a whole the model and the application developed are just a contriution to a bigger and more ambitious project that as mentioned before may be complete by adding some statistic methods for determining the parameters given data, and equilibrium refinements for example.

References

[1] K. Binmore. *Does Game Theory Work? The Bargaining Challenge.* MIT Press, 2007.

[2] P. Bolton and M. Dewatripont. *Contract Theory.* MIT Press, 2005.

[3] S. Moroni et al. A game-theoretic framework for creating optimal sla/contract. http://www.hpl.hp.com/techreports/2007/HPL-2007-126.html, 2007.

[4] D. Fudenberg and J. Tirole. *Game Theory.* MIT Press, 1991.

[5] Younghwan In. A fictitious-play model of bargaining to implement the nash solution. *National University of Singapore, Department of Economics*, 2005.

[6] M. Osborne and A. Rubinstein. *A course in Game Theory.* MIT Press, 1994.

[7] P. Young. An evolutionary model of bargaining. *J. Econ. Theory*, 59:145–168, 1993.

A Proofs

1.1 Proof 1: Numerability

Proof. Proposition 2. The first thing is to recall the payoff for player 2 $F^2 : A_1 \times A_2 \times \Lambda \to \mathbb{R}$ defined by:

$$F^2((p',q'),(p,q),\lambda_k) = \begin{cases} 0 & \text{Si } u_1((p,q),\lambda_k) < 0 \\ u_2(p,q) & \text{Si } u_1((p,q),\lambda_k) \geq 0 \end{cases}$$

Which in its extended version is given by:

$$F^2((p',q'),\sigma_2((p,q)|(p',q'))) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} \sum_k \sigma_2(\vec{\xi}|(p',q'))F((p',q'),\vec{\xi},\lambda_k)\mu(\lambda_k|(p',q'))d\vec{\xi} & \text{Si}|A_2| > \aleph_0 \\ \displaystyle\sum_j \sum_k \sigma_2((p_j,q_j)|(p',q'))F((p',q'),(p_j,q_j),\lambda_k)\mu(\lambda_k|(p',q')) & \text{Si}|A_2| \leq \aleph_0 \end{cases}$$

Then since the player is rational will be maximizing its expected payoff, by choosing an strategy:

$$\sigma_2^* \in \arg\max F^2((p',q'),\sigma_2((p,q)|(p',q')))$$

To prove the proposition consider an action (p,q) not characterized by the system on the proposition, and examine the cases.

Case 1: It's feasible but not maximum. It can be seen directly that by choosing an action that is maximal the payoff would be greater.

Case 2: It's maximal but does not satisfy the inequality for any client. The payoff will be zero, which is dominated by any action that's feasible for some client, or weakly dominated by any action.

Case 3: It's not on the negotiation area. In this case the action is not feasible

Then the action space is finite and characterized by the systems on the proposition. ■

1.2 Proof 2: Atomic Strategies

Proof. Proposition 3. According to the previous proposition the payoff function for player 2 is:

$$F^2((p',q'),\sigma_2((p,q)|(p',q'))) = \sum_j \sum_k \sigma_2((p_j,q_j)|(p',q'))F((p',q'),(p_j,q_j),\lambda_k)\mu(\lambda_k|(p',q'))$$

Reagruping:

$$F^2((p',q'),\sigma_2((p,q)|(p',q'))) = \sum_j \sigma_2((p_j,q_j)|(p',q'))\{\sum_k F((p',q'),(p_j,q_j),\lambda_k)\mu(\lambda_k|(p',q'))\}$$

By calling $g((p_j,q_j)) = \sum_k F((p',q'),(p_j,q_j),\lambda_k)\mu(\lambda_k|(p',q'))$. If we prove that $\max_j g((p_j,q_j))$ is achieved by just one point, the result would be direct. Given the assumptions if the value is greater than zero, the result follows from the conditions of strict increase and convexity. ■

The previous proposition allows to identify an strategy form player 2 with a function $h : A_1 \to A_2$ such that $h((p,q)) = (r,s)$ with (r,s) satisfying $\sigma_2^*((r,s)|(p,q)) = 1$.

1.3 Proof 3: PBE Convergence

Dem. Proposición 4. We must verify that when we have $\sigma_1^n \to_n \hat{\sigma_1}$ we are on a PBE. Given tha algorithm (B) and (P2) follow directly since it's the invariant of the algorithm. (P1) is still to be checked. Assume (P1) is not satisfied, that means:

$$\exists i, \sigma_1^*(\cdot|\lambda_i) \notin \arg\max_{\alpha_1 \in \mathcal{A}_1} u_1(\alpha_1, \sigma_2^*, \lambda_i)$$

If we have this it means that there is another strategy that delivers strictly more utility. Such strategy can not have the same support as $\hat{\sigma_1}$, if it had the same support and also the condition it means:

$$\begin{aligned} F^1(\sigma_1^*, \sigma_2^*, \lambda_i) &> F^1(\sigma_1^n, \sigma_2^*, \lambda_i) \\ \sum_{\substack{k \in A_1 \\ j \in A_2(p_k, q_k)}} \sigma_1^*((p_k, q_k)|\lambda_i) F^1((p_k, q_k), (\hat{p}_j, \hat{q}_j), \lambda_i) &> \sum_{\substack{k \in A_1 \\ j \in A_2(p_k, q_k)}} \sigma_1^n((p_k, q_k)|\lambda_i) F^1((p_k, q_k), (\hat{p}_j, \hat{q}_j), \lambda_i) \\ \sum_{k \in A_1} \sigma_1^*((p_k, q_k)|\lambda_i) F^1((p_k, q_k), (\hat{p}_{g(k)}, \hat{q}_{g(k)}), \lambda_i) &> \sum_{k \in A_1} \sigma_1^n((p_k, q_k)|\lambda_i) F^1((p_k, q_k), (\hat{p}_{g(k)}, \hat{q}_{g(k)}), \lambda_i) \end{aligned}$$

Since $\hat{\sigma_1}$ is the limit strategy using (**??**), it satisfies:

$$\hat{\sigma_1}((r,s)|\lambda_i) = \hat{\sigma_1}((r,s)|\lambda_i) \frac{F^1((r,s), \sigma_2^*, \lambda_i)}{F^1(\hat{\sigma_1}, \sigma_2^*, \lambda_i)} \tag{28}$$

This implies that:

$$\hat{\sigma_1}((r,s)|\lambda_i) F^1(\hat{\sigma_1}, \sigma_2^*, \lambda_i) = \hat{\sigma_1}((r,s)|\lambda_i) F^1((r,s), \sigma_2^*, \lambda_i) \tag{29}$$

Then:

$$\hat{\sigma_1}((r,s)|\lambda_i) > 0 \Rightarrow F^1(\hat{\sigma_1}, \sigma_2^*, \lambda_i) = F^1((r,s), \sigma_2^*, \lambda_i). \tag{30}$$

So all actions played on the limit strategy pay the same value. Also there is no (p,q) such that $F^1((p,q), \sigma_2^*, \lambda_i) > F^1(\hat{\sigma_1}, \sigma_2^*, \lambda_i)$, since as they have the same support this contradicts (30).

Actually the only relevant thing is that there is $(r,s) \in \arg\max F^1((r,s), \sigma_2^*, \lambda_i)$ such that $\hat{\sigma_1}(r,s)|\lambda_i) > 0$. This follows directly from the fact that σ_2^* is achieved on finite time (still to prove), and in any finite number of iterations N $\sigma_1^0 > 0 \Leftrightarrow \sigma_1^N > 0$. ■

B Incorporating the model in SLA negotiation

The main thing is to determine what the uncertainty will be. This uncertainty is what is referred as type in the model, and it can be a parameter of the utility function, may be related to risk aversion or clients preferences. Given the uncertainty we define the players, their strategies and the payoffs. Then we start looking for equilibria. There are many possibilities: service provider's utility function is uncertain, client's reservation utility is uncertain, service provider's reservation utility is uncertain, client's type/utility function is uncertain, etc.

2.1 First Model: SP's Utility Function is Uncertain

The client may not know how tight is the SP situation in terms of resources. The service provider will most of the time try to give the impression she is in a tight situation in order to get more profit as the SP may claim it's more costly for her to give quality given the resources. Let's define the game in this situation:

The player with private information is the SP then she will be player 1 and the client will be player 2. The strategies for both players could be various SLA's templates and the space of possible types could be a paramater of a particular function family or completely different functions. It is important to point out that even if we consider certain utility functions and assumptions, the client doesn't know the SP's utility function (Most of the time the SP doesn't know his own utility function) but he does know that if the resources are scarce it will be more costly, and has an idea of how costly it can be. So in reality the situation is similar, and the main results established apply in a certain degree.

2.2 Linear Example For The First Model

Consider that the possible utility functions for the SP may be $u((p,q),\theta) = p - \theta q, \quad \theta \in \Theta = \{\theta_1, \theta_2\}$, where $\theta_1 < \theta_2$; that is when the SP utility function is $u(\cdot,\theta_2)$ it means that the marginal cost[4] quality is higher than when her utility function is $u(\cdot,\theta_1)$. For simplicity let us consider that the action sace of the service provider is $A_1 = \{(\underline{p},\underline{q}),(\overline{p},\overline{q})\}$ such that $\underline{p} - \theta_1\underline{q} > 0 > \underline{p} - \theta_2\underline{q}$ and $\overline{p} - \theta_1\overline{q} > \overline{p} - \theta_2\overline{q} > 0$; $(\underline{p},\underline{q})$ represents a contract that the SP is unable to accept when she is tight on resources. We consider that the probability that the SP is under the $u(\cdot,\theta_1)$ utility function is t, and that the service provider accepts any contract that gives her utility greater or equal than zero.

Let's assume that the utility for the client when accepting a contract (p,q) is $V(p,q) = q - p$, and that the client's budget bounded by p^*.

Now we can describe the payoff function of the game:

$$F : \begin{array}{ccc} A_1 \times \mathbb{R}^2 \times \Theta & \longrightarrow & \mathbb{R} \\ ((p,q),(r,s),\theta) & \longmapsto & \begin{pmatrix} \max\{0, u((r,s),\theta\} \\ v(r,s)\mathbb{1}_{\{u(\cdot,\theta)>0\}}(r,s) \end{pmatrix} \end{array}$$

The payoff for the service provider when accepting contract (p,q) is $F^1(\cdot,(p,q),\theta)^5$. The payoff for

[4] The marginal cost is the cost of delivering 1 aditional unit of quality.

[5] Even though it may appear that the initial offer has no relevance, the relevance of the SP initial offer is that it influences the beliefs that the client has about the resources avaliability of the SP

the client is $F^2(\cdot, (p,q), \theta)$.

With everything defined we can describe the game in extensive form:

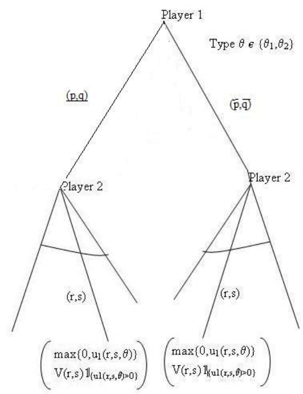

Figure 15: Game in extensive form.

The payoff for player 1, as described before, is the first component of the vector located at the bottom of the figure, player 2's payoff is the second component .

2.2.1 Equilibria of the game

We would like to find $(\mu, \sigma_1^*, \sigma_2^*)$ PBE. Let's study the possibilities:

$(P1)$ Establishes that the SP must maximize her expected utility. That is:

$$\sigma_1^* \in \arg\max_{\alpha_1} \sum_{(p,q)} \sum_{(r,s)} \alpha_1((p,q)|\theta)\sigma_2^*((r,s)|(p,q))(r-\theta s)^+$$

Which excluding all the contracts that the SP won't accept, can be rewritten as:

$$\sigma_1^* \in \arg\max_{\alpha_1} \sum_{(p,q)} \sum_{(r,s)\in\mathcal{F}_\theta} \alpha_1((p,q)|\theta)\sigma_2^*((r,s)|(p,q))(r-\theta s) \tag{31}$$

Where $\mathcal{F}_\theta$ denotes all the feasible contracts for the SP. That is $\mathcal{F}_\theta = \{(p,q)|p > \theta q\}$.

$(P2)$ Establishes that the client must maximize his expected utility given the beliefs for any action (p,q). That is:

$$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|(p,q))F^2((p,q),\alpha_2,\theta)$$

Which can be rewritten as:

$$\sigma_2^* \in \arg\max_{\alpha_2} \mu(\theta_1|(p,q)) \sum_{(r,s)\in\mathcal{F}_{\theta_1}} \alpha_2((r,s)|(p,q))(s-r) + \mu(\theta_2|(p,q)) \sum_{(r,s)\in\mathcal{F}_{\theta_2}} \alpha_2((r,s)|(p,q))(s-r) \quad (32)$$

The last condition is that μ satisfies the Bayes' rule. According to the beliefs induced after the offer we divide the equilibria in separating, pooling and hybrid.

- **Separating Equilibria**

  A separating equilibrium is one that after the SP makes her move her type is revealed. Since there is a contract that is unfeasible for the SP under one type, there is only one reasonable possibility, that is:

  $$\mu(\theta_1|(\underline{p},\underline{q})) = 1; \qquad \mu(\theta_2|(\overline{p},\overline{q})) = 1$$

  What are the consequences/conditions for this type of equilibrium?

  Proposition 5 *Under the assumptions of this model there are no separating equilibria.*

  Proof. $(P2)$ establishes that the client should maximize his utility given the beliefs, therefore:

  $$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{(r,s)\in\mathcal{F}_{\theta_1}} \alpha_2((r,s)|(\underline{p},\underline{q}))(s-r) \quad (33)$$

  $$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{(r,s)\in\mathcal{F}_{\theta_2}} \alpha_2((r,s)|(\overline{p},\overline{q}))(s-r) \quad (34)$$

  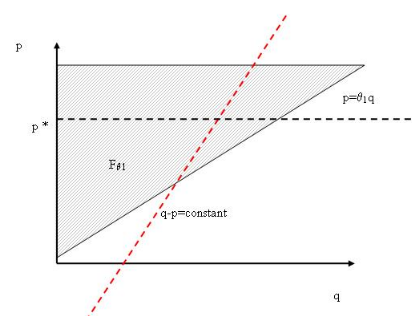

  Figure 16: Maximization problem described by (33)

  Therefore we deduce that if $\theta_1 > 1$ then $\sigma_2^*((0,0)|(\overline{p},\overline{q})) = 1$ that is, there is no deal. We'll suppose from now on, that $\theta_1 < 1$. Then the previous condition states that $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1$ and $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1$.

Let's see the conditions $(P1)$ given this strategy σ_2^*:

$$\sigma_1^* \in \arg\max_{\alpha_1} \alpha_1((\overline{p},\overline{q})|\theta_1)(p^* - \theta_1 \tfrac{p^*}{\theta_2}) + \alpha_1((\underline{p},\underline{q})|\theta_1)(p^* - \theta_1 \tfrac{p^*}{\theta_1}) \quad (35)$$

$$\Rightarrow \quad \sigma_1^*((\overline{p},\overline{q})|\theta_1) = 1 \quad (36)$$

$$\sigma_1^* \in \arg\max_{\alpha_1} \alpha_1((\overline{p},\overline{q})|\theta_2)(p^* - \theta_2 \tfrac{p^*}{\theta_2}) + \alpha_1((\underline{p},\underline{q})|\theta_2) \underbrace{(p^* - \theta_2 \frac{p^*}{\theta_1})^+}_{0} \quad (37)$$

$$\Rightarrow \quad \sigma_1^*((\overline{p},\overline{q})|\theta_2) \in [0,1] \quad (38)$$

Finally the Bayes' rule condition must be studied by cases:

If $\sum_i \sigma_1^*((\underline{p},\underline{q})|\theta_i) > 0$ then (B) is satisfied only by the direct revelation strategy $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1, \sigma_1^*((\underline{p},\underline{q})|\theta_2) = 0$. This goes against condition (36). Therefore there is no separating equilibrium such that $\sigma_1^*((\underline{p},\underline{q})|\theta_1) > 0$ or $\sigma_1^*((\underline{p},\underline{q})|\theta_1) > 0$.

If $\sum \sigma_1^*((\overline{p},\overline{q})|\theta_i) > 0$ then is satisfied only by the direct revelation strategy $\sigma_1^*((\overline{p},\overline{q})|\theta_2) = 1$ and $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = 0$, but this also goes against condition (36).

Then we conclude that there are no separating equilibria. ■

- **Pooling Equilibria**

  Pooling equilibria are the ones that after the SP makes her move, the client does not update his beliefs. That is:

$$\mu(\theta_1|(\overline{p},\overline{q})) = t; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = t.$$

Proposition 6 *Under the assumptions of this model the strategy of the client will be deterministic depending on the prior probability t. More specifically,* $\sigma_2(p^*, \frac{p^*}{\theta_1}) = 1$ *if* $t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$, *and* $\sigma_2(p^*, \frac{p^*}{\theta_1}) = 0$ *otherwise.*

Proof. Let's study the conditions now, let's call $\xi(r,s)$ the variable representing $\alpha_2((r,s)|(\overline{p},\overline{q}))$, then:

$$\sigma_2^* \in \arg\max_{\xi} t \sum_{(r,s)\in\mathcal{F}_{\theta_1}} \xi(r,s)(s-r) + (1-t) \sum_{(r,s)\in\mathcal{F}_{\theta_2}} \xi(r,s)(s-r) \quad (39)$$

Linearity implies that the solution of the maximization problem (39) is such that $\xi(p^*, \frac{p^*}{\theta_1}) + \xi(p^*, \frac{p^*}{\theta_2}) = 1$ since other allocations give strictly less payoff. now the problem is to find the optimal distribution over these 2 points. Let $\phi = \xi(p^*, \frac{p^*}{\theta_1})$, then the maximization problem we have to solve is:

$$\max_{0\le\phi\le1} t[(1-\phi)(\frac{p^*}{\theta_2} - p^*) + \phi(\frac{p^*}{\theta_1} - p^*)] + (1-t)(1-\phi)(\frac{p^*}{\theta_2} - p^*) \quad (40)$$

The solution to this problem is $\phi = 1$ if $t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$, and $\phi = 0$ otherwise.

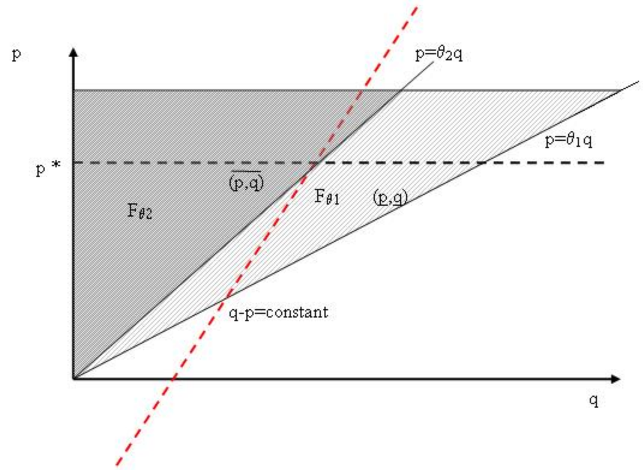

Figure 17: Maximization problem for the client

Since the equations are the same for $\alpha_2((r,s)|(\underline{p},\underline{q}))$ we have the same result: the strategy is deterministic depending on t. ■

This means that if the probability of a relaxed service provider is high enough the client will always counteroffer according to the relaxed type. This is consequent with the fact that the service providers signal does not allow the client to update his beliefs, therefore his decision is not based in the signal.

Let's see the other conditions that make this equilibrium sustainable.

Proposition 7 *The pooling equilibria of this game are infinetly many, and are characterized by:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = s \qquad \forall s \in [0,1] \tag{41}$$

$$\sigma_2(p^*, \frac{p^*}{\theta_1}) = 1 \text{ } if \text{ } t > \frac{\frac{1}{\theta_2} - 1}{\frac{1}{\theta_1} - 1} \qquad \sigma_2(p^*, \frac{p^*}{\theta_1}) = 0 \text{ } otherwise \tag{42}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = t \qquad \mu(\theta_1|(\underline{p},\underline{q})) = t \tag{43}$$

Proof. The characterization of σ_2 had already been proven, and the last equation is the definition of pooling equilibrium, let's see the characterization of $sigma_1$.

Conditions ($P1$): Since player 2 strategy is deterministic, depending only on the prior probabilities it is not hard to see that if $\sigma_2(p^*, \frac{p^*}{\theta_1}) = 1$ the expected utility is always zero independent of the type (since when she accepts the counteroffer gets zero, and otherwise she rejects the counteroffer), and if $\sigma_2(p^*, \frac{p^*}{\theta_1}) = 0$ she gets zero in case her type is θ_2 and constant equal to $p^* - \frac{\theta_1}{\theta_2}p^*$ in case her type is θ_1. Then there are no restrictions for σ_1^* regarding ($P1$).

The Bayes condition states that:

$$\begin{pmatrix} t & 1-t \\ t & 1-t \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} \tag{44}$$

That is SP's strategy must be an eigenvector asociated to the eigenvalue 1 for the matrix above. Then to satisfy this system it is neccesary that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2)$. Then we conclude it can be any strategy as long as it doesn't depend on the type. ■

Since we can use any strategy as long as it doesn't depend on the type it would be nice to find the optimal in mean for the service provider. But since the client does not care what the service providers signal, he only cares about the probability *ex ante*, the Service provider expected utility is $t(p^* - \frac{\theta_1}{\theta_2}p^*)$ only if $t < \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$ regardless of the strategy.

- **Semi-Pooling Equilibria**

In the case that the client believe the service provider never offers a contract that does not give him positive utility, we have that:

$$\mu(\theta_1|(\overline{p},\overline{q})) = t; \qquad \mu(\theta_2|(\underline{p},\underline{q})) = 0$$

Proposition 8 *The semi pooling equilibria of this game are infinetly many, and are characterized by:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = s \qquad s \in \{0,1\} \tag{45}$$

$$\sigma_2(p^*, \frac{p^*}{\theta_1}) = 1 \textit{ if } t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1} \qquad \sigma_2(p^*, \frac{p^*}{\theta_1}) = 0 \textit{ otherwise} \tag{46}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = t \qquad \mu(\theta_2|(\underline{p},\underline{q})) = 0 \tag{47}$$

Proof. (P2) Condition results for $\sigma_2^*((r,s)|(\overline{p},\overline{q}))$ are basically the same, the change is that now $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1$.

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) > 0$ the Bayes' rule implies that $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1$ which has as a consequence that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = 0$, also if $\sum \sigma_1^*((\overline{p},\overline{q})|\theta_i) > 0$, the other condition (44) implies that $\sigma_1^*((\overline{p},\overline{q})|\theta_2) = 0$ which contradicts $\sum \sigma_1^*((\overline{p},\overline{q})|\theta_i) > 0$. Then the service providers strategy will be to always offer $(\underline{p},\underline{q})$.

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) = 0$, means that $\sigma_1^*((\overline{p},\overline{q})|\theta_i) = 1$ which satisfies condition (44), then the other equilibrium is to always offer $(\overline{p},\overline{q})$. ■

- **Hybrid Equilibria**

First we will study the case when the client believes the service provider is not willing to lose money. In this context we have 1 extra parameter:

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = 1$$

Proposition 9 *There are no equilibria with the beliefs* $\mu(\theta_1|(\overline{p},\overline{q})) = \eta; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = 1$ *other than the case* $t = \eta$

Proof. Algebraically is the same problem for the client as the case before, thus $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1$ if $\eta > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$, and $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1$ otherwise.

Since strategy of $(P2)$ is very similar as in the case before (algebracally the same), $(P1)$ does not add any restrictions just like in the case before.

The Hybrid equilibria desirable for the service provider are those where $\eta < \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$. To achieve this Bayes' rule implies that:

$$\begin{pmatrix} \eta & \frac{1-t}{t}\eta \\ \frac{t}{1-t}(1-\eta) & 1-\eta \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix}$$

Which implies that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2)$

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) > 0$, then the other Bayes' rule condition states that $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1$ and $\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 0$, therefore $1 = \sigma_1^*((\overline{p},\overline{q})|\theta_1) \neq \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) = 0$. Then this equilibrium is not sustainable.

If $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \sigma_1^*((\underline{p},\underline{q})|\theta_2) = 0$, then the Bayes' rule does not aply, but this directly implies that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = 1$. Which contradicts the previous Bayes' condition unless $t = \eta$. Then the equilibria is not sustainable. ■

Now if we allow that the client believes the service provider is willing to offer contracts not . In this context we have 2 extra parameters:

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = \pi$$

Proposition 10 *The following* $(\sigma_1^*, \sigma_2^*, \mu)$*:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_2) = \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{48}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{49}$$

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{50}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}(1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}}) \tag{51}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1 \qquad \textit{If } \frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}) \tag{52}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1 \qquad \textit{If } \frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}) \tag{53}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1 \qquad \textit{If } \frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} \tag{54}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1 \qquad \textit{If } \frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} \tag{55}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta \qquad \mu(\theta_1|(\underline{p},\underline{q})) = \pi \tag{56}$$

Is an equilibrium.

Proof. Bayes' rule condition (if the strategy is mixed) implies that:

$$\begin{pmatrix} \eta & \frac{1-t}{t}\eta \\ \frac{t}{1-t}(1-\eta) & 1-\eta \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} \tag{57}$$

$$\begin{pmatrix} \pi & \frac{1-t}{t}\pi \\ \frac{t}{1-t}(1-\pi) & 1-\pi \end{pmatrix} \begin{pmatrix} \sigma_1^*((\underline{p},\underline{q})|\theta_1) \\ \sigma_1^*((\underline{p},\underline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\underline{p},\underline{q})|\theta_1) \\ \sigma_1^*((\underline{p},\underline{q})|\theta_2) \end{pmatrix} \tag{58}$$

These equations imply that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2)$ and $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}\sigma_1^*((\underline{p},\underline{q})|\theta_2)$.
The solution to $(P2)$ maximization problem is:
If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1$, otherwise $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1$, and if $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1$, otherwise $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1$.
Now instead of looking at condition $(P1)$ let's see how the utility function of the service provider changes as she tries to implement this equilibria. That is let's assume that the service provider reacts according to Bayes' rule, and see how much utility can she get depending on the beliefs.

Finally there is a restriction that completely defines σ_1^*:

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) + \sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1 \quad (59)$$

$$\frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) + \frac{(1-t)\pi}{t(1-\pi)}\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1 \quad (60)$$

$$\frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) + \frac{(1-t)\pi}{t(1-\pi)}(1-\sigma_1^*((\overline{p},\overline{q})|\theta_2)) = 1 \quad (61)$$

The last equation implies that:

$$\sigma_1^*((\overline{p},\overline{q})|\theta_2) = \frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}} \quad (62)$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1-\frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}} \quad (63)$$

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}} \quad (64)$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}(1-\frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}}) \quad (65)$$

■

Therefore, there are 4 cases:

If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^*-\theta_1\frac{p^*}{\theta_2})$.

If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^*-\theta_1\frac{p^*}{\theta_2})\frac{(1-t)\eta}{t(1-\eta)}\frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}}$.

If $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^*-\theta_1\frac{p^*}{\theta_2})\frac{(1-t)\pi}{t(1-\pi)}(1-\frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}})$.

If $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is 0.

Observation : Of course the service provider would like to achieve the first equilibrium, the one when $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$, to simplify notation let's consider $\frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} = c \in [0,1]$. In order to satisfy these conditions it is necesary that :

$$(1-c)\eta + c\pi < c \quad \text{and} \quad 1-c < c\pi + (1-c)\eta \quad (66)$$

Rewritting this last equation we have:

$$c\pi + (1-c)\eta \in (c, 1-c) \quad (67)$$

To obtain this equilibrium it is necesary that $\frac{1}{2} < c$. We can do the same with the other equilibria:

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in [0, \min\{c, 1-c\}) \tag{68}$$

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in (\max\{c, 1-c\}, 1] \tag{69}$$

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in (c, 1-c) \tag{70}$$

To be able to achieve the last equilibrium it's necesary that $\frac{1}{2} > c$, [6]

The case (68) and (69) are always accesible, and they give smaller revenue than the one that can be achieved if $\frac{1}{2} < c$.

So if $\frac{1}{2} < c$, we can obtain maximum revenue by finding feasible beliefs such that $c\pi+(1-c)\eta \in (c, 1-c)$. If $\frac{1}{2} > c$ the service provider has to solve a maximization problem that yields what beliefs are optimal.

2.3 Second Model: Client's type/utility function is uncertain

Now let's assume the service provider doesn't know a parameter of the client's utility function. For example if the client's utility function is $V((p,q),\lambda) = a_\lambda - c_\lambda e^{-\lambda q} - p$, where $\{a_\lambda, c_\lambda\}$ are known for each λ, and the SP is uncertain of what's the λ of the client he is negotiating with, although he might have an initial idea. We model this by assuming that there is a prior probability distribution over a set of posible values of λ ($\mathbb{P}(\lambda = \lambda_i) = r_i$), that is common knowledge for both, the client and the service provider.

The situation we are modelling is the following: The client goes to the service provider and asks her what are the possible contracts she has to offer. The client then chooses one of the contracts and modifyies it a little. The service provider's looks at what the client chose as a contract and updates her beliefs about what kind of client she is dealing with, and then offers a final contract, taking into account the client's counteroffer. The client will then accept or reject this counteroffer.

The changes in the model are as follows:

Now the first player will be the client and his action space is restricted to:

$$A_1 = \{(p,q) | \exists (\tilde{p},\tilde{q}) \in \mathcal{M} \quad \|(\tilde{p},\tilde{q}) - (p,q)\| < \varepsilon\}.$$

Where $\mathcal{M}$ is the menu of contracts the service provider initially has to offer.

The counteroffer from the service provider when the client offers $(\tilde{p},\tilde{q})$ will also be restricted to the set:

$$A_2(\tilde{p},\tilde{q}) = \{(p,q) | \quad \|(\tilde{p},\tilde{q}) - (p,q)\| < \eta\}.$$

[6]Inducing the belief $\eta = \pi = \frac{1}{2}$ can not be achieved unless $t = \frac{1}{2}$. Since it violates (59) -(61).

We'll consider ε, η proportional to the price of the contract being modified.
The proportionality constant will be referred to as the client's and service provider's negotiation factor respectively.

2.4 Linear Example For The Second Model

First let's consider as an example when the utility functions of the service provider and the client are linear. Then the client's utility function is $V((p,q),\lambda) = \lambda q - p$ where $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$ (this is a simplification of the example mentioned before). The service provider utility function will be $u(p,q) = p - zq$ where z is some fixed constant. The service provider initially offers 3 menus: $(p_b, q_b), (p_s, q_s)$ and (p_g, q_g). Now let's describe the game:

First the client chooses (r,s) such that $\|(r,s) - (p_b, q_b)\| < \varepsilon$, $\|(r,s) - (p_s, q_s)\| < \varepsilon$ or $\|(r,s) - (p_g, q_g)\| < \varepsilon$. Of course, the client has many choices, and for simplicity we'll usually focus on 9 options: the ones that give maximum utility to the client for any kind of type.

Assumption 11 *By solving the maximization problem:*

$$\max \quad \lambda q - p \tag{71}$$

$$s.t. \quad \|(p,q) - (p_j, q_j)\| \leq \varepsilon \tag{72}$$

where $j \in \{b, s, g\}$, we obtain the different actions that the player may take.

The solution to this problem is:

$$p = p_j - \frac{\varepsilon}{(1+\lambda^2)^{\frac{1}{2}}} \tag{73}$$

$$q = q_j + \frac{\lambda\varepsilon}{(1+\lambda^2)^{\frac{1}{2}}} \tag{74}$$

Which is what can be expected, they counteroffer less price and more quality within the range.

Next the service provider will take that offer, update her beliefs, and make a counteroffer that will also be around the client's offer. That is, the counteroffer (p,q) is such that $\|(p,q)-(r,s)\| < \eta$. The payoffs are similar to the game described before. If no contract is accepted then they both get zero, and if the contract (p,q) is accepted the client type λ and the service provider get $V((p,q),\lambda)$ and $u(p,q)$ respectively.We will also assume that the client accepts any contract such that $V((p,q),\lambda) \geq 0$, the service provider will accept any contract such that $u((p,q)) \geq 0$.

Proposition 12 *When $z \neq \lambda_k, \quad k = 1,2,3$, the SP actions are:*

$$p = p^* + \frac{\eta}{(1+z^2)^{\frac{1}{2}}} \tag{75}$$

$$q = q^* - \frac{z\eta}{(1+z^2)^{\frac{1}{2}}} \tag{76}$$

Where (p^*, q^*) *is the client's offer, or*

$$q = \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)} \tag{77}$$

$$p = \lambda \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)} \tag{78}$$

Where (p^*, q^*) *is the client's offer and* $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$.

Proof. Since we know that the client's offers are limited we may do the same with the service provider. The service provider will face the problem:

$$\max \quad p - zq \tag{79}$$

$$\text{s.t.} \quad \|(p,q) - (p^*, q^*)\| \leq \eta \tag{80}$$

Where (p^*, q^*) is the client's offer. Then the solution to this problem will be:

$$p = p^* + \frac{\eta}{(1+z^2)^{\frac{1}{2}}} \tag{81}$$

$$q = q^* - \frac{z\eta}{(1+z^2)^{\frac{1}{2}}} \tag{82}$$

With this we have reduced the action space the players.

But there is a chance that the contract is not feasible for the client in that case the service provider will go for a contract that's good for him and also feasible for the client. A contract defined by:

$$\max \quad p - zq \tag{83}$$

$$\text{s.t.} \quad \|(p,q) - (p^*, q^*)\| \leq \eta \tag{84}$$

$$\lambda q - p \geq 0 \tag{85}$$

The solution to this problem is:

$$q = \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)} \tag{86}$$

$$p = \lambda q \tag{87}$$

Whether is plus or minus is determined by which one gives greater utility to the service provider.
■

The game played will be $G = (A_1, A_2, F)$ where A_1 and A_2 represent action space for player 1 and player 2 respectively and F is the payoff function of the game. The action space for the client will be

$$A_1 = \{(p,q) \in \mathbb{R}^2 | \|(p,q) - (p_j, q_j)\| \leq p_j\varepsilon \text{ for some } (p_j, q_j) \in \mathcal{M}\}. \tag{88}$$

For the service provider or player 2, the action space will depend on the action taken by player 1:

$$A_2(p', q') = \{(p, q) \in \mathbb{R}^2 | \|(p, q) - (p', q')\| \leq p'\eta\}. \tag{89}$$

The payoff function $F : A_1 \times A_2 \times \Lambda \to \mathbb{R}^2$ is as follows:

$$F^1((p', q'), (p, q), \lambda_k) = (V((p, q), \lambda_k))^+ \tag{90}$$

$$F^2((p', q'), (p, q), \lambda_k) = \begin{cases} 0 & \text{Si } V((p, q), \lambda_k) < 0 \\ u(p, q) & \text{Si } V((p, q), \lambda_k) \geq 0 \end{cases} \tag{91}$$

2.4.1 Equilibria

Here we have more parameters than in the case before, let's just study the equilibria we'd think more likely: hybrid equilibria. Let:

$$\mu(\lambda_i | o_j) = a_{ij}$$

Where o_j is the optimal offer for client type $\lambda_{j-[\frac{j}{3}]\cdot 3}$ when choosing the $(p_{[\frac{j}{3}]+1}, q_{[\frac{j}{3}]+1})$ contract.

To study $(P2)$ condition let's recall what was mentioned earlier about the action space of the service provider. Now that we know the strategies, payoff and beliefs for the service provider. This show us that depending on the belief of the value of λ the service provider has 3 choices for each value, and 7 choices total. Considering this the solution to $(P2)$ is direct: The service provider will play the contract that gives him the most expected utility. Explicitly the condition is:

$$\sigma_2^*(\cdot | o_j) \in \arg\max_{\alpha} \sum_k [F^2(o_j, \varphi_k, \lambda_1) a_{1j} + F^2(o_j, \varphi_k, \lambda_2) a_{2j} + F^2(o_j, \varphi_k, \lambda_3) a_{3j}] \alpha(\varphi_k) \tag{92}$$

Where φ_k represents one of the possible offers made by the service provider, mentioned in the previous paragraph. As mentioned before the solution to this problem is $\sigma_2^*(\varphi_{k(j)} | o_j) = 1$, where $k(j)$ is such that $[F^2(o_j, \varphi_{k(j)}, \lambda_1) a_{1j} + F^2(o_j, \varphi_{k(j)}, \lambda_2) a_{2j} + F^2(o_j, \varphi_{k(j)}, \lambda_3) a_{3j}]$ is maximum.

Now with σ_2^* defined (with the beliefs as a parameter), we can find the client's expected payoff given an strategy. Given a strategy $\sigma_1(\cdot | \lambda_i)$ the client's payoff is:

$$\begin{aligned} F^1(\sigma_1(\cdot | \lambda_i), \sigma_2^*, \lambda_i) &= \sum_j \sigma_1(o_j | \lambda_i) \sum_k \sigma_2^*(\varphi_k | o_j)(\lambda_i q_k - p_k) & (93) \\ &= \sum_j \sigma_1(o_j | \lambda_i)(\lambda_i q_{k(j)} - p_{k(j)}) & (94) \end{aligned}$$

Where q_k and p_k are the quality and price stated in contract φ_k. Therefore if the client is rational, given the beliefs he will maximize his payoff while trying to keep the beliefs.

Even thoughnstudying this game would be interesting we will focus on the next model, which is a more general than this model.