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HPL-2008-161

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External Posting Date: October 21, 2008 [Fulltext] External Copyright Consideration

Internal Posting Date: October 21, 2008 [Fulltext]

Submitted to Operations Research

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1. Introduction

The newsboy problem has played a central role at the conceptual foundations of stochastic inventory theory, and variants of it have been used in analysis of decision problems – such as capacity, allocation and overbooking – under demand uncertainty. In the classical newsboy problem, a firm facing uncertain demand orders a quantity of a perishable item prior to observing demand. If the demand realization is less than the ordered quantity, then the firm will have excess inventory in hand that will perish. If demand turns out to be more than the ordered quantity, then the firm will miss the opportunity of additional profit. In the well–known characterization, the optimal order quantity, which balances the marginal expected cost of ordering one more unit against the marginal expected revenue from satisfying an additional demand, is a critical quantile of the demand distribution.

In the standard newsboy model, strategic interactions are assumed away by taking the demand faced by a firm as a model primitive. In many practical situations, however, the details of the market interaction does matter for the order quantity decisions. Some or all of a firm's unsatisfied demand can be served by other firms offering substitutes; and, vice versa, a firm may be able to sell more than its initial market share in case the rival firm is understocked. Under such conditions, a firm's payoff depends on rival firms', as well as its own, order quantities and appropriate analysis of optimal inventory decisions requires a game theoretic approach. The resulting model, dubbed the competitive newsboy model, has been studied in the literature starting with the seminal works of Parlar (1988), studying the case where the firms' initial demands are statistically independent, and Lippman and McCardle (1997), studying the cases where the demands faced by competing firms are derived from a general class of rationing rules applied to the total industry demand.

A natural extension of the competitive newsboy analysis involves incorporating information asymmetry. Asymmetric information adds a new dimension to the competitive newsboy problem. Firms may be asymmetrically informed in a competitive newsboy setting due to two broad reasons. The firms may be privately informed about their cost and/or revenue structures. Alternatively, there may be asymmetric information regarding the market demand. Alternative specifications for the key structural elements – e.g., the nature of information asymmetry, the structure of the market and firm demands – span a number of interesting classes of models. Among these are models of newsboy oligopoly, and models that allow arbitrary statistical dependence in firm demands, and in cost structures.

In this paper, we study the competitive newsboy problem with asymmetric cost information. The competitive newsboy model we study is built on Parlar (1988) and Lippman and McCardle (1997). The industry demand is random. There are two firms among whom the industry demand is split. Each firm has private information about their costs. If the demand that is allocated to one firm exceeds the order quantity of that firm, a portion of the excess demand spills over to the rival firm. As standard in analysis of games of incomplete information, we use the Bayesian–Nash equilibrium as the solution concept. In a Bayesian–Nash equilibrium each player's strategy is a best response against the strategies of the competing players.

The rest of this paper is organized as follows. In Section 2, we review the related literature. In Section 3, we introduce a model of inventory competition under asymmetric information. Section 4 presents our main results on the characterization of equilibrium and comparative statics analysis. We present the full characterization of equilibrium in a parametric version of the model under uniform demand distribution and a linear split rule in Section 5. We conclude and suggest some avenues for future research in Section 6. All proofs as well as detailed derivations are contained in the Appendix.

2. Literature Review

The literature on multiple item inventory problem with substitution dates back to the paper by Mcgillivray and Silver (1978). However, the role of competition has not been studied until the pioneering work of Parlar (1988). Parlar studies a competitive newsboy problem with two firms managing two substitutable items facing independent demands. A deterministic fraction of unsatisfied demand for each item can be substituted to the other item, if that item has excess stock. It is shown that a unique Nash equilibrium exists. It is also shown that total profits of two competing firms are less than that would have been obtained if they were to cooperate. Wang and Parlar (1994) and Karjalainen (1992) generalize the results of Parlar for the 3 and n firms cases, respectively.

Lippman and McCardle (1997) consider the competitive newsboy problem under a general setting with respect to how initial demands are generated and how excess demand is reallocated. It is assumed that each firm's initial demand is a result of an allocation of the industry demand which is a random variable. In deterministic rules, a specific deterministic function of the industry demand is allocated to each firm in competition. In stochastic rules, a firm's initial allocation depends on the outcome of a random variable (independent demands as in Parlar (1988) can be shown to be a special case of stochastic splitting). If a firm's initial demand exceeds its order quantity, a non–decreasing function of the excess demand is reallocated to each other firm. Lippman and McCardle (1997) show the existence of an equilibrium in the general setting. For the case of symmetric firms and continuous distributions of effective demand for each firm, they also show the uniqueness of the equilibrium. For the case of two firms, they show that competition leads to higher inventory in the system.

Netessine and Rudi (2003) characterize the equilibrium for the case of n firms when the initial demands follow a multi–variate continuous distribution and excess demands spill over fractionally to other firms. The uniqueness of the equilibrium is shown with further conditions and a comparison of centralized and competitive order quantities is provided.

Mahajan and van Ryzin (2003) study a model where the firms' demands are generated by a dynamic process – heterogeneous consumers arrive sequentially and choose a vendor based on a utility maximization criterion and availability at the time of their arrival. They characterize the equilibrium and show its uniqueness for the case of symmetric firms. They also show that competition leads to overstocking.

Serin (2007) considers the possibility of a Stackelberg game in the competitive newsboy problem. She considers both Nash equilibrium solutions and Stackelberg equilibrium solution and gives conditions under which these two lead to the same inventory levels.

Anupindi and Bassok (1999) study the impact of competition and centralization among two retailers on the performance of a supplier in the upper echelon. Under the optimal wholesale pricing mechanism, they show that there is a threshold for the level of substitution, above which the supplier may prefer a decentralized system.

There are other papers in operations literature where competition carries on for multiple periods and backordering is possible. In Hall and Porteus (2000) and Liu et al. (2007), two firms compete on product availability which impacts the market share in future periods. However, within each period that is modeled as a newsboy problem, no substitution occurs. Netessine et al. (2006) model substitution to a competing firm in the current period as well as backordering in future periods.

We restricted our literature review on the horizontal inventory competition. There is a growing body of operations literature where inventory competition takes place between different echelons in the supply chain. Examples include Cachon (2001) and Cachon and Zipkin (1999).

Our model in spirit is similar to Parlar (1988) and Lippman and McCardle (1997). We extend the model in Lippman and McCardle (1997) for the case of non–identical firms and asymmetric information. We show the existence of an equilibrium and show its uniqueness under fairly general assumptions. To our knowledge this is the first study that incorporates the important impact of private information on equilibrium behavior of firms competing on inventory or product availability. Also to our knowledge, this is the first study in operations literature that models horizontal competition under asymmetric information.

The asymmetric information newsboy duopoly game we study can be transformed to a supermodular game. Supermodular games were first introduced by Topkis (1979) who show that there exists at least one pure strategy Nash equilibrium in a full information supermodular game. Milgrom and Roberts (1990) show that a large class of games in economics literature are supermodular and thus have equilibrium. Supermodularity is also used recently to study games in operations literature. Examples include Lippman and McCardle (1997), Bernstein and Federgruen (2003) and Cachon (2001). Vives (1990) uses supermodularity to show the existence of pure strategy Nash equilibrium for compact action spaces and complete separable metric type spaces. This work is recently extended by Athey (2001) to include a larger class of type and strategy spaces which satisfy the single crossing condition. Van Zandt and Vives (2007) shows the existence of Bayesian–Nash equilibrium for supermodular asymmetric information games when type sets are discrete and action sets are continua. Our model of asymmetric information newsboy duopoly is an instance of the general class of incomplete information games studied in Van Zandt and Vives (2007).

3. A Model of Newsboy Duopoly

We consider an industry served by two firms $i = 1, 2$ that offer two substitutable items. Throughout, we assume that the two firms are risk–neutral.

3.1. Industry and Firm Demands

The total industry demand D is a continuous positive random variable with an everywhere positive density function $g()$. Thus, the distribution function $G()$, and the survival function $\overline{G}()$, where $\overline{G}(x) = 1 - G(x) = Pr(D \geq x)$, are strictly monotonic.

As in Lippman and McCardle (1997), demand faced by each firm is determined in a two-step rationing process. First, for any realization, d, of random market demand, initial market shares of the two firms are determined by a deterministic function s such that firm 1's initial market share is $s(d)$ and that of firm 2 is $\hat{s}(d) = d - s(d)$. The share function s satisfies $0 \leq s(d) \leq d$ for all d. To guarantee that both market shares are non–decreasing in market demand realization, we assume $0 \leq s'(d) \leq 1$.

A given initial market share function s induces random demands faced by firm 1, $D_1 = s(D)$, and firm 2, $D_2 = \hat{s}(D) = D - s(D)$. By construction, the initial demands faced by the two firms, (D_1, D_2), are comonotonic since both are deterministic monotone functions of the industry demand.

In the second step, given realized market demand and the order quantities of the two firms, if firm j is stocked out, then some portion, a_i, of firm j's underage goes to firm i. Thus, the effective demand R_i for firm i is the sum of initial allocation and the reallocation:

$$R_i(Q_j) = D_i + a_i(D_j - Q_j)^+.$$

where $(x)^+$ denotes $\max\{x, 0\}$ and $a_i \in [0,1]$ for $i = 1,2$ is the demand substitution rate from firm j to i and is assumed to be deterministic. For notational simplicity, we suppress the dependence of the effective demand on other arguments. The effective demand of firm i, R_i, is a continuous random variable and its distribution is induced by the distributions of initial demands. Effective demands $(R_1(Q_2), R_2(Q_1))$ are also comonotonic random variables for all values of Q_1 and Q_2.

As a first attempt to incorporate private information into the competitive newsboy problem, we take the two items produced by the two firms as perfect substitutes: $a_1 = a_2 = 1$. Despite obvious reduction in model dimensions and notational economy that come with this assumption, this is not without loss of generality. We leave many interesting and important issues related to finer details of the substitution possibilities to future work. However, our main findings (equilibrium existence and qualitative features of the equilibrium) are not affected by this assumption[1].

3.2. Cost and Information Structures

Firm i pays a unit cost for the items that he purchases. We take the type set of firm i, denoted $\mathcal{C}_i$, as the set of values his unit cost can take. Firm i's type is governed by a probability measure p_i over $\mathcal{C}_i$. Type distributions of the two firms are independent. Each firm observes his own cost prior to deciding his order quantity, but he does not observe the other firm's cost. From firm j's perspective, firm i's unit cost is a random variable C_i with support $\mathcal{C}_i$ and distribution p_i.

In this paper, we focus on the case with discrete type sets. Specifically, the unit cost of each firm can take one of two values, i.e., $\mathcal{C}_i = \{c_{iL}, c_{iH}\}$ with $c_{iL} < c_{iH}$. We assume that firm 1's unit cost is c_{1H} with probability $p_1(c_{1H}) = p$ and c_{1L} with probability $p_1(c_{1L}) = 1 - p_1(c_{1H}) = (1-p)$ and firm 2's unit cost is c_{2H} with probability $p_2(c_{2H}) = q$ and c_{2L} with probability $p_2(c_{2L}) = 1 - p_2(c_{2H}) = (1-q)$. With appropriate relabeling of the players, we take $c_{1H} \leq c_{2H}$.

We assume that salvage prices and back–order costs are 0. (The analysis can easily be extended to relax this assumption.) We also assume, without loss of generality, that each firm earns a normalized revenue of 1 per unit of good he sells. This normalization can be achieved by changing the unit of measurement for costs. Under this normalization, we have $c_{2H} \leq 1$. In fact, all our results remain unchanged if one were to take per unit revenues, instead of unit costs, as the source of private information.

Finally, all elements of the model except the cost realizations are common knowledge at the time the order quantity decisions are made.

[1] For example, by taking share functions parameterized by the substitution parameters, $z_1(D, a_1) = s(D) + a_1\hat{s}(D)$ and $z_2(D, a_2) = \hat{s}(D) + a_2 s(D)$, the analysis below can be extended to the more general case.

3.3. Actions, Strategies and Payoffs

For each player i the order quantities are the action sets, $\mathcal{Q}_i = [0, \overline{Q}_i]$, where $\overline{Q}_i$ is the optimal order quantity of firm i assuming that he gets all of the industry demand D with the smallest possible value of c_i. Finally, firm i's expected payoff is $\Pi_i : \mathcal{Q} \times \mathcal{C} \rightarrow \Re$ where $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$.

A pure strategy for player i is a function which maps his type into his action set, $Q_i : \mathcal{C}_i \rightarrow \mathcal{Q}_i$ where $Q_i(c_i)$ is the strategy choice for type c_i of player i. Player i's *interim*[2] expected payoff Π_i is his expected profit conditional on his realized type c_i and order quantity Q, when his rival follows the strategy $Q_j()$:

$$\Pi_i(c_i, Q) = E_{C_j}[\pi_i(Q, Q_j(C_j), c_i)] = \sum_{c_j \in \mathcal{C}_j} p_j(c_j) \pi_i(Q, Q_j(c_j), c_i),$$

where, conditional on $C_j = c_j$,

$$\pi_i(Q, Q_j(c_j), c_i) = E_{R_i(Q_j(c_j))}\left[\min\{R_i(Q_j(c_j)), Q\}\right] - c_i Q$$

is the player's *ex post* profit when his unit cost is c_i and his order quantity Q.

4. Equilibrium Order Quantities

A strategy profile $Q^* = (Q_1^*(), Q_2^*())$ is a Bayesian–Nash equilibrium if, for each player i, and each type $c_i \in \mathcal{C}_i$ of player i,

$$Q_i^*(c_i) \in \arg\max_{Q \in \mathcal{Q}_i} \sum_{c_j \in \mathcal{C}_j} p_j(c_j) \pi_i(Q, Q_j(c_j), c_i).$$

Let $Q_{iL} = Q_i(c_{iL})$ be the order quantity of player i if his cost is c_{iL} and let $Q_{iH} = Q_i(c_{iH})$ be the order quantity of player i if his cost is c_{iH}. Let $(Q_{1L}^*, Q_{1H}^*, Q_{2L}^*, Q_{2H}^*)$ denote a Bayesian–Nash equilibrium. *Interim* expected payoffs conditional on own cost realizations are:

$$\begin{aligned}
\Pi_1(c_{1L}, Q_{1L}) &= q\, E[\min\{R_1(Q_{2H}), Q_{1L}\}] + (1-q)\, E[\min\{R_1(Q_{2L}), Q_{1L}\}] - c_{1L} Q_{1L}, \\
\Pi_1(c_{1H}, Q_{1H}) &= q\, E[\min\{R_1(Q_{2H}), Q_{1H}\}] + (1-q)\, E[\min\{R_1(Q_{2L}), Q_{1H}\}] - c_{1H} Q_{1H}, \\
\Pi_2(c_{2L}, Q_{2L}) &= p\, E[\min\{R_2(Q_{1H}), Q_{2L}\}] + (1-p)\, E[\min\{R_2(Q_{1L}), Q_{2L}\}] - c_{2L} Q_{2L}, \\
\Pi_2(c_{2H}, Q_{2H}) &= p\, E[\min\{R_2(Q_{1H}), Q_{2H}\}] + (1-q)\, E[\min\{R_2(Q_{1L}), Q_{2H}\}] - c_{2H} Q_{2H}.
\end{aligned}$$

A standard property of newsboy models is that $\partial E_R[\min\{R, Q\}]/\partial Q = Pr(R \geq Q)$. Thus, taking the derivative of each type's payoff with respect to his action, the Bayesian–Nash equilibrium order quantities $(Q_{1L}^*, Q_{1H}^*, Q_{2L}^*, Q_{2H}^*)$ satisfy the following conditions:

$$\begin{aligned}
q\, Pr(R_1(Q_{2H}) \geq Q_{1L}) + (1-q) Pr(R_1(Q_{2L}) \geq Q_{1L}) - c_{1L} &= 0, && (1) \\
q\, Pr(R_1(Q_{2H}) \geq Q_{1H}) + (1-q) Pr(R_1(Q_{2L}) \geq Q_{1H}) - c_{1H} &= 0, && (2) \\
p\, Pr(R_2(Q_{1H}) \geq Q_{2L}) + (1-p) Pr(R_2(Q_{1L}) \geq Q_{2L}) - c_{2L} &= 0, && (3) \\
p\, Pr(R_2(Q_{1H}) \geq Q_{2H}) + (1-p) Pr(R_2(Q_{1L}) \geq Q_{2H}) - c_{2H} &= 0. && (4)
\end{aligned}$$

[2] The terms *ex ante, interim* and *ex post* refer to conditioning with respect to the realizations of firm types. Throughout, demand remains uncertain. That is, no new information becomes available about market demand, and, thus, all expressions are *ex ante* with respect to demand.

4.1. Equilibrium Existence

Equilibrium exists under more general assumptions than we make. For instance, the theorem below is valid for arbitrary type sets, not only discrete types. Furthermore, as noted by Lippman and McCardle (1997) in their model of complete information, the existence of equilibrium does not require any assumption on the split functions, or on the joint distribution of the initial demands.

Van Zandt and Vives (2007) shows the existence of Bayesian–Nash equilibrium for supermodular asymmetric information games when type sets are discrete and action sets are continua. Our model of asymmetric information newsboy duopoly is an instance of the general class of incomplete information games studied in Van Zandt and Vives (2007). To establish the existence of pure strategy equilibrium we verify that the equilibrium existence conditions in Van Zandt and Vives (2007) are satisfied in our setting. These conditions are: (i) the payoff function π_i is supermodular in Q_i, (ii) it has increasing differences in (Q_i, Q_j), and (iii) it has increasing differences in (Q_i, t_i), where $t_i = -c_i$.

THEOREM 1. *A pure strategy Nash equilibrium exists for the newsboy duopoly game with asymmetric information.*

4.2. Preliminary Observations on the Equilibrium

In characterizing the structure of equilibrium, some preliminary remarks will be useful. We start with some observations on the best response functions. We then examine optimal order quantities in the absence of strategic interactions to establish a baseline.

Our first claim exploits the assumption that the split functions $s(\cdot)$ and $\hat{s}(\cdot)$ are deterministic and increasing, thus invertible.

CLAIM 1. $\min\{s^{-1}(x), \hat{s}^{-1}(y)\} \leq x + y \leq \max\{s^{-1}(x), \hat{s}^{-1}(y)\}$.

The best response functions of the two types of firm 1, $(Q^*_{1L}(Q_{2L}, Q_{2H}), Q^*_{1H}(Q_{2L}, Q_{2H}))$, and those of firm 2, $(Q^*_{2L}(Q_{1L}, Q_{1H}), Q^*_{2H}(Q_{1L}, Q_{1H}))$, solve:

$$\begin{aligned}
q\,Pr(R_1(Q_{2H}) \geq Q^*_{1L}) + (1-q)\,Pr(R_1(Q_{2L}) \geq Q^*_{1L}) - c_{1L} &= 0,\\
q\,Pr(R_1(Q_{2H}) \geq Q^*_{1H}) + (1-q)\,Pr(R_1(Q_{2L}) \geq Q^*_{1H}) - c_{1H} &= 0,\\
p\,Pr(R_2(Q_{1H}) \geq Q^*_{2L}) + (1-p)\,Pr(R_2(Q_{1L}) \geq Q^*_{2L}) - c_{2L} &= 0,\\
p\,Pr(R_2(Q_{1H}) \geq Q^*_{2H}) + (1-p)\,Pr(R_2(Q_{1L}) \geq Q^*_{2H}) - c_{2H} &= 0.
\end{aligned}$$

Since $R_i(Q)$ and, hence, $Pr(R_i(Q) \geq Q_i)$ are non–increasing in Q, best response functions for both types of both players are non–increasing in both arguments.

Stand–alone order quantities in the absence of competitive interactions will play a useful role as a baseline. We denote by $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ the vector of optimal order quantities for the case with no spillovers (i.e., no competitive interaction).

Lemma 1. *The vector of stand–alone order quantities* $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ *is the unique solution to the system of equations:*

$$\begin{aligned} Pr(D_1 \geq Q_{1L}) &= c_{1L}, \qquad & Pr(D_1 \geq Q_{1H}) &= c_{1H}, \\ Pr(D_2 \geq Q_{2L}) &= c_{2L}, \qquad & Pr(D_2 \geq Q_{2H}) &= c_{2H}. \end{aligned}$$

The ranking of optimal order quantities of the two types of a player is straightforward – the higher a firms' unit cost the lower his stand–alone order quantity: $Q^o_{1L} \geq Q^o_{1H}$ and $Q^o_{2L} \geq Q^o_{2H}$.

In contrast, comparison of the order quantities across firms is complicated by the fact that relative rankings of the firms' market shares and unit costs are not a priori restricted. In general, depending on the relative orderings of market shares and unit costs, all rankings of the four order quantities $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ that are compatible with the orderings $Q^o_{1L} \geq Q^o_{1H}$ and $Q^o_{2L} \geq Q^o_{2H}$ are possible.

One needs further assumptions on market shares and unit costs to be able to rank the stand–alone order quantities of the two firms. For example, if unit costs and initial market shares are perfectly negatively correlated (so that the initial market share of the firm with the lower unit cost exceeds that of the firm with higher unit cost for all demand realizations) then stand–alone order quantities are ordered in the same way as initial market shares.

Note, on the other hand, that stock–out levels, $(Pr(D_i \geq Q^o_{ix}) : i \in \{1,2\}, x \in \{L,H\})$, are ordered the same way as the unit costs. This simple observation, combined with our assumption that initial demands of the two firms are monotone functions of a common market demand, allows a complete ordering of the transformed order quantities:

Claim 2. *For* $x, y \in \{L,H\}$, $c_{1x} \leq c_{2y}$ *if and only if* $s^{-1}(Q^o_{1x}) \geq \hat{s}^{-1}(Q^o_{2y})$.

Returning to the analysis of the equilibrium conditions, we first note an observation on the stock–out probability of firm i with order level Q_i. For firm 1:

$$\begin{aligned} Pr(R_1(Q_2) \geq Q_1) = Pr(D_1 + (D_2 - Q_2)^+ \geq Q_1) = Pr(s(D) + (\hat{s}(D) - Q_2)^+ \geq Q_1) = \\ Pr(D \geq \hat{s}^{-1}(Q_2), D \geq Q_1 + Q_2) + Pr(D \leq \hat{s}^{-1}(Q_2), D \geq s^{-1}(Q_1)). \end{aligned}$$

Similarly, for firm 2:

$$\begin{aligned} Pr(R_2(Q_1) \geq Q_2) = Pr(D_2 + (D_1 - Q_1)^+ \geq Q_2) = Pr(\hat{s}(D) + (s(D) - Q_1)^+ \geq Q_2) = \\ Pr(D \geq s^{-1}(Q_1), D \geq Q_2 + Q_1) + Pr(D \leq s^{-1}(Q_1), D \geq \hat{s}^{-1}(Q_2)). \end{aligned}$$

Second, we observe that low–cost type of each player orders a larger quantity than his high–cost type in equilibrium.

Claim 3. *(i)* $Q^*_{1L} > Q^*_{1H}$, *(ii)* $Q^*_{2L} > Q^*_{2H}$.

Using stand–alone order quantities as a baseline, the next claim shows that order quantities strictly less than the stand–alone order quantities are dominated. Thus, presence of spillovers leads to order quantities that are no less than the order quantities without spillovers. This means that competition does not lead to a decrease in total industry inventory.

Claim 4. *(i)* $Q^*_{1L} \geq Q^o_{1L}$*, (ii)* $Q^*_{1H} \geq Q^o_{1H}$*, (iii)* $Q^*_{2L} \geq Q^o_{2L}$*, (iv)* $Q^*_{2H} \geq Q^o_{2H}$*.*

The following lemma identifies a useful boundary condition that ties the equilibrium order quantity of one of the players to the stand–alone order quantity for the high–cost type of that player.

Lemma 2. *In a Bayesian–Nash equilibrium either (i)* $Q^*_{2H} = Q^o_{2H}$ *or (ii)* $Q^*_{1H} = Q^o_{1H}$*.*

Next, equilibrium order quantities of high–cost types of the two firms are ordered up to transformation by initial market shares:

Lemma 3. *If* $c_{1H} \leq c_{2H}$*, then* $s^{-1}(Q^*_{1H}) \geq \hat{s}^{-1}(Q^*_{2H})$*.*

Finally, in equilibrium, the firm with highest possible unit cost orders his optimal quantity under no competition.

Lemma 4. *If* $c_{1H} \leq c_{2H}$*, then* $Q^*_{2H} = Q^o_{2H}$*.*

When $c_{1H} = c_{2H}$, high–cost types of both firms order their optimal quantities under no competition, i.e., $Q^*_{2H} = Q^o_{2H}$ and $Q^*_{1H} = Q^o_{1H}$.

As a final observation, we note that the best response function of the second firm's high–cost type is flat at its stand–alone level when the order quantities of the first firm's two types exceed their respective stand–alone levels:

Lemma 5. *For* $c_{1H} \leq c_{2H}$*,* $Q^*_{2H}(x, y) = Q^o_{2H}$ *for all* $(x, y) \geq (Q^o_{1L}, Q^o_{1H})$*.*

4.3. Structure of the Equilibrium

Summarizing the observations in the previous sub–section, under the player labeling with $c_{1H} \leq c_{2H}$, the conditions for equilibrium can be stated as follows:

$$q\,Pr(R_1(\hat{s}(\overline{G}^{-1}(c_{2H}))) \geq Q^*_{1L}) + (1-q)Pr(R_1(Q^*_{2L}) \geq Q^*_{1L}) = c_{1L},$$

$$q\,Pr(R_1(\hat{s}(\overline{G}^{-1}(c_{2H}))) \geq Q^*_{1H}) + (1-q)Pr(R_1(Q^*_{2L}) \geq Q^*_{1H}) = c_{1H},$$

$$p\,Pr(R_2(Q^*_{1H}) \geq Q^*_{2L}) + (1-p)Pr(R_2(Q^*_{1L}) \geq Q^*_{2L}) = c_{2L},$$

$$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H})).$$

We can now state the main theorem of this paper that characterizes the structure of equilibrium order quantities.

THEOREM 2. *Assume, without loss of generality, that $c_{1H} \le c_{2H}$. $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

1) $$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H}))$$

2) Q^*_{1L}, Q^*_{1H} *and* Q^*_{2L} *satisfy one of the following sets of conditions:*

(i)
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) = c_{1L} \qquad (i_1)$$
$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H} \qquad (i_2)$$
$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L} \qquad (i_3)$$
$$\hat{s}^{-1}(Q^*_{2L}) \ \ge\ s^{-1}(Q^*_{1L}) \qquad (i_4)$$

(ii)
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{1L} \qquad (ii_1)$$
$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H} \qquad (ii_2)$$
$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L})) = c_{2L} \qquad (ii_3)$$
$$s^{-1}(Q^*_{1L}) \ >\ \hat{s}^{-1}(Q^*_{2L}) \ \ge\ (s^{-1}(Q^*_{1H}) \qquad (ii_4)$$

(iii)
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1L} \qquad (iii_1)$$
$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1H} \qquad (iii_2)$$
$$Q^*_{2L} = \hat{s}(\overline{G}^{-1}(c_{2L})) \qquad (iii_3)$$
$$s^{-1}(Q^*_{1H}) \ >\ \hat{s}^{-1}(Q^*_{2L}) \qquad (iii_4)$$

Before we proceed with discussion of properties of the equilibrium, we first show that it is unique.

THEOREM 3. *The vector of order quantities $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ in Theorem 2 is unique.*

Uniqueness of solutions for each block of equations is a straightforward consequence of the continuity of the demand distribution. To establish uniqueness of the equilibrium, we rule out the possibility that the two or more blocks of equations may have solutions that also satisfy the corresponding inequality. This is done in the Appendix A.4.

A notable pattern in the equilibria across the model space is the recursive structure of the order quantities. This pattern greatly simplifies the computation of equilibrium order quantities. The order quantity of the player type with highest unit cost is determined based on the demand distribution, the split function and his unit cost, independently of other parameters of the game. The remaining equilibrium quantities are obtained recursively. At each step, substituting for the previously computed equilibrium values, a single equation is solved for a single unknown equilibrium quantity.

The recursive pattern of the equilibrium quantities reflect the fact that the equilibrium is partially dominance–solvable, which in turn is a consequence of the supermodular structure of the game. By Claim 4 above, any quantity strictly less than the stand–alone order quantity is strictly dominated by the stand–alone order quantity for every type. Given this fact and Lemma 5, order quantities strictly greater than the stand–alone order quantity are also dominated by the stand–alone order quantity for the highest cost type (c_{2H}). Thus, a two–step reasoning pins the equilibrium behavior of the highest cost type.

4.4. Corollaries

In this sub–section we consider several corollaries of Theorem 2 for special cases of the general model. The first corollary considers a model with *ex ante* symmetric cost structures without restricting the initial market shares. In the second corollary, we impose a restriction on the initial market share function so that one of the firms has larger initial market share for all demand realizations. Corollary 3 presents the equilibrium for the case with fully symmetric firms where both initial market shares and *ex ante* cost structures are identical. In Corollary 4, we remove the restrictions on the initial market shares and consider an extreme form of *ex ante* cost asymmetry: one firm's unit costs are uniformly higher than the other firm's unit costs for all type realizations. Finally, in Corollary 5, we consider a model with symmetric initial market shares and unrestricted *ex ante* asymmetries in the cost structures. As these corollaries are obtained through straightforward substitutions, we omit the proofs.

COROLLARY 1. *Assume that the two firms are* ex ante *symmetric with respect to costs. That is,* $c_{1H}=c_{2H}=c_H$, $c_{1L}=c_{2L}=c_L$, *and* $p=q$. *Then* $(Q^*_{1L},Q^*_{1H},Q^*_{2L},Q^*_{2H})$ *is a Bayesian–Nash equilibrium if and only if*

1) $Q^*_{2H}=\hat{s}(\overline{G}^{-1}(c_H))$

2) Q^*_{1L},Q^*_{1H} *and* Q^*_{2L} *satisfy one of the following sets of conditions:*

(i)
$$q\,\overline{G}(Q^*_{1L}+\hat{s}(\overline{G}^{-1}(c_H)))+(1-q)\,\overline{G}(s^{-1}(Q^*_{1L}))=c_L \qquad (i_1)$$
$$q\,\overline{G}(Q^*_{1H}+\hat{s}(\overline{G}^{-1}(c_H)))+(1-q)\,\overline{G}(s^{-1}(Q^*_{1H}))=c_H \qquad (i_2)$$
$$p\,\overline{G}(Q^*_{2L}+Q^*_{1H})+(1-p)\,\overline{G}(Q^*_{2L}+Q^*_{1L})=c_L \qquad (i_3)$$
$$\hat{s}(Q^*_{2L})\ \geq\ s^{-1}(Q^*_{1L}) \qquad (i_4)$$

(ii)
$$q\,\overline{G}(Q^*_{1L}+\hat{s}(\overline{G}^{-1}(c_H)))+(1-q)\,\overline{G}(Q^*_{2L}+Q^*_{1L})=c_L \qquad (ii_1)$$
$$q\,\overline{G}(Q^*_{1H}+\hat{s}(\overline{G}^{-1}(c_H)))+(1-q)\,\overline{G}(s^{-1}(Q^*_{1H}))=c_H \qquad (ii_2)$$
$$p\,\overline{G}(Q^*_{2L}+Q^*_{1H})+(1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L}))=c_L \qquad (ii_3)$$
$$s^{-1}(Q^*_{1L})\ >\ \hat{s}(Q^*_{2L})\ \geq\ s^{-1}(Q^*_{1H}) \qquad (ii_4)$$

Further simplification is possible under the assumption that initial market shares of the two firms are uniformly ranked, i.e., one firm's initial market share is higher than the other's for all demand realizations. By relabeling firms if necessary, we can take initial market shares to favor firm 1: $s(d) \geq d/2$.

Corollary 2. *Assume that the two firms are* ex ante *symmetric with respect to costs. That is, $c_{1H} = c_{2H} = c_H$, $c_{1L} = c_{2L} = c_L$, and $p = q$. Furthermore, assume $s(d) \geq d/2$ for all demand levels d. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_H)) \text{ and } (Q^*_{1L}, Q^*_{1H}, Q^*_{2L}) \text{ solves:}$$

$$\begin{aligned} q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_H))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &= c_L, \\ q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_H))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1H})) &= c_H, \\ p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) &= c_L. \end{aligned}$$

When the two firms are fully symmetric in terms of cost structures and initial market shares, we get a fully symmetric equilibrium.

Corollary 3. *Assume that the two firms are* ex ante *symmetric with respect to costs. That is, $c_{1H} = c_{2H} = c_H$, $c_{1L} = c_{2L} = c_L$, and $p = q$. Furthermore, let $s(d) = \hat{s}(d) = d/2$ for all demand levels d. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$Q^*_{1H} = Q^*_{2H} = Q^*_H = (1/2)(\overline{G}^{-1}(c_H)) \text{ and } Q^*_{1L} = Q^*_{2L} = Q^*_L \text{ where } Q^*_L \text{ solves}$$

$$q\,\overline{G}(Q^*_L + (1/2)\overline{G}^{-1}(c_H)) + (1-q)\,\overline{G}(2Q^*_L) = c_L.$$

The next corollary looks at the case where one firm has a cost disadvantage for all cost realizations.

Corollary 4. *Assume that $c_{1H} \leq c_{2L}$. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$\begin{gathered} Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H})) \\ Q^*_{2L} = \hat{s}(\overline{G}^{-1}(c_{2L})) \\ q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1L} \\ q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1H}. \end{gathered}$$

As a final corollary, we present the equilibrium order quantities for symmetric initial market shares. In this special case, the equilibrium conditions can be stated explicitly in terms of the exogenous cost parameters, in contrast to the implicit characterization in Theorem 2. For each of the three possible orderings of the unit cost parameters, we have a different set of equilibrium conditions.

COROLLARY 5. *Assume that* $s(d) = \hat{s}(d) = d/2$ *and, without loss of generality, that* $c_{1H} \leq c_{2H}$. *Then* $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ *is a Bayesian–Nash equilibrium if and only if*

1) $Q^*_{2H} = (1/2)\overline{G}^{-1}(c_{2H})$

2) Q^*_{1L},Q^*_{1H} *and* Q^*_{2L} *satisfy one of the following sets of conditions:*

(i) *If* $c_{2L} \leq c_{1L} \leq c_{1H} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1L}) = c_{1L} \qquad (i_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1H}) = c_{1H} \qquad (i_2)$$

$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L} \qquad (i_3)$$

(ii) *If* $c_{1L} \leq c_{2L} \leq c_{1H} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{1L} \qquad (ii_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1H}) = c_{1H} \qquad (ii_2)$$

$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(2Q^*_{2L}) = c_{2L} \qquad (ii_3)$$

(iii) *If* $c_{1L} \leq c_{1H} \leq c_{2L} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2L})) = c_{1L} \qquad (iii_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2L})) = c_{1H} \qquad (iii_2)$$

$$Q^*_{2L} = (1/2)\overline{G}^{-1}(c_{2L}) \qquad (iii_3)$$

4.5. Intra–equilibrium Comparisons

As noted in Claim 3 above, equilibrium is monotone: low–cost type of a firm orders a larger quantity than his high–cost type. Without further restrictions on the initial market shares and the level of unit costs, this is about the extent of what can be said regarding intra–equilibrium comparisons. That is, no general ranking of order quantities across firms is possible without imposing further structure on the model. Furthermore, even under normalization an analog of Claim 2 does not hold for equilibrium order quantities. The only possible ranking is the one provided in Lemma 3 that ranks the normalized equilibrium order quantities of the high–cost types of the two firms.

An interesting observation can be made using the characterization in Corollary 4 in the previous section to illustrate a general phenomenon of inter–type externality. The equilibrium characterization there remains valid for a range of unit costs with $c_{2L} < c_{1H} < c_{2H}$. In this equilibrium, both types of firm 2 choose an order quantity equal to his stand–alone quantity while it is common knowledge that firm 1 may have larger unit cost. That is, low–cost type firm 2 ignores spillover from the less efficient type of the rival firm. This is due to the fact that high–cost type of firm 1, while less efficient than the low–cost type firm 2, selects a large order quantity expecting spillover demand from the less efficient type of firm 2. The increased order quantity of the firm $1H$ forces firm $2L$ to stick to Q^o_{2L}.

4.6. Comparative Statics

Comparative static analysis of the equilibrium and payoffs with respect to the exogenous parameters of the model is done in two parts. We first establish general comparative statics results with respect to two exogenous functions in the model, namely, the demand and the market share function. Then we derive explicit comparative static expressions for the scalar parameters.

THEOREM 4. *Let D_A and D_B be two positive random variables such that D_A dominates D_B under first order stochastic dominance. Then, the equilibrium order quantities with industry demand D_A are larger than the equilibrium order quantities with industry demand D_B.*

THEOREM 5. *If $s_A(d) > s_B(d)$ for all positive real numbers d, then the equilibrium order quantities of both types of firm 1 (firm 2) are larger (respectively, smaller) when the split function is s_A than the order quantities under s_B.)*

In Table 1, we provide the signs of all first order derivatives of equilibrium order quantities with respect to the exogenous scalar parameters, c_{1L}, c_{1H}, p, c_{2L}, c_{2H} and q. The explicit expressions for the comparative statics derivatives themselves are provided in Appendix A.7. Cases (i), (ii) and (iii) correspond to the cases in Theorem 2.

Table 1 Comparative Statics

Cases	Quantities	Conditions	$\mathbf{c_{1L}}$	$\mathbf{c_{1H}}$	$\mathbf{p}$	$\mathbf{c_{2L}}$	$\mathbf{c_{2H}}$	$\mathbf{q}$
	Q^*_{2H}		0	0	0	0	−	0
(i)	Q^*_{1L}		−	0	0	0	+	+
	Q^*_{1H}		0	−	0	0	+	+
	Q^*_{2L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})>0$	+	+	+	−	−	−
	Q^*_{2L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})=0$	0	+	+	−	−	−
(ii)	Q^*_{1L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})>0$	−	−	−	+	+	+
	Q^*_{1L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})=0$	−	0	0	0	+	+
	Q^*_{1H}		0	−	0	0	+	+
	Q^*_{2L}		0	+	+	−	−	−
(iii)	Q^*_{1L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})>0$	−	0	0	+	+	+
	Q^*_{1L}	$\overline{G}(Q^*_{1L}+Q^*_{2L})=0$	−	0	0	0	+	+
	Q^*_{1H}	$\overline{G}(Q^*_{1H}+Q^*_{2L})>0$	0	−	0	+	+	+
	Q^*_{1H}	$\overline{G}(Q^*_{1H}+Q^*_{2L})=0$	0	−	0	0	+	+
	Q^*_{2L}		0	0	0	−	0	0

As expected, the equilibrium order quantities for both players are non–increasing with respect to their own costs and non–decreasing with respect to their rival's costs. In equilibrium, each player orders more as his rival's probability of being high type increases. Conversely, each player orders less as his own probability of being high type increases. This is due to information asymmetry between players and can be explained as follows. Suppose the probability of being high type for firm 1 is increasing. In this case, firm 2 will be ordering more since he will anticipate a higher chance of low order quantity from firm 1. This will lead firm 1 to expect less spillover from firm 2 and hence order less himself. Whether these monotonicities are strict or not depend on specific cases and conditions as given in Table 1. The only exception to these results is that firm 2's (the firm with larger high cost) equilibrium order quantity when his type is high only depends on its own cost as shown in Theorem 2.

5. A Special Case: Uniform Demand and Linear Market Shares

In this section, we present the full explicit characterization of the equilibrium and the corresponding payoff functions for uniformly distributed demand and linear market share functions: $D \sim \text{Uniform}(0,1)$, and $s(D) = sD$ and $\hat{s}(D) = (1-s)D$. Under uniform demand and linear market shares, an instance of the model is represented by 7 parameters: $(c_{1L}, c_{1H}, c_{2L}, c_{2H}, p, q, s)$.

As shown in Section 4, while $Q^*_{2H} = (1-s)(1-c_{2H})$, solution to Q^*_{1L}, Q^*_{1H} and Q^*_{2L} (and the corresponding payoffs) requires a detailed analysis.

5.1. A Partition of the Parameter Space

Detailed analysis, provided in Appendix A.8, lead to 8 regions in the parameter space. In each of the 8 regions, different equilibrium quantities and payoff functions are valid. In other words, in each of these regions the equilibrium structure (functional form) of at least one of endogenous variable is different from its from in other regions. The conditions that determine the partition of the parameter space are as follows: Denoting $\hat{p} = sp$ and $\hat{q} = (1-s)q$,

$$\begin{array}{rrcllr}
 & (1-\hat{q})\,c_{2L} & < & c_{1H} & -\hat{q}\,c_{2H} & (C_A) \\
c_{1L} & & < & & \hat{q}\,c_{2H} & (C_B) \\
 & (1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} & -\hat{p}\,\hat{q}\,c_{2H} & (C_C) \\
 & \hat{q}\,c_{2L} & < & -c_{1H} & +\hat{q}\,c_{2H} & (C_D) \\
-(1-\hat{p})\,c_{1L} & +(1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} & -\hat{q}\,c_{2H} & (C_E) \\
\hat{p}\,c_{1L} & +(1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} & & (C_F) \\
(1-\hat{p})\,(1-\hat{q})\,c_{1L} & +\hat{q}\,(1-\hat{q})\,c_{2L} & < & \hat{p}\,\hat{q}\,c_{1H} & +\hat{q}\,(1-\hat{p}-\hat{q})\,c_{2H} & (C_G) \\
c_{1L} & +\hat{q}\,c_{2L} & < & & \hat{q}\,c_{2H} & (C_H)
\end{array}$$

The 8 different regions that these equilibrium conditions lead to are given in Figure 1.

5.2. Equilibrium Order Quantities

Q^*_{1L},Q^*_{1H} and Q^*_{2L} and payoffs $\pi_1(c_{1L}, c_{2L})$, $\pi_1(c_{1H}, c_{2L})$, $\pi_2(c_{1L}, c_{2L})$ and $\pi_2(c_{1H}, c_{2L})$ in these regions can be found using the following table:

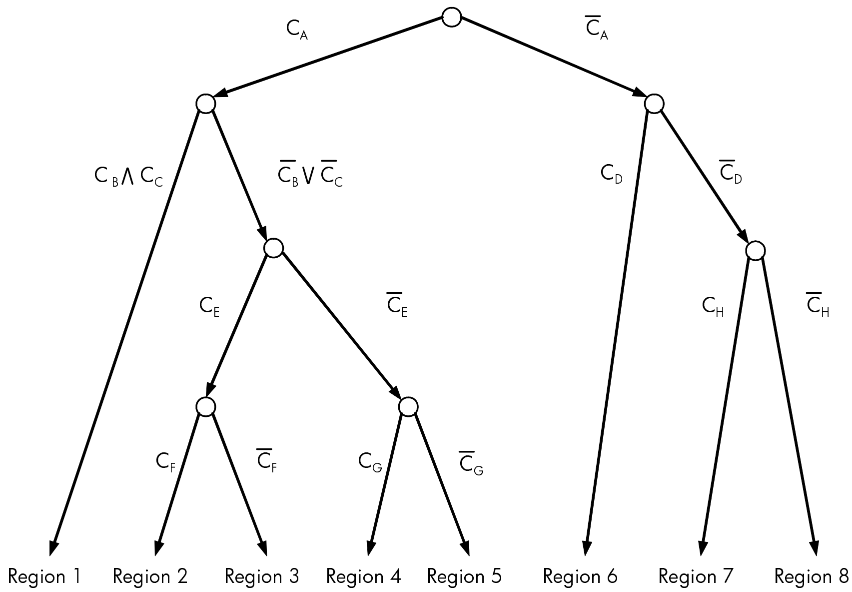

Figure 1 Conditions characterizing the partition of the parameter space

Table 2 Functional forms of endogenous variables by parameter region

Region	$\mathbf{Q_{1L}}$	$\mathbf{Q_{1H}}$	$\mathbf{Q_{2L}}$	$\pi_1(\mathbf{c_{1L}, c_{2L}})$	$\pi_1(\mathbf{c_{1H}, c_{2L}})$	$\pi_2(\mathbf{c_{1L}, c_{2L}})$	$\pi_2(\mathbf{c_{1H}, c_{2L}})$
1	Q_{1L}^{α}	Q_{1H}^{α}	Q_{2L}^{α}	$\pi_1^{\alpha}(c_{1L}, c_{2L})$	$\pi_1^{\alpha}(c_{1H}, c_{2L})$	$\pi_2^{\alpha}(c_{1L}, c_{2L})$	$\pi_2^{\alpha}(c_{1H}, c_{2L})$
2	Q_{1L}^{β}	Q_{1H}^{α}	Q_{2L}^{α}	$\pi_1^{\beta}(c_{1L}, c_{2L})$	$\pi_1^{\alpha}(c_{1H}, c_{2L})$	$\pi_2^{\beta}(c_{1L}, c_{2L})$	$\pi_2^{\alpha}(c_{1H}, c_{2L})$
3	Q_{1L}^{β}	Q_{1H}^{α}	Q_{2L}^{β}	$\pi_1^{\beta}(c_{1L}, c_{2L})$	$\pi_1^{\alpha}(c_{1H}, c_{2L})$	$\pi_2^{\gamma}(c_{1L}, c_{2L})$	$\pi_2^{\alpha}(c_{1H}, c_{2L})$
4	Q_{1L}^{α}	Q_{1H}^{α}	Q_{2L}^{γ}	$\pi_1^{\gamma}(c_{1L}, c_{2L})$	$\pi_1^{\alpha}(c_{1H}, c_{2L})$	$\pi_2^{\delta}(c_{1L}, c_{2L})$	$\pi_2^{\alpha}(c_{1H}, c_{2L})$
5	Q_{1L}^{γ}	Q_{1H}^{α}	Q_{2L}^{γ}	$\pi_1^{\delta}(c_{1L}, c_{2L})$	$\pi_1^{\alpha}(c_{1H}, c_{2L})$	$\pi_2^{\delta}(c_{1L}, c_{2L})$	$\pi_2^{\alpha}(c_{1H}, c_{2L})$
6	Q_{1L}^{α}	Q_{1H}^{β}	Q_{2L}^{δ}	$\pi_1^{\gamma}(c_{1L}, c_{2L})$	$\pi_1^{\beta}(c_{1H}, c_{2L})$	$\pi_2^{\delta}(c_{1L}, c_{2L})$	$\pi_2^{\beta}(c_{1H}, c_{2L})$
7	Q_{1L}^{α}	Q_{1H}^{γ}	Q_{2L}^{δ}	$\pi_1^{\gamma}(c_{1L}, c_{2L})$	$\pi_1^{\gamma}(c_{1H}, c_{2L})$	$\pi_2^{\delta}(c_{1L}, c_{2L})$	$\pi_2^{\beta}(c_{1H}, c_{2L})$
8	Q_{1L}^{δ}	Q_{1H}^{γ}	Q_{2L}^{δ}	$\pi_1^{\delta}(c_{1L}, c_{2L})$	$\pi_1^{\gamma}(c_{1H}, c_{2L})$	$\pi_2^{\delta}(c_{1L}, c_{2L})$	$\pi_2^{\beta}(c_{1H}, c_{2L})$

The equilibrium order quantity for firm 1 when his type is low takes four different functional forms:

$$
\begin{aligned}
Q_{1L}^{\alpha} &= 1 - \frac{c_{1L}}{q} - (1-s)(1-c_{2H}), \\
Q_{1L}^{\beta} &= \frac{(1 - c_{1L} - q(1-s)(1-c_{2H}))}{(q + (1-q)/s)}, \\
Q_{1L}^{\gamma} &= 1 - c_{1L} - q(1-s)(1-c_{2H}) - \frac{(1-q)(1-c_{2L})}{(p + (1-p)/(1-s))}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-q)p(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)(p+(1-p)/(1-s))},\\
Q^{\delta}_{1L} &= 1-c_{1L}-q(1-s)(1-c_{2H})-(1-q)(1-s)(1-c_{2L}).
\end{aligned}
$$

When firms 1's type is high, his equilibrium order quantity takes three possible forms:

$$
\begin{aligned}
Q^{\alpha}_{1H} &= \frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)},\\
Q^{\beta}_{1H} &= 1-\frac{c_{1H}}{q}-(1-s)(1-c_{2H}),\\
Q^{\gamma}_{1H} &= 1-c_{1H}-q(1-s)(1-c_{2H})-(1-q)(1-s)(1-c_{2L}).
\end{aligned}
$$

Finally, the low type of firm 2 has four different functional forms for his equilibrium order quantity:

$$
\begin{aligned}
Q^{\alpha}_{2L} &= 1-\frac{c_{2L}}{p}-\frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)},\\
Q^{\beta}_{2L} &= 1-c_{2L}-\frac{p\,(1-c_{1H}-q\,(1-s)(1-c_{2H}))}{(q+(1-q)/s)}-\frac{(1-p)(1-c_{1L}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)},\\
Q^{\gamma}_{2L} &= \frac{(1-c_{2L})}{(p+(1-p)/(1-s))}-\frac{p\,(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)(p+(1-p)/(1-s))},\\
Q^{\delta}_{2L} &= (1-s)(1-c_{2L}).
\end{aligned}
$$

5.3. Equilibrium Payoffs

When both firms have low costs, Firm 1's *ex post* payoff can take four different functional forms:

$$
\begin{aligned}
\pi^{\alpha}_{1}(c_{1L},c_{2L}) &= \frac{1}{2}\,s-c_{1L}Q_{1L},\\
\pi^{\beta}_{1}(c_{1L},c_{2L}) &= Q_{1L}(1-c_{1L})-\frac{(Q_{1L})^2}{2s},\\
\pi^{\gamma}_{1}(c_{1L},c_{2L}) &= \frac{1}{2}+\frac{(Q_{2L})^2}{2(1-s)}-Q_{2L}-c_{1L}Q_{1L},\\
\pi^{\delta}_{1}(c_{1L},c_{2L}) &= Q_{1L}(1-c_{1L})+\frac{(Q_{2L})^2}{2(1-s)}-\frac{(Q_{1L}+Q_{2L})^2}{2}.
\end{aligned}
$$

Firm 2's payoff, similarly, has four possible functional forms when both firms have low cost:

$$
\begin{aligned}
\pi^{\alpha}_{2}(c_{1L},c_{2L}) &= \frac{1}{2}\,(1-s)-c_{2L}Q_{2L},\\
\pi^{\beta}_{2}(c_{1L},c_{2L}) &= \frac{1}{2}+\frac{(Q_{1L})^2}{2s}-Q_{1L}-c_{2L}Q_{2L},\\
\pi^{\gamma}_{2}(c_{1L},c_{2L}) &= Q_{2L}(1-c_{2L})+\frac{(Q_{1L})^2}{2s}-\frac{(Q_{1L}+Q_{2L})^2}{2},\\
\pi^{\delta}_{2}(c_{1L},c_{2L}) &= Q_{2L}(1-c_{2L})-\frac{(Q_{2L})^2}{2(1-s)}.
\end{aligned}
$$

When firms 1 and 2 have low and high costs, respectively, we have three possibilities for the payoff for firm 1's payoff:

$$
\begin{aligned}
\pi_1^\alpha(c_{1H}, c_{2L}) &= Q_{1H}(1-c_{1H}) - \frac{(Q_{1H})^2}{2s}, \\
\pi_1^\beta(c_{1H}, c_{2L}) &= \frac{1}{2} + \frac{(Q_{2L})^2}{2(1-s)} - Q_{2L} - c_{1H}Q_{1H}, \\
\pi_1^\gamma(c_{1H}, c_{2L}) &= Q_{1H}(1-c_{1H}) + \frac{(Q_{2L})^2}{2(1-s)} - \frac{(Q_{1H}+Q_{2L})^2}{2};
\end{aligned}
$$

and two possible forms for the payoff for firm 2:

$$
\begin{aligned}
\pi_2^\alpha(c_{1H}, c_{2L}) &= Q_{2L}(1-c_{2L}) + \frac{(Q_{1H})^2}{2s} - \frac{(Q_{1H}+Q_{2L})^2}{2}, \\
\pi_2^\beta(c_{1H}, c_{2L}) &= Q_{2L}(1-c_{2L}) - \frac{(Q_{2L})^2}{2(1-s)}.
\end{aligned}
$$

When firm 2 has a high cost, the payoffs of the two players are same in all regions:

$$
\begin{aligned}
\pi_1(c_{1L}, c_{2H}) &= Q_{1L}(1-c_{1L}) + \frac{Q_{2H}^2}{2(1-s)} - \frac{(Q_{1L}+Q_{2H})^2}{2}, \\
\pi_1(c_{1H}, c_{2H}) &= Q_{1H}(1-c_{1H}) + \frac{Q_{2H}^2}{2(1-s)} - \frac{(Q_{1H}+Q_{2H})^2}{2}, \\
\pi_2(c_{1L}, c_{2H}) &= \pi_2(c_{1H}, c_{2H}) = (1-s)(1-c_{2H})^2/2.
\end{aligned}
$$

5.4. Comparative Statics

We present the explicit expressions for comparative static derivatives for the equilibrium order quantities for the uniform demand and linear split case in Appendix A.9. Comparative static sign patterns are summarized in Table 3. This is a specific version of Table 1 for the uniform demand and linear split function. Since s characterize the whole split function in this case, we also provide the comparative statics with respect to s in this table.

Table 3 **Comparative Statics for Uniform Demand Case**

	$\mathbf{Q_{1L}^\alpha}$	$\mathbf{Q_{1L}^\beta}$	$\mathbf{Q_{1L}^\gamma}$	$\mathbf{Q_{1L}^\delta}$	$\mathbf{Q_{2L}^\alpha}$	$\mathbf{Q_{2L}^\beta}$	$\mathbf{Q_{2L}^\gamma}$	$\mathbf{Q_{2L}^\delta}$	$\mathbf{Q_{1H}^\alpha}$	$\mathbf{Q_{1H}^\beta}$	$\mathbf{Q_{1H}^\gamma}$	$\mathbf{Q_{2H}}$
$\mathbf{c_{1L}}$	−	−	−	−	0	+	0	0	0	0	0	0
$\mathbf{c_{1H}}$	0	0	−	0	+	+	+	0	−	−	−	0
$\mathbf{p}$	0	0	−	0	+	+	+	0	0	−	−	0
$\mathbf{c_{2L}}$	0	0	+	+	−	−	−	−	0	0	+	0
$\mathbf{c_{2H}}$	+	+	+	+	−	−	−	0	+	+	+	−
$\mathbf{q}$	+	+	+	+	−	−	−	0	+	+	+	0
$\mathbf{s}$	+	+	+	+	−	−	−	−	+	+	+	−

6. Concluding Remarks

We studied a model of inventory competition in a newsboy duopoly under asymmetric cost information. We showed that a pure strategy Bayesian–Nash equilibrium exists under fairly general assumptions. We characterized the equilibrium for the case where the industry demand is allocated between two firms using a deterministic split function and show its uniqueness. We showed that presence of strategic interactions creates incentives to increase order quantities for all firm types except the type that has the highest possible unit cost, who orders the same quantity as he would as a monopolist newsboy facing scaled version of the market demand. Therefore, competition leads to higher total inventory in the industry. The equilibrium conditions have an interesting recursive structure that enables an easy computation of the equilibrium order quantities. Comparative statics analysis shows that a stochastic increase in market demand or an increase in one firm's initial allocation of the total industry demand lead to higher inventory for that firm. We finally derived a complete characterization of the equilibrium and its comparative statics for the case of uniform demand and linear split rule.

Certain extensions of the current model are relatively straightforward and not likely to change the structure of the equilibrium qualitatively. For instance, allowing more than two levels for the unit costs, will lead to more complicated but qualitatively similar equilibrium characterization in that many of the claims, the recursive structure of the equilibrium order quantities, and, particularly, the behavior of the highest–cost type will remain valid with this extension.

Information asymmetry adds a new dimension to the competitive newsboy problem. Alternative specifications for the key structural elements of the current model – e.g., the the nature of information asymmetry, and the structure of the market and firm demands – span a number of interesting classes of models we intend to explore in the future. Among these are models of newsboy oligopoly, and models that allow arbitrary statistical dependence in firm demands, and in cost structures.

Acknowledgments
We acknowledge helpful comments and suggestions of Atila Abdulkadiroglu, Abdullah Dasci, Nesim Erkip and Patrick Scholler without implicating them for the remaining errors and shortcomings for which we are solely responsible.

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Appendix A:

A.1. Proof of Theorem 1

First, define $\mathcal{Y}_2 = -\mathcal{Q}_2$ so that $\mathcal{Q}_1 \times \mathcal{Y}_2$ is a lattice (This order change is necessary to form a supermodular game). Moreover, let $t_1 = -c_1$, $t_2 = c_2$ and define effective demand functions as $R_i : t_j \to \Re$. Then for

$$\pi_1(Q_1, y_2, t_1, t_2) = E[\min\{R_1(t_2), Q_1\}] + t_1 Q_1,$$
$$\pi_2(Q_1, y_2, t_1, t_2) = E[\min\{R_2(t_1), -y_2\}] + t_2 y_2.$$

The supermodularity and continuity of these functions and the increasing differences in (Q_1, y_2) are proved in Lippman and McCardle (1997). The only thing remains is to show that π_1 has increasing differences in (Q_1, t_1) and π_2 has increasing differences in (y_2, t_2) (Again, π_i is not directly dependent on the type of firm j. Hence, increasing differences for (Q_1, t_2) and (y_2, t_1) are trivially satisfied.). Let $\varsigma_1(t_1) = \pi_1(Q_1', y_2, t_1, t_2) - \pi_1(Q_1, y_2, t_1, t_2)$ where $Q_1' \geq Q_1$ for given y_2, t_2. Then

$$\varsigma_1(t_1) = E[\min\{R_1(t_2), Q_1'\}] - E[\min\{R_1(t_2), Q_1\}] + t_1[Q_1' - Q_1].$$

Define t_1' such that $t_1' \geq t_1$. It follows that $\varsigma(t_1') - \varsigma(t_1) = [t_1' - t_1][Q_1' - Q_1] \geq 0$. Thus π_1 has increasing differences in (Q_1, t_1). Similarly, $\varsigma_2(t_2) = \pi_2(Q_1, y_2', t_1, t_2) - \pi_2(Q_1, y_2, t_1, t_2)$ where $y_2' \geq y_2$ for given Q_1, t_1. Then

$$\varsigma_2(t_2) = E[\min\{R_2(t_1), -y_2'\}] - E[\min\{R_2(t_1), -y_2\}] + t_2[y_2' - y_2].$$

Define t_2' such that $t_2' \geq t_2$. It follows that $\varsigma(t_2') - \varsigma(t_2) = [t_2' - t_2][y_2' - y_2] \geq 0$. Thus π_2 has increasing differences in (y_2, t_2). Since our priors over the types are independent, the condition for priors to be increasing with respect to types is trivially satisfied. The existence of pure strategy Nash equilibrium follows. □

A.2. Proof of Claims 1-4 and Lemmas 2–5

Proof of Claim 1: Let $\min\{s^{-1}(x), \hat{s}^{-1}(y)\} = \hat{s}^{-1}(y)$, i.e., $s^{-1}(x) \geq \hat{s}^{-1}(y)$. Suppose, to get a contradiction, that $s^{-1}(x) < x + y$. Then $x < s(x+y) = x + y - \hat{s}(x+y)$, since $\hat{s}(x) = x - s(x)$. Thus, $\hat{s}(x+y) < y$, and $x + y < \hat{s}^{-1}(y)$. Therefore, $s^{-1}(x) < \hat{s}^{-1}(y)$, yielding a contradiction. The second inequality is established similarly. □

Proof of Claim 2: $Pr(D_1 \geq Q_{1H}^o) = Pr(D \geq s^{-1}(Q_{1H}^o)) = c_{1H} \leq c_{2H} = Pr(D \geq \hat{s}^{-1}(Q_{2H}^o))$.
Hence, $s^{-1}(Q_{1H}^o) \geq \hat{s}^{-1}(Q_{2H}^o)$. □

Proof of Claim 3: (i) (1) evaluated at $Q_{1L} = Q_{1H}^*$ is positive.
(ii) Similar argument with (i). □

Proof of Claim 4: We will only show (i). Other cases are established similarly. Evaluating the left hand side of (1) at $Q_{1L} = Q_{1L}^o$ gives:

$$q\,Pr(D_1 + (D_2 - Q_{2H})^+ \geq Q_{1L}^o) + (1-q)Pr(D_1 + (D_2 - Q_{2L})^+ \geq Q_{1L}^o) - c_{1L}$$
$$\geq q\,Pr(D_1 \geq Q_{1L}^o) + (1-q)Pr(D_1 \geq Q_{1L}^o) - c_{1L} = Pr(D_1 \geq Q_{1L}^o) - c_{1L} = 0$$

Thus, $Q_{1L}^* \geq Q_{1L}^o$. □

Proof of Lemma 2: Assume that $s^{-1}(Q_{1H}^*) > \hat{s}^{-1}(Q_{2H}^*)$. First note that,

$$Pr(D_2 + (D_1 - Q_1)^+ \geq Q_2) = Pr(D \geq s^{-1}(Q_1), D \geq Q_1 + Q_2) + Pr(D \leq s^{-1}(Q_1), D \geq \hat{s}^{-1}(Q_2)).$$

By substituting this in (4) we obtain:

$$p\,Pr(D \geq s^{-1}(Q_{1H}^*), D \geq Q_{2H}^* + Q_{1H}^*) + p\,Pr(D \leq s^{-1}(Q_{1H}^*), D \geq \hat{s}^{-1}(Q_{2H}^*)) +$$
$$(1-p)Pr(D \geq s^{-1}(Q_{1L}^*), D \geq Q_{2H}^* + Q_{1L}^*) + (1-p)Pr(D \leq s^{-1}(Q_{1L}^*), D \geq \hat{s}^{-1}(Q_{2H}^*)) - c_{2H} = 0$$

Since $s^{-1}(Q_{1H}^*) > \hat{s}^{-1}(Q_{2H}^*)$, $s^{-1}(Q_{1L}^*) > \hat{s}^{-1}(Q_{2H}^*)$ by Claim 3. By Claim 1, $Pr(D \geq s^{-1}(Q_{1H}^*), D \geq Q_{2H}^* + Q_{1H}^*) = Pr(D \geq s^{-1}(Q_{1H}^*)$. In addition, $Pr(D \geq s^{-1}(Q_{1H}^*)) + Pr(\hat{s}^{-1}(Q_{2H}^*) \leq D \leq s^{-1}(Q_{1H}^*)) = Pr(D \geq \hat{s}^{-1}(Q_{2H}^*))$. Therefore,

$$p\,Pr(D \geq \hat{s}^{-1}(Q_{2H}^*)) + (1-p)Pr(D \geq \hat{s}^{-1}(Q_{2H}^*)) - c_{2H} = Pr(D \geq \hat{s}^{-1}(Q_{2H}^*)) - c_{2H} = 0.$$

Thus, $Q_{2H}^* = Q_{2H}^o$. Using $s^{-1}(Q_{1H}^*) \leq \hat{s}^{-1}(Q_{2H}^*) < \hat{s}^{-1}(Q_{2L}^*)$ in (2) in a similar fashion gives the result $Q_{1H}^* = Q_{1H}^o$. □

Proof of Lemma 3: Assume to the contrary that for $c_{1H} \leq c_{2H}$, $s^{-1}(Q_{1H}^*) < \hat{s}^{-1}(Q_{2H}^*)$. Then, by Lemma 1, $Q_{1H}^* = Q_{1H}^o$. By Claims 2 and 3, we get $s^{-1}(Q_{1H}^*) \geq s^{-1}(Q_{1H}^o) \geq \hat{s}^{-1}(Q_{2H}^o)$ and

$$Pr(D \geq Q_{2H}^* + Q_{1H}^*) \leq Pr(D \geq Q_{2H}^o + Q_{1H}^o) < Pr(D \geq Q_{2H}^o + \hat{s}(s^{-1}(Q_{2H}^o))) = Pr(\hat{s}(D) \geq Q_{2H}^o) = c_{2H}. \quad (*)$$

Now, we have either $s^{-1}(Q_{1L}^*) > \hat{s}^{-1}(Q_{2H}^*)$ or $s^{-1}(Q_{1L}^*) \leq \hat{s}^{-1}(Q_{2H}^*)$. In the first case equilibrium condition (4) simplifies to:

$$c_{2H} = p\,Pr(D \geq Q_{2H}^* + Q_{1H}^*) + (1-p)Pr(\hat{s}(D) \geq Q_{2H}^*) \leq p\,Pr(D \geq Q_{2H}^* + Q_{1H}^*) + (1-p)c_{2H},$$

since $Pr(\hat{s}(D) \geq Q^*_{2H}) \leq Pr(\hat{s}(D) \geq Q^o_{2H})$ by Claim 4 and $Pr(\hat{s}(D) \geq Q^o_{2H}) = c_{2H}$ by definition. This leads to

$$c_{2H} \ \leq \ p\,Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)c_{2H} \ \leq \ Pr(D \geq Q^*_{2H} + Q^*_{1H}),$$

which is a contradiction to $(*)$.

For the second case, the equilibrium condition (4) simplifies to:

$$c_{2H} = p\,Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)Pr(D \geq Q^*_{2H} + Q^*_{1L}) < Pr(D \geq Q^*_{2H} + Q^*_{1H}),$$

since $Q^*_{1L} > Q^*_{1H}$ by Claim 3. Again this contradicts $(*)$. □

Proof of Lemma 4: By Lemma 2, $c_{1H} \leq c_{2H}$ implies $s^{-1}(Q^*_{1H}) \geq \hat{s}^{-1}(Q^*_{2H})$. Using this condition in Lemma 1 yields the desired result. □

Proof of Lemma 5: First note that Q^o_{iL} and Q^o_{iH} are stand-alone order levels for firms $i = 1, 2$. It is important to notice that each firm will at least play his stand-alone order quantity in the equilibrium. Now, define Q^1_{2H} as the order level of high type of firm 2 when firm 1 plays his stand-alone quantities for both his types in the equilibrium i.e.,

$$p\,Pr(\hat{s}(D) + (s(D) - Q^o_{1H})^+ \geq Q^1_{2H}) + (1-p)Pr(\hat{s}(D) + (s(D) - Q^o_{1L})^+ \geq Q^1_{2H}) - c_{2H} = 0.$$

and $Q^1_{2H} \geq Q^o_{2H}$ since firm 2 will play at least his stand-alone order level. Rewriting the equilibrium condition gives,

$$\begin{aligned} &p\,Pr(D \geq s^{-1}(Q^o_1), D \geq Q^o_{1H} + Q^1_{2H}) + p\,Pr(D \leq s^{-1}(Q^o_{1H}), D \geq \hat{s}^{-1}(Q^1_{2H})) \\ &\quad +(1-p)Pr(D \geq s^{-1}(Q^o_{1L}), D \geq Q^o_{1L} + Q^1_{2H}) + (1-p)Pr(D \leq s^{-1}(Q^o_{1L}), D \geq \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = 0. \end{aligned}$$

For this equilibrium condition, we have three possibilities: $\hat{s}^{-1}(Q^1_{2H}) \leq s^{-1}(Q^o_{1H})$, $s^{-1}(Q^o_{1H}) < \hat{s}^{-1}(Q^1_{2H}) \leq s^{-1}(Q^o_{1L})$ and $s^{-1}(Q^o_{1L}) < \hat{s}^{-1}(Q^1_{2H})$. First assume $\hat{s}^{-1}(Q^1_{2H}) \leq s^{-1}(Q^o_{1H})$, then the equilibrium condition becomes:

$$p\,Pr(D \geq \hat{s}^{-1}(Q^1_{2H})) + (1-p)Pr(D \geq \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = Pr(D \geq \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = 0.$$

Thus, $Q^1_{2H} = Q^o_{2H}$. Now, we assume that $s^{-1}(Q^o_{1H}) < \hat{s}^{-1}(Q^1_{2H}) < s^{-1}(Q^o_{1L})$. Moreover, if we use the fact that $s^{-1}(Q^o_{1H}) < Q^o_{1H} + Q^1_{2H}$ (by Claim 1), the condition becomes

$$\begin{aligned} 0 \ &= \ p\,Pr(D \geq Q^o_{1H} + Q^1_{2H}) + (1-p)Pr(D \geq \hat{s}^{-1}(Q^1_{2H})) - c_{2H} \\ &< \ p\,Pr(D \geq s^{-1}(Q^o_{1H})) + (1-p)Pr(D \geq s^{-1}(Q^o_{1H})) - c_{2H} \\ &= \ Pr(D \geq s^{-1}(Q^o_{1H})) - c_{2H} = c_{1H} - c_{2H} \end{aligned}$$

Thus, $c_{1H} > c_{2H}$ which is a contradiction to our assumption that $c_{1H} \leq c_{2H}$. A similar proof can be obtained for $s^{-1}(Q^o_{1L}) \leq \hat{s}^{-1}(Q^1_{2H})$. Hence, $Q^1_{2H} = Q^o_{2H}$ which implies that any order quantity of high type of firm 2 satisfies $Q_{2H} \leq Q^o_{2H}$. Combining this with the fact that $Q_{2H} \geq Q^o_{2H}$, we obtain $Q_{2H} = Q^o_{2H}$. □

A.3. Proof of Theorem 2

Under an increasing and deterministic split function, we know that there is a unique Bayesian–Nash equilibrium and using Lemma 3, our unique equilibrium conditions take the form:

$$\begin{aligned} q\,Pr(D \geq Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D_1 + (D_2 - Q^*_{2L})^+ \geq Q^*_{1L}) &= c_{1L}, \\ q\,Pr(D \geq Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D_1 + (D_2 - Q^*_{2L})^+ \geq Q^*_{1H}) &= c_{1H}, \\ p\,Pr(D_2 + (D_1 - Q^*_{1H})^+ \geq Q^*_{2L}) + (1-p)Pr(D_2 + (D_1 - Q^*_{1L})^+ \geq Q^*_{2L}) &= c_{2L}, \\ Q^*_{2H} &= \hat{s}(\overline{G}^{-1}(c_{2H})). \end{aligned}$$

Now, if we use $D_1 = s(D)$ and $D_2 = \hat{s}(D)$ and use the fact that,

$$\begin{aligned} Pr(D_1 + (D_2 - Q_2)^+ \geq Q_1) &= Pr(D \geq \hat{s}^{-1}(Q_2), D \geq Q_2 + Q_1) + Pr(D \leq \hat{s}^{-1}(Q_2), D \geq s^{-1}(Q_1)), \\ Pr(D_2 + (D_1 - Q_1)^+ \geq Q_2) &= Pr(D \geq s^{-1}(Q_1), D \geq Q_1 + Q_2) + Pr(D \leq s^{-1}(Q_1), D \geq \hat{s}^{-1}(Q_2)), \end{aligned}$$

which can be obtained using a simple conditional probability argument, equilibrium conditions will become:

$$\begin{aligned} &q\,Pr(D \geq Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q^*_{2L}), D \geq Q^*_{2L} + Q^*_{1L}) \\ &\quad +(1-q)Pr(D \leq \hat{s}^{-1}(Q^*_{2L}), D \geq s^{-1}(Q^*_{1L})) = c_{1L}, \qquad (A_1) \end{aligned}$$

$$\begin{aligned} &q\,Pr(D \geq Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q^*_{2L}), D \geq Q^*_{2L} + Q^*_{1H}) \\ &\quad +(1-q)Pr(D \leq \hat{s}^{-1}(Q^*_{2L}), D \geq s^{-1}(Q^*_{1H})) = c_{1H}, \qquad (A_2) \end{aligned}$$

$$\begin{aligned} &pPr(D \geq s^{-1}(Q^*_{1H}), D \geq Q^*_{2L} + Q^*_{1H}) + pPr(D \leq s^{-1}(Q^*_{1H}), D \geq \hat{s}^{-1}(Q^*_{2L})) \\ &\quad +(1-p)Pr(D \geq s^{-1}(Q^*_{1L}), D \geq Q^*_{2L} + Q^*_{1L}) \\ &\quad +(1-p)Pr(D \leq s^{-1}(Q^*_{1L}), D \geq \hat{s}^{-1}(Q^*_{2L})) = c_{2L}, \qquad (A_3) \end{aligned}$$

$$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H})). \qquad (A_4)$$

The proof of part 1 follows since $Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H}))$ is obviously an equilibrium condition.

Part 2 has three separate subsets. To prove (i), let $\hat{s}^{-1}(Q^*_{2L}) \geq s^{-1}(Q^*_{1L})$. (A_1) becomes (i_1):

$$\begin{aligned} &q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q^*_{2L}), D \geq Q^*_{2L} + Q^*_{1L}) + (1-q)Pr(D \leq \hat{s}^{-1}(Q^*_{2L}), D \geq s^{-1}(Q^*_{1L})) \\ &= q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq s^{-1}(Q^*_{1L})) = q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}^{-1}(s^{-1}(Q^*_{1L})) = c_{1L}. \end{aligned}$$

Similarly, using the fact that $\hat{s}^{-1}(Q^*_{2L}) \geq s^{-1}(Q^*_{1L})$ implies $\hat{s}^{-1}(Q^*_{2L}) \geq s^{-1}(Q^*_{1H})$, (A_2) becomes (i_2):

$$\begin{aligned} &q\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q^*_{2L}), D \geq Q^*_{2L} + Q^*_{1H}) + (1-q)Pr(D \leq \hat{s}^{-1}(Q^*_{2L}), D \geq s^{-1}(Q^*_{1H})) \\ &= q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq s^{-1}(Q^*_{1H})) = q\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}^{-1}(s^{-1}(Q^*_{1H})) = c_{1H}. \end{aligned}$$

And combining two inequalities, (A_3) becomes (i_3):

$$\begin{aligned} &pPr(D \geq s^{-1}(Q^*_{1H}), D \geq Q^*_{2L} + Q^*_{1H}) + pPr(D \leq s^{-1}(Q^*_{1H}), D \geq \hat{s}^{-1}(Q^*_{2L})) \\ &+(1-p)Pr(D \geq s^{-1}(Q^*_{1L}), D \geq Q^*_{2L} + Q^*_{1L}) + (1-p)Pr(D \leq s^{-1}(Q^*_{1L}), D \geq \hat{s}^{-1}(Q^*_{2L})) \\ &= pPr(D \geq Q^*_{2L} + Q^*_{1H}) + (1-p)Pr(D \geq Q^*_{2L} + Q^*_{1L}) \\ &= p\overline{G}^{-1}(D \geq Q^*_{2L} + Q^*_{1H}) + (1-p)\overline{G}^{-1}(D \geq Q^*_{2L} + Q^*_{1L}) = c_{2L}. \end{aligned}$$

The proof for (ii) and (iii) follows similarly under $s^{-1}(Q^*_{1L}) > \hat{s}^{-1}(Q^*_{2L}) \geq s^{-1}(Q^*_{1H})$ and $s^{-1}(Q^*_{1H}) > \hat{s}^{-1}(Q^*_{2L})$. □

A.4. Proof of Theorem 3

First, since the demand has a continuous distribution, the inverse of distribution function G and $\overline{G}$ are well-defined. Only one of the (i),(ii) or (iii) given in Theorem 2 can be satisfied since a vector of order quantities satisfying one of the inequality conditions (i_4),(ii_4) or (iii_4) cannot satisfy others.

Take the region (i). There can be only one Q^*_{1L} satisfying condition (i_1) which is:

$$q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}(s^{-1}(Q^*_{1L})) = c_{1L},$$

since s^{-1}, $\hat{s}^{-1}$ and $\overline{G}^{-1}$ gives unique results and it does not depend on any other variables. Similarly, only one Q^*_{1H} satisfies (i_2):

$$q\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H}.$$

Since both Q^*_{1L} and Q^*_{1H} are unique, (i_3) i.e.,

$$p\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L},$$

also gives a unique Q^*_{2L}. Thus, the set of order quantities satisfying region (i) is unique.

Similar arguments are valid for regions (ii) and (iii). The argument so far does not rule out multiple equilibria each of which is the unique solution of one of three blocks of equalities. Finally, we need to show that only one of that three cases can arise.

Assume to the contrary that case (i) and (ii) gives different solutions. Now, let $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ and $(\hat{Q}_{1L}, \hat{Q}_{1H}, \hat{Q}_{2L}, \hat{Q}_{2H})$ be the solutions of cases (i) and (ii) respectively. First notice that $Q^*_{1H} = \hat{Q}_{1H} = Q_{1H}$ and $Q^*_{2H} = \hat{Q}_{2H} = Q_{2H}$ since they require the same conditions. However, low type quantities should satisfy:

$$\begin{aligned} q\,\overline{G}(Q^*_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &= q\,\overline{G}(\hat{Q}_{1L} + Q_{2H}) + (1-q)\,\overline{G}(\hat{Q}_{2L} + \hat{Q}_{1L}) \\ p\,\overline{G}(Q^*_{2L} + Q_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) &= p\,\overline{G}(\hat{Q}_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(\hat{Q}_{2L})) \\ \hat{s}^{-1}(Q^*_{2L}) \geq Q^*_{1L} &+ Q^*_{2L} \geq s^{-1}(Q^*_{1L}) \\ \hat{s}^{-1}(\hat{Q}_{2L}) < \hat{Q}_{1L} &+ \hat{Q}_{2L} < s^{-1}(\hat{Q}_{1L}) \end{aligned}$$

where inequalities come from Claim 4. Thus, we have

$$\begin{aligned} q\,\overline{G}(Q^*_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &> q\,\overline{G}(\hat{Q}_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(\hat{Q}_{1L})) \\ p\,\overline{G}(Q^*_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L})) &< p\,\overline{G}(\hat{Q}_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(\hat{Q}_{2L})) \end{aligned}$$

which implies $Q^*_{1L} < \hat{Q}_{1L}$ and $Q^*_{2L} > \hat{Q}_{2L}$ (Remember that $\overline{G}$ is a decreasing function.). If we use this in equilibrium conditions,

$$\begin{aligned} \overline{G}(s^{-1}(Q^*_{1L})) &< \overline{G}(\hat{Q}_{2L} + \hat{Q}_{1L}) \\ \overline{G}(Q^*_{2L} + Q^*_{1L}) &> \overline{G}(\hat{s}^{-1}(\hat{Q}_{2L})) \end{aligned}$$

meaning that both $Q^*_{1L} + Q^*_{2L} > s^{-1}(Q^*_{1L}) > \hat{Q}_{2L} + \hat{Q}_{1L}$ and $\hat{Q}_{2L} + \hat{Q}_{1L} > \hat{s}^{-1}(\hat{Q}_{2L}) > Q^*_{2L} + Q^*_{1L}$ should be true, which is a contradiction. The proof for other cases are similar.

Thus, the solution given by Theorem 2 is unique. □

A.5. Proof of Theorem 4

Let G_A and G_B be the distribution functions of D_A and D_B, respectively. D_A stochastically dominates D_B. Thus, $G_A(x) \le G_B(x)$ and $\overline{G}_A(x) \ge \overline{G}_B(x)$ for all x. Since $\overline{G}_A$ and $\overline{G}_B$ are decreasing functions, $\overline{G}_A^{-1}(y) \ge \overline{G}_B^{-1}(y)$ for all y. We define $(Q_{1L}^A, Q_{1H}^A, Q_{2L}^A, Q_{2H}^A)$ and $(Q_{1L}^B, Q_{1H}^B, Q_{2L}^B, Q_{2H}^B)$ as the equilibrium order quantities for D_A and D_B, respectively.

Returning to the result of Theorem 2, we have three possible cases. Consider the equilibrium conditions in case (i). Now, since $\hat{s}$ is an increasing function, there exists $\delta_{2H} = Q_{2H}^A - Q_{2H}^B = \hat{s}(\overline{G}_A^{-1}(c_{2H})) - \hat{s}(\overline{G}_B^{-1}(c_{2H})) \ge 0$. Note that, the stock–out probability of firm 2 under high type does not change.

Now, by (i_2),

$$q\,\overline{G}_A(Q_{1L}^A + Q_{2H}^A) + (1-q)\,\overline{G}_A(s^{-1}(Q_{1L}^A)) = q\,\overline{G}_B(Q_{1L}^B + Q_{2H}^B) + (1-q)\,\overline{G}_B(s^{-1}(Q_{1L}^B)).$$

Since the stock–out probability of firm 2 under high type does not change and low type of firm 1 gets spillover only from high type of firm 2, the probability of firm 1's getting a spillover should not change.

Let $\delta_{1L} = Q_{1L}^A - Q_{1L}^B$. We can rewrite the equilibrium condition as,

$$q\,\overline{G}_A(Q_{1L}^A + Q_{2H}^A) + (1-q)\,\overline{G}_A(s^{-1}(Q_{1L}^A)) = q\,\overline{G}_B(Q_{1L}^A + Q_{2H}^A - \delta_{1L} - \delta_{2H}) + (1-q)\,\overline{G}_B(s^{-1}(Q_{1L}^A - \delta_{1L})),$$

We know that for any $\{x_1, x_2\}$, if $\overline{G}_A(x_1) = \overline{G}_B(x_2)$ then $x_1 \ge x_2$. Moreover, since the spillover probability does not change, $\overline{G}_A(s^{-1}(Q_{1L}^A)) \ge \overline{G}_B(s^{-1}(Q_{1L}^A))$ should be satisfied. Thus, the difference between order quantities is positive, i.e., $\delta_{1L} \ge 0$ and $Q_{1L}^A \ge Q_{1L}^B$.

By a similar argument for (i_2), $\delta_{1H} = Q_{1H}^A - Q_{1H}^B \ge 0$.

For (i_3), we have

$$p\;\overline{G}_A(Q_{2L}^A + Q_{1H}^A) + (1-p)\,\overline{G}_A(Q_{2L}^A + Q_{1L}^*) = p\;\overline{G}_B(Q_{2L}^B + Q_{1H}^A - \delta_{1H}) + (1-p)\,\overline{G}_B(Q_{2L}^B + Q_{1L}^A - \delta_{1L}).$$

From previous argument, we know that the stock–out probability of firm 1 does not change with a stochastic increase in demand distribution. (Equilibrium order quantities increase to compensate the change in demand distribution.) Using a similar argument for (i_3), $\delta_{2L} = Q_{2L}^A - Q_{2L}^B \ge 0$. Thus all the equilibrium order quantities increase.

Similar proof for cases (ii) and (iii). □

A.6. Proof of Theorem 5

As s increases uniformly, $\hat{s}^{-1}$ increase, $\hat{s}$ and s^{-1} decreases. From Theorem 2, as s increases, Q_{2H}^* decreases.

From (i_1),

$$q\,\overline{G}^A(Q_{1L}^* + Q_{2H}^*) + (1-q)\,\overline{G}^A(s^{-1}(Q_{1L}^*)) = c_{1L}.$$

If s increases uniformly, s^{-1} decreases. Hence, Q_{1L} should increase to satisfy the equilibrium condition. Similarly, Q_{1H}^* increases as s increases.

From (i_3),

$$p\;\overline{G}^A(Q_{2L}^* + Q_{1H}^*) + (1-p)\,\overline{G}^A(Q_{2L}^* + Q_{1L}^*) = c_{2L}.$$

Since Q_{1L}^* and Q_{1H}^* increase, Q_{2L}^* should decrease to compensate. Similar argument applies for cases (ii) and (iii). □

A.7. Comparative Statics

This section summarizes the comparative statics results for general demand distributions. But we need the following results.

First note that $s' = \partial s(D)/\partial D > 0$ and $\hat{s}' = \partial \hat{s}(D)/\partial D > 0$ since we assume both s and $\hat{s}$ are increasing and deterministic functions. Then the derivative of the inverses of the split functions can be found by

$$(s^{-1})' = \frac{\partial s^{-1}(Q)}{\partial Q} = \frac{1}{s'(s^{-1}())} > 0$$
$$(\hat{s}^{-1})' = \frac{\partial \hat{s}^{-1}(Q)}{\partial Q} = \frac{1}{\hat{s}'(\hat{s}^{-1}())} > 0$$

We use these results to find the signs of derivatives of order quantities with respect to each parameter in the model.

Table 4 **Derivatives w.r.t.** c_{1L}

	Q	Conditions	$\mathbf{c_{1L}}$	Sign
	Q_{2H}		0	
(i)	Q_{1L}	$\overline{G}(s^{-1}(Q_{1L}))>0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1L}))}$	<0
		$\overline{G}(s^{-1}(Q_{1L}))=0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	<0
	Q_{1H}		0	
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{(1-p)g(Q_{1L}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{1L}}{\partial c_{1L}})$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	0	
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	<0
	Q_{1H}		0	
	Q_{2L}		0	
(iii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	<0
	Q_{1H}		0	
	Q_{2L}		0	

Table 5 **Derivatives w.r.t.** c_{1H}

	Q	Conditions	$\mathbf{c_{1H}}$	Sign
	Q_{2H}		0	
(i)	Q_{1L}		0	
	Q_{1H}		$-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	<0
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{1H}}{\partial c_{1H}})$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{\partial Q_{1H}}{\partial c_{1H}}$	>0
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial c_{1H}})$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	0	
	Q_{1H}		$-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	<0
	Q_{2L}	$\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$	$-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial c_{1H}})$	>0
		$\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$	$-\frac{\partial Q_{1H}}{\partial c_{1H}}$	>0
(iii)	Q_{1L}		0	
	Q_{1H}	$\overline{G}(Q_{1H}+Q_{2L})>0$	$-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	<0
		$\overline{G}(Q_{1H}+Q_{2L})=0$	$-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	<0
	Q_{2L}		0	

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Table 6 Derivatives w.r.t. c_{2L}

	Q	Conditions	$\mathbf{c_{2L}}$	Sign
	Q_{2H}		0	
(i)	Q_{1L}		0	
	Q_{1H}		0	
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{1}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{1}{pg(Q_{1H}+Q_{2L})}$	<0
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial c_{2L}})$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	0	
	Q_{1H}		0	
	Q_{2L}	$\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$	$-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	<0
		$\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$	$-\frac{1}{pg(Q_{1H}+Q_{2L})}$	<0
(iii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{(1-q)\hat{s}'g(Q_{1L}+Q_{2L})/g(\overline{G}^{-1}(c_{2L}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	0	
	Q_{1H}	$\overline{G}(Q_{1H}+Q_{2L})>0$	$\frac{(1-q)\hat{s}'g(Q_{1H}+Q_{2L})/g(\overline{G}^{-1}(c_{2L}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1H}+Q_{2L})=0$	0	
	Q_{2L}		$-\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2L}))}$	<0

Table 7 Derivatives w.r.t. c_{2H}

	Q	Conditions	$\mathbf{c_{2H}}$	Sign
	Q_{2H}		$-\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$	<0
(i)	Q_{1L}	$\overline{G}(s^{-1}(Q_{1L}))>0$	$\frac{\hat{s}'g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1L}))}$	>0
		$\overline{G}(s^{-1}(Q_{1L}))=0$	$\frac{\hat{s}'}{qg(\overline{G}^{-1}(c_{2H}))}$	>0
	Q_{1H}		$\frac{\hat{s}'g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	>0
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{pg(Q_{1H}+Q_{2L})\partial Q_{1H}/\partial c_{2H}+(1-p)g(Q_{1L}+Q_{2L})\partial Q_{1L}/\partial c_{2H}}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{\partial Q_{1H}}{\partial c_{1H}}$	<0
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{q\hat{s}'g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))+(1-q)g(Q_{1L}+Q_{2L})(\partial Q_{2L}/\partial c_{2H})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$	>0
	Q_{1H}		$\frac{\hat{s}'g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	>0
	Q_{2L}	$\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$	$-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial c_{2H}})$	<0
		$\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$	$-\frac{\partial Q_{1H}}{\partial c_{2H}}$	<0
(iii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{q\hat{s}'g(Q_{1L}+Q_{2H})/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$	>0
	Q_{1H}	$\overline{G}(Q_{1H}+Q_{2L})>0$	$\frac{q\hat{s}'g(Q_{1H}+Q_{2H})/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1H}+Q_{2L})=0$	$\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$	>0
	Q_{2L}		0	

.

Table 8 **Derivatives w.r.t.** p

	Q	Conditions	**p**	Sign
	Q_{2H}		0	
(i)	Q_{1L}		0	
	Q_{1H}		0	
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{\overline{G}(Q_{1H}+Q_{2L})-\overline{G}(Q_{1L}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$\frac{\overline{G}(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})}$	>0
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial p})$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	0	
	Q_{1H}		0	
	Q_{2L}	$\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$	$\frac{\overline{G}(Q_{1H}+Q_{2L})-\overline{G}(\hat{s}^{-1}(Q_{2L}))}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}$	>0
		$\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$	$\frac{\overline{G}(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})}$	>0
(iii)	Q_{1L}		0	
	Q_{1H}		0	
	Q_{2L}		0	

Table 9 **Derivatives w.r.t.** q

	Q	Conditions	**q**	Sign
	Q_{2H}		0	
(i)	Q_{1L}	$\overline{G}(s^{-1}(Q_{1L}))>0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1L}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1L}))}$	>0
		$\overline{G}(s^{-1}(Q_{1L}))=0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	>0
	Q_{1H}		$\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	>0
	Q_{2L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$-\frac{pg(Q_{1H}+Q_{2L})\partial Q_{1H}/\partial q+(1-p)g(Q_{1L}+Q_{2L})\partial Q_{1L}/\partial q}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$	<0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$-\frac{\partial Q_{1H}}{\partial q}$	<0
(ii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1L}+Q_{2L})-(1-q)g(Q_{1L}+Q_{2L})(\partial Q_{2L}/\partial q)}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	>0
	Q_{1H}		$\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$	>0
	Q_{2L}	$\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$	$-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial q})$	<0
		$\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$	$-\frac{\partial Q_{1H}}{\partial q}$	<0
(iii)	Q_{1L}	$\overline{G}(Q_{1L}+Q_{2L})>0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2L})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1L}+Q_{2L})=0$	$\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	>0
	Q_{1H}	$\overline{G}(Q_{1H}+Q_{2L})>0$	$\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2L})))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$	>0
		$\overline{G}(Q_{1H}+Q_{2L})=0$	$\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$	>0
	Q_{2L}		0	

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A.8. Equilibrium under Uniform Demand and Linear Market Shares

The equilibrium conditions under the assumption $D \sim \text{Uniform}(0,1)$ are as follows:

$$\begin{aligned} q(1-\min\{1,Q_{1H}+Q_{2H}\})+(1-q)(1-\min\{1,\max\{Q_{2L}/(1-s),Q_{1H}+Q_{2L}\}\}) \\ +(1-q)(\max\{\min\{1,Q_{2L}/(1-s)\}-\min\{1,Q_{1H}/s\},0\}) = c_{1H} \end{aligned}$$

$$\begin{aligned} q(1-\min\{1,Q_{1L}+Q_{2H}\})+(1-q)(1-\min\{1,\max\{Q_{2L}/(1-s),Q_{1L}+Q_{2L}\}) \\ +(1-q)(\max\{\min\{1,Q_{2L}/(1-s)\}-\min\{1,Q_{1L}/s\},0\}) = c_{1L} \end{aligned}$$

$$Q_{2H}/(1-s) = 1-c_{2H}$$

$$\begin{aligned} p(1-\min\{1,\max\{Q_{1H}/s,Q_{1H}+Q_{2L}\}\}+\max\{\min\{1,Q_{1H}/s\}-\min\{1,Q_{2L}/(1-s)\},0\}) \\ +(1-p)(1-\min\{1,\max\{Q_{1L}/s,Q_{1L}+Q_{2L}\}\}) \\ +(1-p)(\max\{\min\{1,Q_{1L}/s\}-\min\{1,Q_{2L}/(1-s)\},0\}) = c_{2L} \end{aligned}$$

Solution for $Q_{2H} = (1-s)(1-c_{2H})$ is straight forward. However in order to obtain the solutions for Q_{1L},Q_{1H} and Q_{2L} we have to know the ordering for Q_{1L}/s, Q_{1H}/s, $Q_{2L}/(1-s)$, 1 and whether $Q_{1L}+Q_{2L}$, $Q_{1H}+Q_{2L}$,$Q_{1L}+Q_{2H}$ and $Q_{1H}+Q_{2H}$ are greater than 1 or not. We can summarize all the possibilities as:

$$\begin{array}{lll} \{\frac{Q_{1L}}{s}>1, \frac{Q_{1L}}{s}\le 1\} & \{\frac{Q_{1H}}{s}>1, \frac{Q_{1H}}{s}\le 1\} & \{\frac{Q_{2L}}{(1-s)}>1, \frac{Q_{2L}}{(1-s)}\le 1\} \\ \{\frac{Q_{1L}}{s}>\frac{Q_{2L}}{(1-s)}, \frac{Q_{1L}}{s}\le\frac{Q_{2L}}{(1-s)}\} & \{\frac{Q_{1H}}{s}>\frac{Q_{2L}}{(1-s)}, \frac{Q_{1H}}{s}\le\frac{Q_{2L}}{(1-s)}\} & \\ \{Q_{1L}+Q_{2L}>1, Q_{1L}+Q_{2L}\le 1\} & \{Q_{1H}+Q_{2L}>1, Q_{1H}+Q_{2L}\le 1\} & \\ \{Q_{1L}+Q_{2H}>1, Q_{1L}+Q_{2H}\le 1\} & \{Q_{1H}+Q_{2H}>1, Q_{1H}+Q_{2H}\le 1.\} & \end{array}$$

We have 512 different possibilities for Q_{1L},Q_{1H} and Q_{2L} each leading to a different region in the 7 dimensional space. However, the number of regions can be reduced to 8 regions as shown below.

First, if both of the players have a high type, then the total inventory cannot exceed 1 and if second firm has high type since he does not expect any spillover. This is simply due to the suboptimality of all values greater than 1. Second, some of the conditions imply the others. For example, if $Q_{1L}/s > 1$ and $Q_{2L}/(1-s) > 1$ then $Q_{1L}+Q_{2L} > 1$. Third, $Q_{2L}/(1-s) > Q_{1L}/s$ implies $Q_{2L}/(1-s) > Q_{1H}/s$ since low type of a firm orders as much as high type of the firm due to submodularity. Similarly, $Q_{2L}/(1-s) \le Q_{1H}/s$ implies $Q_{2L}/(1-s) \le Q_{1L}/s$.

Using these kind of arguments we reduce the conditions to form 8 different regions. It can be shown that it is not possible to reduce the conditions further without making additional assumptions on the parameters.

Region	Conditions
1	$\frac{Q_{1L}}{s}>1$, $\frac{Q_{2L}}{(1-s)}>1$
2	$Q_{1L}+Q_{2L}>1$, $\frac{Q_{1L}}{s}\le 1$
3	$Q_{1L}+Q_{2L}\le 1$, $\frac{Q_{1L}}{s}\le\frac{Q_{2L}}{(1-s)}$
4	$Q_{1L}+Q_{2L}>1$, $\frac{Q_{2L}}{(1-s)}\le 1$, $\frac{Q_{1H}}{s}\le\frac{Q_{2L}}{(1-s)}$
5	$Q_{1L}+Q_{2L}\le 1$, $\frac{Q_{1L}}{s}>\frac{Q_{2L}}{(1-s)}$, $\frac{Q_{1H}}{s}\le\frac{Q_{2L}}{(1-s)}$
6	$Q_{1H}+Q_{2L}>1$, $\frac{Q_{1H}}{s}>\frac{Q_{2L}}{(1-s)}$
7	$Q_{1L}+Q_{2L}>1$, $Q_{1H}+Q_{2L}\le 1$, $\frac{Q_{1H}}{s}>\frac{Q_{2L}}{(1-s)}$
8	$Q_{1L}+Q_{2L}\le 1$, $\frac{Q_{1H}}{s}>\frac{Q_{2L}}{(1-s)}$

In each of the regions, the given inequalities simplify the equilibrium conditions leading to an easy computation of the equilibrium order quantities.

For Region 1, we reduce the equilibrium conditions to the following form:

$$\begin{aligned} q(1-Q_{1H}-Q_{2H})+(1-q)(1-Q_{1H}/s) &= c_{1H}, \\ q(1-Q_{1L}-Q_{2H}) &= c_{1L}, \\ Q_{2H}/(1-s) &= 1-c_{2H}, \\ p(1-Q_{1H}-Q_{2L}) &= c_{2L}. \end{aligned}$$

It is straightforward to find the order quantities for this region:

$$\begin{array}{ll} Q_{1H} = \frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)} & Q_{1L} = 1-\frac{c_{1L}}{q}-(1-s)(1-c_{2H}), \\ Q_{2H} = (1-s)(1-c_{2H}) & Q_{2L} = 1-\frac{c_{2L}}{p}-\frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)}. \end{array}$$

Now, by plugging these quantities into necessary inequalities, we obtain:

$$\begin{aligned}
&\frac{Q_{1L}}{s} > 1 \qquad \Rightarrow \qquad 1 - \frac{c_{1L}}{q} - (1-s)(1-c_{2H}) > s \\
&\Rightarrow \frac{c_{1L}}{q} - (1-s)(1-c_{2H}) < 1-s \qquad \Rightarrow \qquad \frac{c_{1L}}{q} - (1-s)c_{2H} < 0 \\
&\Rightarrow c_{1L} < q(1-s)c_{2H}
\end{aligned}$$

$$\begin{aligned}
&\frac{Q_{2L}}{(1-s)} > 1 \qquad \Rightarrow \qquad 1 - \frac{c_{2L}}{p} - \frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)} > 1-s \\
&\Rightarrow \frac{c_{2L}}{p} + \frac{s(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)} < s \qquad \Rightarrow \qquad \frac{c_{2L}}{p} - \frac{s(c_{1H}-q(1-s)(c_{2H}))}{(1-(1-s)q)} < 0 \\
&\Rightarrow c_{2L} < \frac{sp(c_{1H}-q(1-s)c_{2H})}{1-(1-s)q}
\end{aligned}$$

Thus, Region 1 can be characterized by two inequalities:

$$\begin{aligned}
&c_{1L} < q(1-s)c_{2H}, \\
&c_{2L} < \frac{sp(c_{1H}-q(1-s)c_{2H})}{1-(1-s)q}.
\end{aligned}$$

These conditions are necessary and sufficient, i.e., if these inequalities are satisfied, then equilibrium order quantities take the values in Region 1.

In a similar fashion, we can obtain the conditions for all 8 regions. This is summarized in Figure 1.

A.9. Comparative Statics under Uniform Demand and Linear Market Shares

$Q_{1L} \rightarrow$

	$\mathbf{Q_{1L}^{\alpha}}$	$\mathbf{Q_{1L}^{\beta}}$	$\mathbf{Q_{1L}^{\gamma}}$	$\mathbf{Q_{1L}^{\delta}}$
$\mathbf{c_{1L}}$	$-\frac{1}{q}$	$-\frac{1}{(q+(1-q)/s)}$	-1	-1
$\mathbf{c_{1H}}$	0	0	$-\frac{s(1-s)(1-q)p}{(1-(1-s)q)(1-sp)}$	0
$\mathbf{p}$	0	0	$\frac{s(1-s)(1-q)(c_{2L}-c_{1H}+q(1-s)(c_{2H}-c_{2L}))}{(1-(1-s)q)(1-sp)^2}$	0
$\mathbf{c_{2L}}$	0	0	$\frac{(1-s)(1-q)}{1-sp}$	$(1-q)(1-s)$
$\mathbf{c_{2H}}$	$(1-s)$	$\frac{s(1-s)q}{(1-sp)}$	$\frac{q(1-s)(1-q(1-s)-ps^2)}{(1-(1-s)q)(1-sp)}$	$q(1-s)$
$\mathbf{q}$	$\frac{c_{1L}}{q^2}$	$\frac{s(1-s)(c_{2H}-c_{1L})}{(1-(1-s)q)^2}$	$\frac{(1-s)((1-(1-s)q)^2(c_{2H}-c_{2L})-ps^2(c_{2H}-c_{1H}))}{(1-(1-s)q)^2(1-ps)}$	$(1-s)(c_{2H}-c_{2L})$
$\mathbf{s}$	$(1-c_{2H})$	$1-c_{2H}+\frac{(1-q)(c_{2H}-c_{1L})}{(1-(1-s)q)^2}$	$q(1-c_{2H})+(1-q)(\frac{(1-s)(1-q)(1-c_{2L})}{(1-sp)^2}+\frac{s(1-s)pq(1-c_{2H})}{(1-(1-s)q)(1-sp)}$ $+\frac{p(1-2s+s^2p-(1-s)^2q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2(1-sp)^2})$	$q(1-c_{2H})+(1-q)(1-c_{2L})$

$Q_{2L} \rightarrow$

	$\mathbf{Q_{2L}^{\alpha}}$	$\mathbf{Q_{2L}^{\beta}}$	$\mathbf{Q_{2L}^{\gamma}}$	$\mathbf{Q_{2L}^{\delta}}$
$\mathbf{c_{1L}}$	0	$\frac{s(1-p)}{(1-(1-s)q)}$	0	0
$\mathbf{c_{1H}}$	$\frac{s}{(1-(1-s)q)}$	$\frac{sp}{(1-(1-s)q)}$	$\frac{s(1-s)p}{(1-(1-s)q)(1-sp)}$	0
$\mathbf{p}$	$\frac{c_{2L}}{p^2}$	$\frac{s(c_{1H}-c_{1L})}{(1-(1-s)q)}$	$\frac{s(1-s)(c_{1H}-c_{2L}-q(1-s)(c_{2H}-c_{2L}))}{(1-(1-s)q)(1-sp)^2}$	0
$\mathbf{c_{2L}}$	$-\frac{1}{p}$	-1	$-\frac{(1-s)}{(1-sp)}$	$-(1-s)$
$\mathbf{c_{2H}}$	$-\frac{qs(1-s)}{(1-(1-s)q)}$	$-\frac{qs(1-s)}{(1-(1-s)q)}$	$-\frac{pqs(1-s)^2}{(1-(1-s)q)(1-sp)}$	0
$\mathbf{q}$	$-\frac{s(1-s)(c_{2H}-c_{1H})}{(1-(1-s)q)^2}$	$-\frac{s(1-s)(c_{2H}-c_{1L}-p(c_{1H}-c_{1L}))}{(1-(1-s)q)^2}$	$-\frac{ps(1-s)^2(c_{2H}-c_{1H})}{(1-(1-s)q)^2(1-sp)}$	0
$\mathbf{s}$	$-\frac{(1-q)(1-c_{1H})}{(1-(1-s)q)^2}$ $-\frac{q^2(1-s)^2(1-c_{2H})-q(1-2s)(1-c_{2H})}{(1-(1-s)q)^2}$	$-\frac{(1-q)(1-pc_{1H}-(1-p)c_{1L})}{(1-(1-s)q)^2}$ $+\frac{q^2(1-s)^2(1-c_{2H})-q(1-2s)(1-c_{2H})}{(1-(1-s)q)^2}$	$-\frac{(1-s)(1-q)(1-c_{2L})}{(1-sp)^2}-\frac{s(1-s)pq(1-c_{2H})}{(1-(1-s)q)(1-sp)}$ $-\frac{p(1-2s+s^2p-(1-s)^2q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2(1-sp)^2}$	$-(1-c_{2L})$

Q_{1H} and $Q_{2H} \rightarrow$

	$\mathbf{Q_{1L}^{\alpha}}$	$\mathbf{Q_{1L}^{\beta}}$	$\mathbf{Q_{1L}^{\gamma}}$	$\mathbf{Q_{2H}}$
$\mathbf{c_{1L}}$	0	0	0	0
$\mathbf{c_{1H}}$	$-\frac{s}{(1-(1-s)q)}$	$-\frac{1}{q}$	-1	0
$\mathbf{p}$	0	0	0	0
$\mathbf{c_{2L}}$	0	0	$(1-q)(1-s)$	0
$\mathbf{c_{2H}}$	$\frac{qs(1-s)}{(1-(1-s)q)}$	$(1-s)$	$q(1-s)$	$-(1-s)$
$\mathbf{q}$	$\frac{s(1-s)(c_{2H}-c_{1H})}{(1-(1-s)q)^2}$	$\frac{c_{1H}}{q^2}$	$(1-s)(c_{2H}-c_{2L})$	0
$\mathbf{s}$	$\frac{sq(1-c_{2H})}{(1-(1-s)q)} + \frac{(1-q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2}$	$(1-c_{2H})$	$q(1-c_{2H})+(1-q)(1-c_{2L})$	$-(1-c_{2H})$