Affine Invariant Total Variation Models

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Total Variation,
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image processing

This report relates to the field of image restorations and features extracting from noisy and blurred images. Since their introduction in a classical paper by Rudin, Osher and Fatemi, Total Variation (*TV*) minimising models have become one of the most popular and successful tools for image restorations. Whilst invariance under affine transformations is very important for many image processing tasks, the total variation functional is not invariant under general affine transformation. In the current report we introduce for the first time a new affine invariant regularization functional which has many properties similar to total variation and can be used for affine invariant denoising and restoration tasks. The explicit formula for calculation of this regularization functional is given.

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Abstract

This report relates to the field of image restorations and features extracting from noisy and blurred images. Since their introduction in a classical paper by Rudin, Osher and Fatemi [2], Total Variation (*TV*) minimising models have become one of the most popular and successful tools for image restorations. Whilst invariance under affine transformations is very important for many image processing tasks, the total variation functional is not invariant under general affine transformation. In the current report we introduce for the first time a new affine invariant regularization functional which has many properties similar to total variation and can be used for affine invariant denoising and restoration tasks. The explicit formula for calculation of this regularization functional is given.

Introduction

Variational models have been extremely successful in a wide variety of restoration problems (denoising, deblurring, blind deconvolution, and impainting), and remain one of the most active areas of research in image processing and computer vision. Variational models exhibit the solution of these problems as minimizers of appropriately chosen energy functionals.

Assume that a given image u_0 is noisy and blurred:

$$u_0 = Ku + n.$$

Then the Bayesian restoration energy proposed in [2] is

(1) $$E[u \mid u_0] = TV(u) + \lambda E[u_0 \mid u]$$

for a certain tuning parameter $\lambda > 0$, and $TV(u) := \int |\nabla u|$ denotes the total variation of u. Used as a regularization term, the TV functional is particularly relevant in recovering piecewise smooth functions without smoothing the sharp discontinuities, in contrast with other regularization functionals generally based on a quadratic norm.

The revolutionary aspect of the model (1) is its regularization term $TV(u)$ that allows for discontinuities, but at the same time disfavours oscillations. However, this regularization term is not invariant under general linear transformation. This limits application of this method for image registration problems and for analysis of images obtained under different angles.

Affine invariant total variation energy

Since we are going to introduce an affine invariant energy we will think about R^2 as an abstract two dimensional real vector space V without any preselected basis. Let $u : V \to \mathrm{R}$ be any smooth enough function on V with compact support. V is commutative locally compact group, so it has a well defined invariant Haar measure $\mu(dx)$ defined up to constant. This Haar measure is a multiple the standard Lebesgue measure, but we want to construct everything "coordinate free". Now, using the function u and the measure $\mu(dx)$, we define a norm $\|\cdot\|_u$ on the *same* vector space V. Let $v \in V$ be any vector from V. The derivative $L_v u$ of the function u in direction of the vector v is defined as usual by

$$(L_v u)(x) = \left. \frac{du(x+tv)}{dt} \right|_{t=0}.$$

$L_v u$ does not involve any inner product or norm on V. We defined $\|v\|_u$ as

(2) $$\|v\|_u = \int_V |(L_v u)(x)| \mu(dx).$$

Now $(V, \|\cdot\|_u)$ is the two-dimensional Banach space that we shall associate with u. Its unit ball $B_1(u) = \{v \in V : \|v\|_u \le 1\}$ is a symmetric convex body in V and our new affine invariant total variation energy of u is defined as

$$ATV(u) = \frac{1}{\sqrt{Vol(B_1(u))}},$$

where $Vol(B_1(u))$ is just $\mu-$measure of $B_1(u)$. This ATV energy is obviously invariant under linear measure preserving transformation. Moreover, as was shown in [4] the following Sobolev-type inequality holds

$$ATV(u) \ge const \cdot \|u\|_{L^2},$$

which opens the way for using functional analysis techniques to analyse these ATV-type models. For practical calculation another expression for $Vol(B_1(u))$ is more useful. If we introduce any Euclidean structure on V then

$$Vol(B_1(u)) = \frac{1}{2}\int_{S^1} \|\nabla_v u\|_{L^1}^{-2}\, dv,$$

where S^1 is the unit circle with canonical measure dv and $\nabla_v u = v^1 \frac{\partial u}{\partial x^1} + v^2 \frac{\partial u}{\partial x^2}$ in the orthogonal coordinate system ($x = (x^1, x^2),\ v = (v^1, v^2)$).

Remark. 1) $ATV(u) \le \frac{2}{\pi^{3/2}} TV(u)$. To see this, note that from the Hölder inequality and Fubini's theorem we have

$$\left(\frac{1}{2\pi}\int_{S^1} \|\nabla_v u\|_{L^1}^{-2}\, dv\right)^{-\frac{1}{2}} \le \frac{1}{2\pi}\int_{S^1} \|\nabla_v u\|_{L^1}\, dv = \frac{1}{2\pi}\int_{S^1}\int_{\mathrm{R}^2} \left|\langle \nabla u(x), v\rangle\right| dxdv$$
$$= \frac{1}{2\pi}\int_{\mathrm{R}^2}\int_{S^1} \left|\langle \nabla u(x), v\rangle\right| dvdx = \frac{1}{2\pi}\int_{S^1} |v_0 \cdot v|\, dv = \frac{2}{\pi}\int_{\mathrm{R}^2} |\nabla u(x)|\, dx,$$

where v_0 is any fixed unit vector. From this we have $\frac{1}{\sqrt{\pi}\sqrt{Vol(B_1(u))}} \le \frac{2}{\pi} TV(u)$ which implies $ATV(u) \le \frac{2}{\pi^{3/2}} TV(u)$.

2) If data has additional smoothness then we can define p-version of *ATV* by using L^p norm in (2).

Conclusions

In this report we have introduced new affine invariant total variation energy. Using this energy as a regularization term results in affine invariant total variation models. This *ATV* energy has the following

Advantages:

1) *ATV* energy is bounded from above by total variation energy
   $$\mathrm{ATV(u)} \bullet 2\ \mathrm{p}^{-3/2}\ TV(u)$$
   and thus is capable of handling edges. This is because edges are precisely the case of finite TV.
2) Stability and efficiency of *TV* models heavily depend on Sobolev inequality (Sobolev inequality allows to prove existence of decompositions and to control error in numerical schemes). New proposed *ATV* energy satisfies the same inequality.
3) Since *ATV* is affine invariant by construction, the results of restoration will also be affine invariant.
4) Despite of being affine invariant *ATV* energy can be effectively calculated in any orthogonal coordinate system.

References

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