



Quantum error correction via robust probe modes

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HP Laboratories Bristol
HPL-2005-213
December 5, 2005*

quantum error
correction,
quantum
computation and
communication

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* Internal Accession Date Only

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Approved for External Publication

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(Dated: November 12, 2005)

We propose a new scheme for quantum error correction using robust continuous variable probe modes, rather than fragile ancilla qubits, to detect errors without destroying data qubits. The use of such probe modes reduces the required number of expensive qubits in error correction and allows efficient encoding, error detection and error correction. Moreover, the elimination of the need for direct qubit interactions significantly simplifies the construction of quantum circuits. We will illustrate how the approach implements three existing quantum error correcting codes: the 3-qubit bit-flip (phase-flip) code, the Shor code, and an erasure code.

PACS numbers: 03.67.Pp, 03.67.Lx, 42.65.-k

In recent years, we have seen the development and realization of small scale quantum devices and circuits [1–6]. While these systems currently use only a few qubits, they show great promise for being able to be used in large scale quantum computation. As the size of such systems grows, the investigation of practical schemes for fault-tolerant architectures has become an urgent task. The standard method of error correction is to use several physical qubits to encode quantum information in a redundant fashion and then introduce ancilla qubits to perform error syndrome detection and correction [7–9]. The concatenation of these error correction schemes has been shown to guarantee the computational system to be fault tolerant for error rates below a certain threshold value [10]. Thus the total system in principle satisfies efficiency and scalability. However, in practice, implementation of a system for fault-tolerant quantum computation is a daunting task due to the huge overhead in number of qubits and fundamental quantum logic gates.

In the standard method of error correction, ancilla qubits are used to project the state of the logical qubit onto one of several subspaces, and thus the task of ancilla qubits is rather simple. Hence it might be possible to replace the costly ancilla qubits with something more practically viable such as continuous-variable probe modes. In this paper, we present a new scheme to implement existing error correcting codes, replacing the ancilla qubits with robust probe modes. The introduction of probe modes interacting nonlinearly with qubits facilitates the encoding, error detection and correction procedures. The interaction between qubits and a probe mode enables detection of errors in the qubits by measurement of the probe mode. Such operations can be achieved in a variety of physical systems [11]. For example, for electron spin qubits, the interaction can be achieved with a coherent light interacting with the electrons contained

in optical microcavities [12–14]. In the all-optical implementation, such operations can be achieved with an intense coherent light interacting with photonic qubits via a cross-Kerr nonlinearity [15–17]. In these physical systems, the probe mode is particularly stable in its preparation, and this provides flexibility in arranging qubits in space by connecting them remotely. To illustrate our approach, we demonstrate implementations of the following codes: (1) the 3-qubit bit (or phase) flip code [18], (2) the 9-qubit Shor code [7] and lastly (3) an erasure code [19, 20].

We begin by describing two fundamental two-qubit gates used in our scheme. The essential component in these gates is a weak nonlinear interaction between a qubit and a probe mode (indicated by p). This interaction is described by the Hamiltonian

$$H = \hbar\chi|1\rangle\langle 1| \otimes \hat{n}_p, \quad (1)$$

where χ is the interaction strength, \hat{n}_p is the number operator of the probe mode, and the basis states of the qubit are given by $|0\rangle$ and $|1\rangle$. These basis states could be, for example, the spin states of an electron, or the polarization states of a photon. The process causes a conditional phase shift on the probe mode dependent on the state of the qubit, namely,

$$|0\rangle|\alpha\rangle \rightarrow |0\rangle|\alpha\rangle, \quad |1\rangle|\alpha\rangle \rightarrow |1\rangle|\alpha e^{i\theta}\rangle. \quad (2)$$

The size of the phase shift is given by $\theta = \chi t$ with t being the interaction time. This is the elementary operation to construct the fundamental two-qubit gates.

Our first fundamental gate, the two-qubit parity gate, has been shown to be as useful as the controlled-NOT gate for universal quantum computation [21]. As shown in Fig. 1a), two qubits subsequently interact with the probe mode inducing a phase shift dependent on the two-qubit state. The even parity states ($|00\rangle, |11\rangle$) cause no overall phase shift in the probe mode, whereas the odd parity states ($|01\rangle, |10\rangle$) cause a phase shift $\pm\theta$ as in

$$\left\{ \begin{array}{l} |00\rangle|\alpha\rangle \rightarrow |00\rangle|\alpha\rangle \\ |11\rangle|\alpha\rangle \rightarrow |11\rangle|\alpha\rangle \end{array} \right\}, \quad \left\{ \begin{array}{l} |01\rangle|\alpha\rangle \rightarrow |01\rangle|\alpha e^{-i\theta}\rangle \\ |10\rangle|\alpha\rangle \rightarrow |10\rangle|\alpha e^{i\theta}\rangle \end{array} \right\}. \quad (3)$$

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A projective measurement on the probe mode distinguishes $|\alpha\rangle$ and $|\alpha e^{\pm i\theta}\rangle$ (without distinguishing $|\alpha e^{i\theta}\rangle$ from $|\alpha e^{-i\theta}\rangle$). This projects the two-qubit state onto either the even parity subspace or the odd parity subspace [16]. An odd-parity state can be converted to an even-parity state by a classical feed-forward after the parity measurement. In the construction of the parity gate, the direct interaction between the qubits is not needed. Therefore, the gate can be performed among distributed qubits.

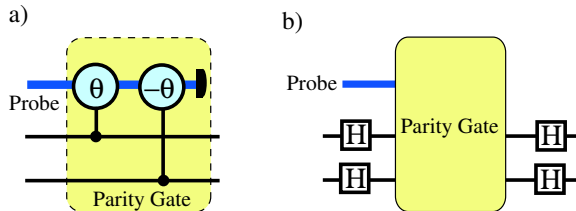


FIG. 1: a) A two-qubit parity gate. The conditional phase shift given by the Hamiltonian Eq. (1) is denoted by θ . After the interactions, the probe mode is measured to project the qubit state onto the even or odd parity subspace. b) We depict a symmetrizer gate. The two-qubit parity gate is applied in $|+\rangle/|-\rangle$ basis instead of $|0\rangle/|1\rangle$ basis.

The second fundamental gate is a two-qubit parity gate with a twist [15]. Starting with the two-qubit state $|xy\rangle$, where $x, y = 0, 1$, the gate generates the symmetrized state, $(|xy\rangle + |\bar{x}\bar{y}\rangle)/\sqrt{2}$. Hence we call this gate the “symmetrizer gate.” Now as shown in Fig. 1b) we first change the basis for the two-qubit state $|xy\rangle$ from the computational basis $|0\rangle/|1\rangle$ to the $|+\rangle/|-\rangle$ basis, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. This basis change is described by the Hadamard transformation H . The two-qubit state $|xy\rangle$ is

$$|xy\rangle = \frac{1}{2} [|++\rangle + (-1)^{x+y} |--\rangle] + \frac{1}{2} (-1)^x [|+-\rangle + (-1)^{x+y} |-+\rangle], \quad (4)$$

in the $|+\rangle/|-\rangle$ basis, where we have used $|x\rangle = (|+\rangle + (-1)^x |-\rangle)/\sqrt{2}$ and $|y\rangle = (|+\rangle + (-1)^y |-\rangle)/\sqrt{2}$. Then we apply the two-qubit parity gate on qubits 1 and 2 in the $|+\rangle/|-\rangle$ basis, and if the measured parity is odd, flip qubit 2 by classical feed-forward. Hence we are left with the even state $(|++\rangle + (-1)^{x+y} |--\rangle)/\sqrt{2}$, which is transferred back to $(|xy\rangle + |\bar{x}\bar{y}\rangle)/\sqrt{2}$ by another basis change to the $|0\rangle/|1\rangle$ basis. This shows the operation of the symmetrizer gate and indicates its potential in the encoding procedures.

We next consider our error correcting scheme. An arbitrary unitary operation on a qubit can be described as a linear superposition of a bit-flip error (X), a phase-flip error (Z) and both. Thus, unless a qubit is lost, an arbitrary quantum error is either a bit-flip, a phase-flip or both [18]. We first describe the 3-qubit bit-flip code, which can detect and correct at most one bit-flip error by

utilizing redundant encoding. The computational basis states for the logical qubit, $|0\rangle_L$ and $|1\rangle_L$, are

$$|0\rangle_L = |0\rangle_1 |0\rangle_2 |0\rangle_3 \equiv |000\rangle, |1\rangle_L = |1\rangle_1 |1\rangle_2 |1\rangle_3 \equiv |111\rangle. \quad (5)$$

The standard method of detecting a bit-flip error is to measure the products of Pauli operators, $Z_1 Z_2$ and $Z_2 Z_3$, with the help of two ancilla qubits and four controlled-NOT operations to the ancilla qubits. The operators $Z_1 Z_2$ and $Z_2 Z_3$ constitute the stabilizer of the code. The code subspace consists of their simultaneous eigenstates with eigenvalue $+1$ ($|000\rangle, |111\rangle$). If a bit-flip error occurs, one or both of the stabilizer generators $Z_1 Z_2$ and $Z_2 Z_3$ becomes -1 , as summarized in Table I.

Bit-flip error	$Z_1 Z_2$	$Z_2 Z_3$	modulo(4)
None	+1	+1	0
on qubit 1	-1	+1	2
on qubit 2	-1	-1	3
on qubit 3	+1	-1	1

TABLE I: Error syndromes both in binary readout and in modulo(4) readout. The binary readout is given by the measurements of $Z_1 Z_2$ and $Z_2 Z_3$ using two probe modes as shown in Fig. 2, while the module(4) readout is given using one probe mode by the circuit shown in Fig. 3.

A circuit for the syndrome measurement gate can be constructed as shown in Fig. 2. If $Z_1 Z_2$ is $+1$ (-1), the first parity gate, involving qubits 1 and 2, measures even (odd) parity. The operator $Z_2 Z_3$ can be measured by the second parity gate involving qubits 2 and 3. Therefore those two parity gates complete error detection. Then the error can be corrected after the detection by classical feed-forward control, or can be recorded for later error correction.

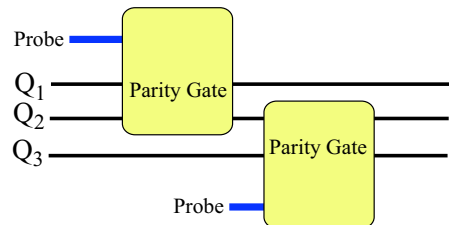


FIG. 2: Circuit for error detection by the 3-qubit bit-flip code. The first parity gate measures $Z_1 Z_2$ and the second parity gate measures $Z_2 Z_3$.

Alternatively, the measurements of $Z_1 Z_2$ and $Z_2 Z_3$ can be combined and performed by using only one probe mode instead of two, as shown in Fig. 3. The interactions of the probe mode $|\alpha\rangle$ with the three qubits are set such that the phase shifts caused by the qubits 1, 2, and 3 are θ_1, θ_2 and θ_3 , where θ_1, θ_2 and θ_3 are all distinct and satisfy $\theta_1 + \theta_2 + \theta_3 = 0$ (for example, $\theta_1 = \theta, \theta_2 = 2\theta, \theta_3 = -3\theta$). The encoded states ($|000\rangle$ or $|111\rangle$) cause no phase shift in the probe mode. If a bit-flip error occurs on qubit 1 ($|100\rangle$ or $|011\rangle$), the probe mode $|\alpha\rangle$ evolves

into $|\alpha e^{\pm i\theta_1}\rangle$. The probe mode gains a phase shift $\pm\theta_2$ or $\pm\theta_3$ in the case of a bit-flip error on qubit 2 or 3. By measuring the phase of the probe mode, distinguishing $|\alpha\rangle$, $|\alpha e^{\pm i\theta_1}\rangle$, $|\alpha e^{\pm i\theta_2}\rangle$ and $|\alpha e^{\pm i\theta_3}\rangle$, we can identify an error. Those four error syndromes correspond to different values in modulo(4) in Table I. This modification is equivalent to replacement of two binary readouts by one modulo(4) readout, and allows the use of the larger Hilbert space of the probe mode. As a result, the error correcting procedure is significantly simplified.

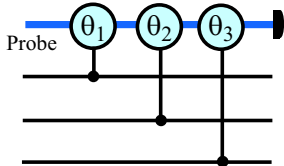


FIG. 3: Circuit for error detection by the 3-qubit bit-flip code using one probe mode. Phase shifts (θ_1 , θ_2 and θ_3) of the probe mode by qubits 1, 2, and 3 are all distinct and satisfy $\theta_1 + \theta_2 + \theta_3 = 0$. This gate works as a modulo(4) error detection so as to give the readouts described in Table I.

The 3-qubit bit-flip code can be converted to a code to correct at most one phase-flip error by changing its basis from $|0\rangle/|1\rangle$ to $|+\rangle/|-\rangle$.

We now consider the encoding procedure. The conventional circuit to encode a unknown qubit state $c_0|0\rangle + c_1|1\rangle$, initially stored in qubit 1, by the 3-qubit bit-flip code consists of two controlled-NOT gates. Instead, we simply use the error correcting procedure. To do so, we first initialize qubits 2 and 3 in the state $(|0\rangle + |1\rangle)/\sqrt{2}$. The three-qubit state is a sum of the four states corresponding to the four error syndromes shown in Table I, and hence by running the error correcting procedure described above we can reduce the state to $c_0|000\rangle + c_1|111\rangle$. The one slight difference from the error correcting procedure is that there should be no error on qubit 1 in the encoding procedure. When the syndrome measurement indicates an error on qubit 1, the result needs to be interpreted as errors on qubits 2 and 3 and the classical feed forward has to correct those two errors.

The scheme for the 3-qubit error correcting code above can be extended for the 9-qubit Shor code [7]. This code is constructed by concatenating the 3-qubit bit-flip code into the 3-qubit phase-flip code, and can correct arbitrary errors: bit-flip errors, phase-flip errors or both. With the logical basis states defined by $|0\rangle_L = (|000\rangle + |111\rangle)^{\otimes 3}/2\sqrt{2}$ and $|1\rangle_L = (|000\rangle - |111\rangle)^{\otimes 3}/2\sqrt{2}$ we can create the encoded state $c_0|0\rangle_L + c_1|1\rangle_L$ from qubit 1, $c_0|0\rangle + c_1|1\rangle$, as follows. We first initialize qubits 2 to 8 in the state $|0\rangle$ and apply the 3-qubit phase-flip code on qubits 1/4/7. This prepares the state $c_0|+\rangle_1|+\rangle_4|+\rangle_7 + c_1|-\rangle_1|-\rangle_4|-\rangle_7$. Then we apply the 3-qubit bit-flip code on each of the three sets of qubits 1/2/3, 4/5/6, and 7/8/9. Such an operation leads to the encoded state $c_0|0\rangle_L + c_1|1\rangle_L$.

Next, in order to detect and correct errors we need to

perform the above procedure in roughly the reverse order. First we detect and correct a bit-flip error if present. The 3-qubit bit-flip code is applied to each of the three sets of qubits, 1/2/3, 4/5/6, and 7/8/9, and any errors are then corrected. Following this, we deal with phase-flip errors. The phase-flip error correction requires an additional step compared with the bit-flip error correction. We first apply the symmetrizer gate on each of the pairs of qubits, 2/3, 5/6, and 8/9. The symmetrizer gate performs the following action,

$$|000\rangle \pm |111\rangle \rightarrow |\pm\rangle(|00\rangle + |11\rangle). \quad (6)$$

and thus disentangles qubits 2/3, 5/6, and 8/9 from qubits 1/4/7. We now apply the 3-qubit phase-flip code on the remaining qubits 1/4/7 to correct phase-flip errors that may have occurred. After the error detection and correction, we then need to re-encode our quantum state. This can be done as described in the encoding procedure.

The last error correcting code to be considered here will recover an error caused by qubit leakage at a known location (qubit leakage is the dominant error for instance in optical implementations). The knowledge of the position of a qubit loss is assumed for the erasure code to work, but this can be satisfied again with the use of a probe mode. The probe mode can be used in a nondestructive operation to determine if the qubit is present or not without measuring its quantum state. There are a number of erasure codes we could consider, but we will focus here on one presented recently by Ralph *et al.* [19, 20] for linear optical quantum computation. It is applicable to all qubit leakage situations. The logical basis states are defined by tensor products of Bell states as $|0\rangle_L^{(n)} = (|00\rangle + |11\rangle)^{\otimes n}/2^{n/2}$ and $|1\rangle_L^{(n)} = (|01\rangle + |10\rangle)^{\otimes n}/2^{n/2}$. Here n is the number of Bell states used in the encoding. An arbitrary state can then be written as $c_0|0\rangle_L^{(n)} + c_1|1\rangle_L^{(n)}$ and can be created from $c_0|0\rangle + c_1|1\rangle$ using only local operations and parity gates. For simplicity, consider the $n = 2$ (4 qubits) code where qubit 1 is prepared as $c_0|0\rangle + c_1|1\rangle$ with the remaining three in the state $|0\rangle$. The encoding procedure begins by performing a Hadamard transformation on qubit 3 and then applying the parity gate between qubits 1 and 3. This results in the state $c_0|0000\rangle + c_1|1010\rangle$, after a bit-flip operation if the odd-parity state occurred. We then apply the symmetrizer gate between qubits 1/2 and 3/4. This transforms the state to $c_0(|00\rangle + |11\rangle)(|00\rangle + |11\rangle)/2 + c_1(|01\rangle + |10\rangle)(|01\rangle + |10\rangle)/2$, which is our encoded state $c_0|0\rangle_L^{(2)} + c_1|1\rangle_L^{(2)}$. Larger n states can be prepared in a similar fashion. Now consider that a qubit is lost and we have identified its location. That causes our encoded state to become mixed, but we can easily rectify this. We measure in the $|+\rangle/|-\rangle$ basis the remaining qubit of the Bell pair in which a qubit is lost, and then perform a phase-flip operation if the $|-\rangle$ state is measured. The effect of this operation is to transform the encoded state according to $c_0|0\rangle_L^{(n)} + c_1|1\rangle_L^{(n)} \rightarrow c_0|0\rangle_L^{(n-1)} + c_1|1\rangle_L^{(n-1)}$, that is, we have lost one Bell pair (2 qubits) worth of en-

coding. We further perform measurement on the second qubit of one of the remaining Bell states in the $|0\rangle/|1\rangle$ basis. If the measurement result is $|0\rangle$, the state is projected onto

$$c_0|0\rangle_L^{(n-2)}|00\rangle + c_1|1\rangle_L^{(n-2)}|10\rangle. \quad (7)$$

A similar state is obtained if the measurement result is $|1\rangle$, which can be transformed via local operations to Eq. (7). By adding two qubits prepared in the state $|0\rangle$ and applying the symmetrizer gates as in the encoding procedure, we can recover the fully encoded state $c_0|0\rangle_L^{(n)} + c_1|1\rangle_L^{(n)}$.

We now analyze error propagation during the error correcting procedures. Consider first the conventional error correcting scheme using ancilla qubits. Suppose we use the 3-qubit bit-flip code and measure Z_1Z_2 using an ancilla qubit at an error rate ϵ (due to storage error and gate error) and two controlled-NOT gates to the ancilla qubit conditioned on qubit 1 and 2, respectively. Then an error on the ancilla qubit propagates to both qubits 1 and 2, inducing two errors in the data qubits with ϵ . Our proposed error correcting scheme, the parity gate is the fundamental component. In this gate, there are two types of errors on the probe mode that propagate back to the data qubits: (1) an intrinsic measurement error and (2) errors due to loss, decoherence or noise on the probe mode.

The first type of errors arises from the fact that the states $|\alpha\rangle$ and $|\alpha e^{\pm i\theta}\rangle$ of the probe mode are not orthogonal and a measurement result in one parity subspace could have come from the opposite parity state. This intrinsic error results in a wrong error syndrome, which may introduce a bit-flip error in the error correction procedure. However, the error rate $P_{\text{err}}(\theta) = \text{erfc}[|\alpha| \sin \theta / \sqrt{2}] / 2$ can be suppressed to be small when $|\alpha|\theta \gg 1$.

The second type of errors includes the loss due to decoherence causing dephasing, corresponding to phase flip errors, in the original two-qubit state. The degree of dephasing is characterized by $\gamma = (1 - T)|\alpha|^2\theta^2$ for small θ , where T is the transmission coefficient of the probe

mode through the parity gate. In a case where the data qubits are close to each other and $\gamma \ll 1$, the dephasing in the data qubits is negligible. This generally requires that $|\alpha|\theta$ is not too large. Other sources of errors are due to an error in the rotation angle θ and its fluctuation $\Delta\theta$ during the error correcting procedures, causing the phase errors. We can keep those errors small when $|\alpha|\theta \gg 1$ and $\Delta\theta/\theta \ll 1$.

We have presented the error correcting procedures and circuits based on weak nonlinearity between qubits and robust continuous variable probe modes. Our error correcting procedures have several distinct differences over the conventional method of quantum error correction [7–10]. First, as fragile ancilla qubits are replaced by robust continuous variable probe modes, costly preparation of ancilla qubits is substituted by the easy preparation of the probe modes. The use of such probe modes also gives us freedom in constructing quantum circuits since direct interaction between qubits is not necessary.

Secondly, we have also shown that in general the error correcting circuits can be used for encoding an unknown state as well with a slight modification. This property is general hence applicable to the standard error-correction encoding procedure. Such a way of encoding might have an advantage even with ancilla qubits where ancilla qubits are used with probe modes or the use of controlled-NOT gates on qubits are restricted. The easy preparation and initialization of probe mode together with the efficient parity gates make the error detection and correction procedure significantly simpler than the standard error correction scheme. Our new approach can be applied to the existing error correcting codes including the 3-qubit bit-flip (phase-flip) code, Shor and erasure codes and so has wide applicability in many physical implementation of quantum computation ranging from the solid-state to optics.

Acknowledgments: We would like to thank I. L. Chuang, T. D. Ladd, R. Laflamme, P. van Loock, R. Van Meter and Y. Yamamoto for valuable discussions. This work was supported in part by the Quantum Entanglement Project, SORST of JST, JSPS, MIC, and Asahi-Glass research grants.

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- [1] E. Knill, R. Laflamme, and G. J. Milburn, *Nature* **409**, 46 (2001).
 - [2] D. Kielpinski, C. Monroe, and D. J. Wineland, *Nature* **417**, 709 (2002).
 - [3] I. Chiorescu *et al.*, *Nature* **431**, 159 (2004).
 - [4] Y. Nakamura, Y. A. Pashkin, and J. S. Tsai, *Nature* **398**, 786 (1999).
 - [5] D. Vion *et al.*, *Science* **296**, 886 (2002).
 - [6] B. Kane, *Nature* **393**, 133 (1998).
 - [7] P. W. Shor, *Phys. Rev. A* **52**, 2493 (1995).
 - [8] A. Steane, *Proc. Roy. Soc. London Ser. A* **452**, 2551 (1996).
 - [9] A. R. Calderbank and P. Shor, *Phys. Rev. A* **54**, 1098 (1996).
 - [10] P. W. Shor, in *Proceedings of 37th Annual Symposium on Foundations of Computer Science* (IEEE Comput. Soc. Press, 1996), p. 56.
 - [11] T. P. Spiller *et al.*, quant-ph/0509202 (2005).
 - [12] J. P. Reithmaier *et al.*, *Nature* **432**, 197 (2004).
 - [13] T. Yoshie *et al.*, *Nature* **432**, 200 (2004).
 - [14] Y. Akahane, T. Asano, B.-S. Song, and S. Noda, *Nature* **425** (2003).
 - [15] K. Nemoto and W. J. Munro, *Phys. Rev. Lett.* **93**, 250502 (2004).
 - [16] W. J. Munro, K. Nemoto, and T. P. Spiller, *New Journal of Physics* **7**, 137 (2005).

- [17] W. J. Munro, K. Nemoto, R. G. Beausoleil, and T. P. Spiller, Phys. Rev. A **71**, 033819 (2005).
- [18] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
- [19] T. C. Ralph, A. J. F. Hayes, and A. Gilchrist, Phys. Rev. Lett. **95**, 100501 (2005).
- [20] A. Gilchrist, A. J. F. Hayes, and T. C. Ralph, quant-ph/0505125 (2005).
- [21] K. Nemoto and W. J. Munro, Phys. Lett A **344**, 104 (2005).