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Connections between relative entropy of entanglement and geometric measure of entanglement

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As two of the most important entanglement measures—the entanglement of formation and the entanglement of distillation—have so far been limited to bipartite settings, the study of other entanglement measures for multipartite systems appears necessary. Here, connections between two other entanglement measures—the relative entropy of entanglement and the geometric measure of entanglement—are investigated. It is found that for arbitrary pure states the latter gives rise to a lower bound on the former. For certain pure states, some bipartite and some multipartite, this lower bound is saturated, and thus their relative entropy of entanglement can be found analytically in terms of their known geometric measure of entanglement. For certain mixed states, upper bounds on the relative entropy of entanglement are also established. Numerical evidence strongly suggests that these upper bounds are tight, i.e., they are actually the relative entropy of entanglement.

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I. INTRODUCTION

Entanglement has been identified as a resource central to much of quantum information processing [1]. To date, progress in the quantification of entanglement for mixed states has resided primarily in the domain of bipartite systems [2]. For multipartite systems in pure and mixed states the characterization and quantification of entanglement presents even greater challenges. Even for multipartite pure states it is not clear whether there exists a finite minimal reversible entanglement generating set (MREGS) [3] and, if it exists, what the set is. This complicates the task of extending measures such as entanglement of distillation [4] and formation [5, 6] to multipartite systems [7]. Moreover, the characterization of multipartite entanglement remains incomplete.

On the other hand, quantifying multipartite entanglement via other measures, such as relative entropy of entanglement [7, 8], is still a challenging task, even for *pure* states. One reason for the difficulty is the absence, in general, of Schmidt decompositions for multipartite pure states [9]. This implies that for multipartite pure states the entropies of the reduced density matrices can differ, in contrast to bipartite pure states, as the following example shows. Consider a three-qubit pure state $|\psi\rangle_{ABC} \equiv \alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle$, where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The reduced density matrices for parties A, B, and C are, respectively,

$$\rho_A = (|\alpha|^2 + |\beta|^2)|0\rangle\langle 0| + |\gamma|^2|1\rangle\langle 1|, \quad (1a)$$

$$\rho_B = (|\alpha|^2 + |\gamma|^2)|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|, \quad (1b)$$

$$\rho_C = (|\gamma|^2 + |\beta|^2)|0\rangle\langle 0| + |\alpha|^2|1\rangle\langle 1|, \quad (1c)$$

which, in general, have different entropies. Thus, for a multipartite pure state the entropy of the reduced density matrix does not give a consistent entanglement measure. However, even in the case in which all parties have the identical entropy, e.g., $\alpha = \beta = \gamma = 1/\sqrt{3}$ [10], it is in general nontrivial to obtain the relative entropy of entanglement for the state. More generally, for pure multipartite states, it is not yet known how to obtain their relative entropy of entanglement analytically. The situation is even worse for *mixed* multipartite states.

Recently, a multipartite entanglement measure based on the geometry of Hilbert space has been proposed [11–13]. For pure states, this geometric measure of entanglement depends on the maximal overlap between the entangled state and unentangled states, and is easy to compute numerically. The measure has been applied to several bipartite and multipartite pure and mixed states [12, 13], including two distinct multipartite bound entangled states [14]. In the present paper, we explore connections between this measure and the relative entropy of entanglement. For certain pure states, some bipartite and some multipartite, this lower bound is saturated, and thus their relative entropy of entanglement can be found analytically, in terms of their known geometric measure of entanglement. For certain mixed states, upper bounds on the relative entropy of entanglement are also established. Numerical evidence strongly suggests that these upper bounds are tight, i.e., they are actually the relative entropy of entanglement. These results, although not general enough to solve the problem of calculating the relative entropy of entanglement for arbitrary

multipartite states, may offer some insight into, and serve as a testbed for, future analytic progress related to the relative entropy of entanglement.

The structure of the present paper is as follows. In Sec. II we review the two entanglement measures considered in the paper: the relative entropy of entanglement and the geometric measure of entanglement. In Sec. III we explore connections between the two, in both pure- and mixed-state settings. Examples are provided in which bounds and exact values of the relative entropy of entanglement are obtained. In Sec. IV we give some concluding remarks.

II. ENTANGLEMENT MEASURES

In this section we briefly review the two measures considered in the present paper: the relative entropy of entanglement and the geometric measure of entanglement.

A. Relative entropy of entanglement

The relative entropy $S(\rho||\sigma)$ between two states ρ and σ is defined via

$$S(\rho||\sigma) \equiv \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma), \quad (2)$$

which is evidently not symmetric under exchange of ρ and σ , and is non-negative, i.e., $S(\rho||\sigma) \geq 0$. The relative entropy of entanglement (RE) for a mixed state ρ is defined to be the minimal relative entropy of ρ over the set of separable mixed states [8, 22]:

$$E_R(\rho) \equiv \min_{\sigma \in \mathcal{D}} S(\rho||\sigma) = \min_{\sigma \in \mathcal{D}} \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma), \quad (3)$$

where \mathcal{D} denotes the set of all separable states.

In general, the task of finding the RE for arbitrary states ρ involves a minimization over all separable states, and this renders the computation of the RE very difficult. For bipartite pure states, the RE is equal to entanglements of formation and of distillation. But, despite recent progress [15], for mixed states—even in the simplest setting of two qubits—no analog of Wootters’ formula [6] for the entanglement of formation has been found. Things are even worse in multipartite settings. Even for pure states, there has not been a systematic method for computing relative entropies of entanglement. It is thus worthwhile seeking cases in which one can explicitly obtain an expression for the RE. A trivial case arises when there exists a Schmidt decomposition for a multipartite pure state: in this case, the RE is the usual expression

$$-\sum_i \alpha_i^2 \log_2 \alpha_i^2, \quad (4)$$

where the α_i ’s are Schmidt coefficients (with $\sum_i \alpha_i^2 = 1$). We shall see that there exist cases in which the RE can be determined analytically, even though there is no Schmidt decomposition.

We remark that an alternative definition of RE is to replace the set of separable states by the set of positive partial transpose (PPT) states. The RE thus defined, as well as its regularized version, gives a tighter bound on distillable entanglement. There has been important progress in calculating the RE (and its regularized version) with respect to PPT states for certain bipartite mixed states; see Refs. [16] for more detailed discussions. For multipartite settings one could also use this definition, and define the set of states to optimize over to be the set of states that are PPT with respect to all bipartite partitionings. However, we shall use the first definition, i.e., optimization over the set of completely separable states, throughout the discussion of the present paper.

B. Geometric measure of entanglement

We continue by briefly reviewing the formulation of this measure in both pure-state and mixed-state settings. Let us start with a multipartite system comprising n parts, each of which can have a distinct Hilbert space. Consider a general n -partite pure state (expanded in the local bases $\{|e_{p_i}^{(i)}\}$):

$$|\psi\rangle = \sum_{p_1 \cdots p_n} \chi_{p_1 p_2 \cdots p_n} |e_{p_1}^{(1)} e_{p_2}^{(2)} \cdots e_{p_n}^{(n)}\rangle. \quad (5)$$

As shown in Ref. [13], the closest separable pure state,

$$|\phi\rangle \equiv \bigotimes_{i=1}^n |\phi^{(i)}\rangle = \bigotimes_{i=1}^n \left(\sum_{p_i} c_{p_i}^{(i)} |e_{p_i}^{(i)}\rangle \right), \quad (6)$$

satisfies the stationarity conditions

$$\sum_{p_1 \cdots \widehat{p_i} \cdots p_n} \chi_{p_1 p_2 \cdots p_n}^* c_{p_1}^{(1)} \cdots \widehat{c_{p_i}^{(i)}} \cdots c_{p_n}^{(n)} = \Lambda c_{p_i}^{(i)*}, \quad (7a)$$

$$\sum_{p_1 \cdots \widehat{p_i} \cdots p_n} \chi_{p_1 p_2 \cdots p_n} c_{p_1}^{(1)*} \cdots \widehat{c_{p_i}^{(i)*}} \cdots c_{p_n}^{(n)*} = \Lambda c_{p_i}^{(i)}, \quad (7b)$$

in which the eigenvalues Λ are associated with the Lagrange multiplier enforcing the constraint $\langle \phi | \phi \rangle = 1$, and lie in $[-1, 1]$, and the symbol $\widehat{}$ denotes exclusion. Moreover, the spectrum of Λ 's can be interpreted as the cosine of the angle between $|\psi\rangle$ and $|\phi\rangle$; the largest, Λ_{\max} (i.e. $\cos \theta_{\min}$ with the smallest angle θ_{\min}), which we call the *entanglement eigenvalue*, corresponds to the closest separable state, and is the maximal overlap with unentangled states:

$$\Lambda_{\max}(|\psi\rangle) = \max_{\phi} |\langle \phi | \psi \rangle|, \quad (8)$$

where $|\phi\rangle$ is an arbitrary separable pure state. In Ref. [13], the particular form $E_{\sin^2} \equiv 1 - \Lambda_{\max}^2(|\psi\rangle) = \sin^2 \theta_{\min}$ was defined to be the geometric measure of entanglement (GME) for any pure state $|\psi\rangle$. Here, we shall be concerned with the related quantity $E_{\log_2}(\psi) \equiv -2 \log_2 \Lambda_{\max}(|\psi\rangle)$, which we shall show to be a lower bound on the relative entropy of entanglement for $|\psi\rangle$. Although this quantity is not, as we shall see later, an entanglement monotone for mixed states, it is a good measure of *pure-state* entanglement.

Given the definition of entanglement for pure states just formulated, the extension to mixed states ρ can be built upon pure states via the *convex hull* construction (indicated by “co”), as was done for the entanglement of formation; see Ref. [6]. The essence is a minimization over all decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ into pure states:

$$E(\rho) \equiv (\text{co}E_{\text{pure}})(\rho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i E_{\text{pure}}(|\psi_i\rangle). \quad (9)$$

This convex hull construction ensures that the measure gives zero for separable states; however, in general it also complicates the task of determining mixed-state entanglement.

Illustrative examples: We consider several examples involving symmetric states, mostly restricting our attention to n -qubit systems. First, one can classify permutation-invariant pure states, as follows:

$$|S(n, k)\rangle \equiv \sqrt{\frac{k!(n-k)!}{n!}} \sum_{\text{Permutations}} \text{P} | \underbrace{0 \cdots 0}_k \underbrace{1 \cdots 1}_{n-k} \rangle. \quad (10)$$

As the amplitudes are all positive, one can assume that the closest separable (equivalently, Hartree) state is of the form

$$|\phi\rangle = (\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle)^{\otimes n}, \quad (11)$$

for which the maximal overlap (w.r.t. p) gives the entanglement eigenvalue for $|S(n, k)\rangle$:

$$\Lambda_{\max}(n, k) = \sqrt{\frac{n!}{k!(n-k)!}} \left(\frac{k}{n}\right)^{\frac{k}{2}} \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}}. \quad (12)$$

More generally, for n parties each a $(d+1)$ -level system, the state

$$|S(n; \{k\})\rangle \equiv \sqrt{\frac{k_0! k_1! \cdots k_d!}{n!}} \sum_{\text{Permutations}} \text{P} | \underbrace{0 \cdots 0}_{k_0} \underbrace{1 \cdots 1}_{k_1} \cdots \underbrace{d \cdots d}_{k_d} \rangle \quad (13)$$

has the entanglement eigenvalue

$$\Lambda_{\max}(n; \{k\}) = \sqrt{\frac{n!}{\prod_i (k_i!)}} \prod_{i=0}^d \left(\frac{k_i}{n}\right)^{\frac{k_i}{2}}. \quad (14)$$

Now consider the totally antisymmetric state $|\text{Det}_n\rangle$, defined via

$$|\text{Det}_n\rangle \equiv \frac{1}{\sqrt{n!}} \sum_{i_1, \dots, i_n=1}^n \epsilon_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle. \quad (15)$$

It has been shown [17] that $\Lambda_{\max}^2 = 1/n!$. The generalization of the antisymmetric state to the $n = p d^p$ -partite determinant state is via [17]

$$\begin{aligned} \phi(1) &= (0, 0, \dots, 0, 0), \\ \phi(2) &= (0, 0, \dots, 0, 1), \\ &\vdots \\ \phi(d^p - 1) &= (d - 1, d - 1, \dots, d - 1, d - 2), \\ \phi(d^p) &= (d - 1, d - 1, \dots, d - 1, d - 1), \end{aligned}$$

and

$$|\text{Det}_{n,d}\rangle \equiv \frac{1}{\sqrt{(d^p)!}} \sum_{i_1, \dots, i_{d^p}} \epsilon_{i_1, \dots, i_{d^p}} |\phi(i_1), \dots, \phi(i_{d^p})\rangle. \quad (16)$$

In this case, it can be shown that $\Lambda_{\max}^2 = 1/(d^p)!$.

Although the above states were discussed in terms of the GME [13], we shall, in the following section, show the rather surprising fact that the RE of these example states, is given by the corresponding expression: $-2 \log_2 \Lambda_{\max}$.

III. CONNECTION BETWEEN THE TWO MEASURES

In bipartite systems, due to the existence of Schmidt decompositions, the relative entropy of entanglement of a pure state is simply the von Neumann entropy of its reduced density matrix. However, for multipartite systems there is, in general, no such decomposition, and how to calculate the relative entropy of entanglement for an arbitrary pure state remains an open question. We now connect the relative entropy of entanglement to the geometric measure of entanglement for arbitrary pure states by giving a lower bound on the former in terms of the latter or, more specifically, via the entanglement eigenvalue.

A. Pure states: lower bound on relative entropy of entanglement

Let us begin with the following theorem:

Theorem 1. For any pure state $|\psi\rangle$ with entanglement eigenvalue $\Lambda_{\max}(\psi)$ the quantity $-2 \log_2 \Lambda_{\max}(\psi)$ is a lower bound on the relative entropy of entanglement of $|\psi\rangle$, i.e.,

$$E_{\text{R}}(|\psi\rangle\langle\psi|) \geq -2 \log_2 \Lambda_{\max}(\psi). \quad (17)$$

Proof: From the definition (3) of the relative entropy of entanglement we have, for a pure state $|\psi\rangle$,

$$E_{\text{R}}(|\psi\rangle\langle\psi|) = \min_{\sigma \in \mathcal{D}} -\langle\psi| \log_2 \sigma |\psi\rangle = -\max_{\sigma \in \mathcal{D}} \langle\psi| \log_2 \sigma |\psi\rangle. \quad (18)$$

Using the concavity of the log function, we have

$$\langle\psi| \log_2 \sigma |\psi\rangle \leq \log_2(\langle\psi| \sigma |\psi\rangle) \quad (19)$$

and, furthermore,

$$\max_{\sigma \in \mathcal{D}} \langle\psi| \log_2 \sigma |\psi\rangle \leq \max_{\sigma \in \mathcal{D}} \log_2(\langle\psi| \sigma |\psi\rangle), \quad (20)$$

although the σ 's maximizing the left- and right-hand sides are not necessarily identical. We then conclude that

$$E_{\text{R}}(|\psi\rangle\langle\psi|) \geq -\max_{\sigma \in \mathcal{D}} \log_2(\langle\psi| \sigma |\psi\rangle). \quad (21)$$

As any $\sigma \in \mathcal{D}$ can be expanded as $\sigma = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle$'s are separable pure states, one has

$$\langle\psi|\sigma|\psi\rangle = \sum_i p_i |\langle\phi_i|\psi\rangle|^2 \leq \Lambda_{\max}^2(\psi), \quad (22)$$

and hence we arrive at the sought result

$$E_{\text{R}}(|\psi\rangle\langle\psi|) \geq -2 \log_2 \Lambda_{\max}(\psi). \quad (23)$$

We wish to point out that such an inequality was previously established and exploited in Refs. [18].

When does the inequality becomes an equality? The demand that Eq. (19) hold as an equality implies that σ (un-normalized) can be decomposed into either (a)

$$\sigma = \sum_i |i\rangle\langle i|, \quad (24a)$$

where $\{|i\rangle\}$ are mutually orthogonal but *not* orthogonal to $|\psi\rangle$, or (b)

$$\sigma = |\psi\rangle\langle\psi| + \tau^\perp, \quad (24b)$$

where τ^\perp (either pure or mixed) is orthogonal to ψ , i.e., $\langle\psi|\tau^\perp|\psi\rangle = 0$. However, the separable σ that has either property is not necessarily the one that maximizes both sides of the inequality (20), unless $|\psi\rangle$ (and hence σ) has high symmetry. On the other hand, a corollary arises from Theorem 1 which says that for any multipartite pure state $|\psi\rangle$, if one can find a separable mixed state σ such that $S(\rho|\sigma)|_{\rho=|\psi\rangle\langle\psi|} = -2 \log_2 \Lambda_{\max}(|\psi\rangle)$ then $E_{\text{R}} = -2 \log_2 \Lambda_{\max}(|\psi\rangle)$. This result follows directly from the fact that when the lower bound on E_{R} given in Eq. (17) equals an upper bound, the relative entropy of entanglement is immediate. In all the examples we shall consider for which this lower bound is saturated, it turns out that

$$\sigma^* \equiv \sum_i p_i |\phi_i\rangle\langle\phi_i| \quad (25)$$

is a closest separable mixed state, in which $\{|\phi_i\rangle\}$ are separable pure states closest to $|\psi\rangle$. (The distribution p_i is uniform, and can be either discrete or continuous, and $\{|\phi_i\rangle\}$ are not necessarily mutually orthogonal.)

We now examine several illustrative states in the light of the above corollary, thus obtaining E_{R} for each of them. We begin with the permutation-invariant states $|S(n, k)\rangle$ of Eq. (10), for which Λ_{\max} was given in Eq. (12). The above theorem guarantees that $E_{\text{R}}(|S(n, k)\rangle) \geq -2 \log_2 \Lambda_{\max}(n, k)$. To find an *upper* bound we construct a separable mixed state

$$\sigma^* \equiv \int \frac{d\phi}{2\pi} |\xi(\phi)\rangle\langle\xi(\phi)|, \quad (26a)$$

$$|\xi(\phi)\rangle \equiv \left(\sqrt{p}|0\rangle + e^{i\phi} \sqrt{1-p}|1\rangle \right)^{\otimes n}, \quad (26b)$$

with p chosen to maximize $|\langle\xi|S(n, k)\rangle| = \sqrt{C_k^n p^k (1-p)^{n-k}}$, which gives $p = k/n$. Direct evaluation then gives

$$\sigma^* = \sum_{k=0}^n C_k^n p^k (1-p)^{n-k} |S(n, k)\rangle\langle S(n, k)|, \quad (27)$$

and $S(\rho|\sigma) = -2 \log_2 \Lambda_{\max}(n, k)$, where $\rho = |S(n, k)\rangle\langle S(n, k)|$ and $\Lambda_{\max}(n, k)$ is given in Eq. (12). The upper and lower bounds on E_{R} coincide, and hence we have that

$$E_{\text{R}}(|S(n, k)\rangle) = -2 \log_2 \Lambda_{\max}(n, k). \quad (28)$$

The closest separable mixed state σ^* belongs to the case (b), i.e., Eq. (24b). Similar equalities can be established for the generalized permutation-invariant n -party $(d+1)$ -dit states $|S(n, \{k\})\rangle$ of Eq. (13). We remark that the entanglements of the symmetric states $|S(n, k)\rangle$ (which are also known as *Dicke* states) have been analyzed via other approaches; see Ref. [19].

For our next example we consider the totally anti-symmetric states $|\text{Det}_n\rangle$ of Eq. (15). It was shown in Ref. [17] that for these states $\Lambda_{\max}^2 = 1/n!$, and hence it is straightforward to see that each of the $n!$ basis states $|i_1, \dots, i_n\rangle$

is a closest separable pure state. Thus, one can construct a separable mixed state from these separable pure states [cf. Eq. (25)]:

$$\sigma_1 \equiv \frac{1}{n!} \sum_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle \langle i_1, \dots, i_n|. \quad (29)$$

Then, by direct calculation one gets $S(\rho_{\text{Det}_n} || \sigma_1) = \log_2(n!)$, which is identical to $-2 \log_2 \Lambda_{\text{max}}$, as mentioned above. As in our previous examples, upper and lower bounds on E_{R} coincide, and hence we have that $E_{\text{R}}(|\text{Det}_n\rangle) = \log_2(n!)$. The closest separable mixed state σ_1 belongs to the case (a), i.e., Eq. (24a). Similarly, for the generalized determinant state (16) one can show that $E_{\text{R}} = \log_2(d^p!)$.

We now focus our attention on three-qubit settings. Of these, the states $|S(3, 0)\rangle = |000\rangle$ and $|S(3, 3)\rangle = |111\rangle$ are not entangled and are, respectively, the components of the the 3-GHZ state: $|\text{GHZ}\rangle \equiv (|000\rangle + |111\rangle)/\sqrt{2}$. Although the GHZ state is not of the form $|S(n, k)\rangle$, it has $\Lambda_{\text{max}} = 1/\sqrt{2}$, and two of its closest separable pure states are $|000\rangle$ and $|111\rangle$ [13]. From these one can construct a separable mixed state

$$\sigma_2 = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|), \quad (30)$$

From the discussion given after Eq. (23), one concludes that $E_{\text{R}}(\text{GHZ}) = -2 \log_2 \Lambda_{\text{max}} = 1$ and that σ_2 is one of the closest separable mixed states to $|\text{GHZ}\rangle$. This closest separable mixed state σ_2 belongs to the case (a), i.e., Eq. (24a). With some rewriting, it can also be classified as case (b), i.e.,

$$\sigma_2 = \frac{1}{2}|\text{GHZ}\rangle\langle \text{GHZ}| + \frac{1}{2}|\text{GHZ}^-\rangle\langle \text{GHZ}^-|, \quad (31)$$

where $|\text{GHZ}^-\rangle \equiv (|000\rangle - |111\rangle)/\sqrt{2}$.

The states

$$|\text{W}\rangle \equiv |S(3, 2)\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}, \quad (32a)$$

$$|\tilde{\text{W}}\rangle \equiv |S(3, 1)\rangle = (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}, \quad (32b)$$

are equally entangled, and have $\Lambda_{\text{max}} = 2/3$ [13]. Again, from the discussion after Eq. (23) we have $E_{\text{R}} = \log_2(9/4)$, and one of the closest separable mixed states to the W state can be constructed from

$$\sigma_3 \equiv \int \frac{d\phi}{2\pi} |\psi(\phi)\rangle \langle \psi(\phi)|, \quad \text{with} \quad (33)$$

$$|\psi(\phi)\rangle \equiv (\sqrt{2/3}|0\rangle + e^{i\phi}\sqrt{1/3}|1\rangle)^{\otimes 3}, \quad (34)$$

which gives the result

$$\sigma_3 = \frac{4}{9}|\text{W}\rangle\langle \text{W}| + \frac{2}{9}|\tilde{\text{W}}\rangle\langle \tilde{\text{W}}| + \frac{8}{27}|000\rangle\langle 000| + \frac{1}{27}|111\rangle\langle 111|. \quad (35)$$

We remark that the mixed state σ_3 is not the only closest separable mixed state to the W state; the following state σ_4 is another example (as would be any mixture of σ_3 and σ_4):

$$\sigma_4 \equiv \frac{1}{3} \sum_{k=0}^2 |\psi(2\pi k/3)\rangle \langle \psi(2\pi k/3)| = \frac{4}{9}|\text{W}\rangle\langle \text{W}| + \frac{2}{9}|\tilde{\text{W}}\rangle\langle \tilde{\text{W}}| + \frac{1}{3}|\xi\rangle\langle \xi|, \quad (36a)$$

where $3|\xi\rangle \equiv 2\sqrt{2}|000\rangle + |111\rangle$. These closest separable mixed states of W state belong to the case (b), i.e., Eq. (24b).

Having obtained RE for W and $\tilde{\text{W}}$, it is interesting to examine the RE of the following superposition of the two: $|\text{W}\tilde{\text{W}}(s)\rangle \equiv \sqrt{s}|\text{W}\rangle + \sqrt{1-s}|\tilde{\text{W}}\rangle$. We have not been able to find an analytical result for RE, but we can compare the analytical expression for $-2 \log_2 \Lambda_{\text{max}}(\text{W}\tilde{\text{W}}(s))$ with the numerical evaluation of $E_{\text{R}}(\text{W}\tilde{\text{W}}(s))$, and we do this in Fig. 1. As we see in this figure, the qualitative behavior of the two functions is similar, but $-2 \log_2 \Lambda_{\text{max}}$ and E_{R} only coincide at the two end-points, $s = 0$ and $s = 1$.

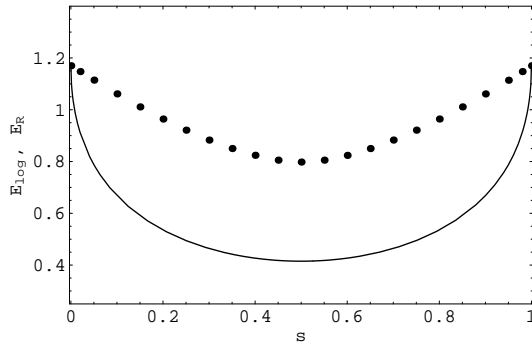


FIG. 1: The solid curve represents $E_{\log_2}(s)$ of the pure state $\sqrt{s}|W\rangle + \sqrt{1-s}|\tilde{W}\rangle$ vs. s . The dots are corresponding relative entropies of entanglement obtained numerically.

B. Mixed states: upper bound on relative entropy of entanglement

In Ref. [13] the procedure was given to find the geometric measure of entanglement, E_{\sin^2} , for the mixed state comprising symmetric states:

$$\rho(\{p\}) = \sum_k p_k |S(n, k)\rangle\langle S(n, k)|. \quad (37)$$

Here, we focus instead on the quantity E_{\log_2} , but the basic procedure is the same. The first step is to find the entanglement eigenvalue $\Lambda_n(\{q\})$ for the pure state

$$\sum_k \sqrt{q_k} |S(n, k)\rangle, \quad (38)$$

thus arriving at the quantity

$$\mathcal{E}(\{q\}) \equiv -2 \log_2 \Lambda_n(\{q\}). \quad (39)$$

Then the quantity E_{\log_2} for the mixed state (37) is actually the convex hull of the expression (39):

$$E_{\log_2}(\rho(\{p\})) = \text{co } \mathcal{E}(\{p\}). \quad (40)$$

This prompts us to ask the question: Can we find RE for the mixture of $|S(n, k)\rangle$ in Eq. (37)? To answer it, we shall first construct an upper bound to RE, and then compare this bound with the numerically evaluated RE. To accomplish the first step, bearing in mind the fact that any separable mixed state will yield an upper bound, we consider the state formed by mixing the separable pure states $|\xi(\theta, \phi)\rangle$ [cf. Eq. (27)]:

$$\sigma(\theta) = \int \frac{d\phi}{2\pi} |\xi(\theta, \phi)\rangle\langle\xi(\theta, \phi)| = \sum_{k=0}^n C_k^n \cos^{2k} \theta \sin^{2(n-k)} \theta |S(n, k)\rangle\langle S(n, k)|, \quad (41)$$

where

$$|\xi(\theta, \phi)\rangle \equiv (\cos \theta |0\rangle + e^{i\phi} \sin \theta |1\rangle)^{\otimes n}. \quad (42)$$

We then minimize the relative entropy between $\rho(\{p\})$ and $\sigma(\theta)$,

$$S(\rho(\{p\})||\sigma(\theta)) = \sum_k p_k \log \frac{p_k}{C_k^n \cos^{2k} \theta \sin^{2(n-k)} \theta}, \quad (43)$$

with respect to θ , obtaining the stationarity condition

$$\tan^2 \theta \equiv \frac{\sum_k p_k (n-k)}{\sum_k p_k k}. \quad (44)$$

Due to the convexity of the relative entropy,

$$S\left(\sum_i q_i \rho_i \parallel \sum_i q_i \sigma_i\right) \leq \sum_i q_i S(\rho_i \parallel \sigma_i), \quad (45)$$

we can further tighten the expression of the relative entropy by taking its convex hull. (Via the convexification process, i.e., the convex hull construction, the corresponding separable state can also be obtained.) Therefore, we arrive at an upper bound for the relative entropy of entanglement of the mixed state $\rho(\{p\})$:

$$E_R(\rho(\{p\})) \leq \text{co}F(\{p\}), \quad (46)$$

where

$$F(\{p\}) \equiv \sum_k p_k \log_2 \frac{p_k}{C_k^n \cos^{2k} \theta \sin^{2(n-k)} \theta} = \sum_k p_k \log_2 \frac{p_k n^n}{C_k^n \alpha^k (n-\alpha)^{n-k}}, \quad (47)$$

where the angle θ satisfies Eq. (44), $C_k^n \equiv n!/(k!(n-k)!)$, and $\alpha \equiv \sum_k p_k k$.

Having established an upper bound for RE for the state $\rho(\{p\})$, we now make the restriction to mixtures of two distinct n -qubit states $|S(n, k_1)\rangle$ and $|S(n, k_2)\rangle$ (with $k_1 \neq k_2$):

$$\rho_{n;k_1,k_2}(s) \equiv s|S(n, k_1)\rangle\langle S(n, k_1)| + (1-s)|S(n, k_2)\rangle\langle S(n, k_2)|. \quad (48)$$

One trivial example is $\rho_{n;0,n}(s)$, which is obviously unentangled as it is the mixture of two separable pure states $|0^{\otimes n}\rangle$ and $|1^{\otimes n}\rangle$. Other mixtures are generally entangled, except possibly at the end-points $s=0$ or $s=1$ when the mixture contains either $|S(n, 0)\rangle$ or $|S(n, n)\rangle$. We first investigate the two-qubit (i.e. $n=2$) case. Besides the trivial mixture, $\rho_{2;0,2}$, there is only one inequivalent mixture, $\rho_{2;0,1}(s)$ [which is equivalent to $\rho_{2;2,1}(s)$], which is—up to local basis change—the so-called *maximally entangled mixed state* [20, 21] (for a certain range of s)

$$\rho_{2;0,1} = s|11\rangle\langle 11| + (1-s)|\Psi^+\rangle\langle \Psi^+|, \quad (49)$$

where $|\Psi^+\rangle \equiv (|01\rangle + |10\rangle)/\sqrt{2}$. The function F for this state [denoted by $F_{2;0,1}(s)$] is

$$F_{2;0,1}(s) = s \log_2 \frac{4s}{(1+s)^2} + (1-s) \log_2 \frac{2}{1+s}, \quad (50)$$

which is convex in s . It is exactly the expression for the relative entropy of entanglement for the state $\rho_{2;0,1}$ found by Vedral and Plenio [22] (see their Eq. (56) with λ replaced by $1-s$).

For $n=3$ there are three other inequivalent mixtures: $\rho_{3;0,1}(s)$ [equivalent to $\rho_{3;3,2}(s)$], $\rho_{3;0,2}(s)$ [to $\rho_{3;3,1}(s)$], and $\rho_{3;1,2}(s)$ [to $\rho_{3;2,1}(s)$]. In Fig. 2 we compare the function F in Eq. (47), its convex hull $\text{co}F$, and numerical values of E_R obtained using the general scheme described in Ref. [22] extended beyond the two-qubit case. The agreement between $\text{co}F$ and the numerical values of E_R appears to be exact.

For $n=4$ there are five inequivalent nontrivial mixtures: $\rho_{4;0,1}(s)$, $\rho_{4;0,2}(s)$, $\rho_{4;0,3}(s)$, $\rho_{4;1,2}(s)$, and $\rho_{4;1,3}(s)$. In Figs. 3 and 4 we again compare the function F in Eq. (47), its convex hull $\text{co}F$, and numerical values of E_R . Again the agreement between $\text{co}F$ and the numerical values of E_R appears to be exact.

From these agreements, we are led to the following conjecture:

Conjecture 1: The relative entropy of entanglement $E_R(\rho(\{p\}))$ for the mixed states $\rho(\{p\})$ is given exactly by $\text{co}F(\{p\})$.

For the states that we have just considered, we now pause to give the formulas for E_R suggested by the conjecture. For the three-qubit mixed state $\rho_{3;2,1}(s)$, its conjectured E_R is

$$s \log_2 \frac{9s}{(1+s)^2(2-s)} + (1-s) \log_2 \frac{9(1-s)}{(2-s)^2(1+s)}. \quad (51a)$$

For $\rho_{3;0,1}(s)$, it is

$$s \log_2 \frac{27s}{(2+s)^3} + (1-s) \log_2 \frac{9}{(2+s)^2}. \quad (51b)$$

For $\rho_{4;0,1}(s)$, it is

$$s \log_2 \frac{256s}{(3+s)^4} + (1-s) \log_2 \frac{64}{(3+s)^3}. \quad (52a)$$

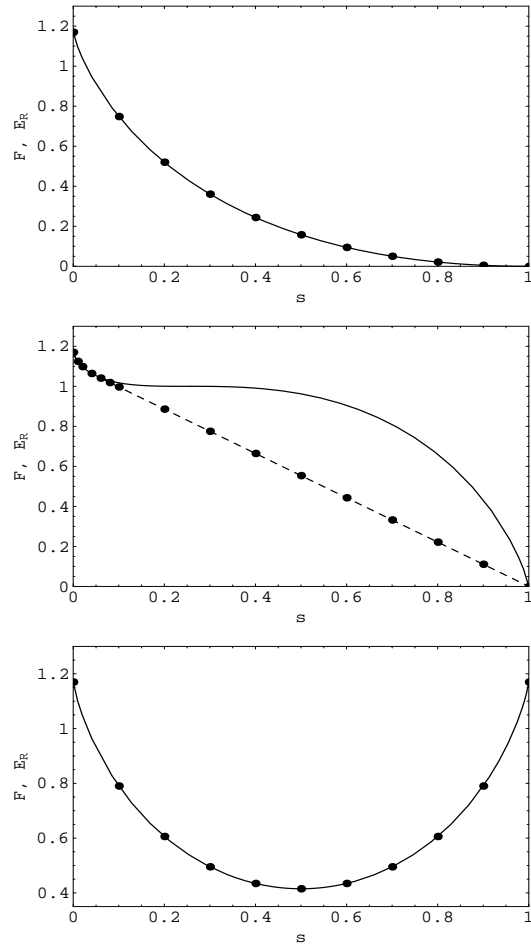


FIG. 2: Comparison of F (solid curve), $\text{co} F$ (convexification indicated by dashed line) and the numerical value of E_R (dots) for the states $\rho_{3;0,1}(s)$, $\rho_{3;0,2}(s)$, and $\rho_{3;1,2}(s)$ (from top to bottom). Note that the log function is implicitly base-2.

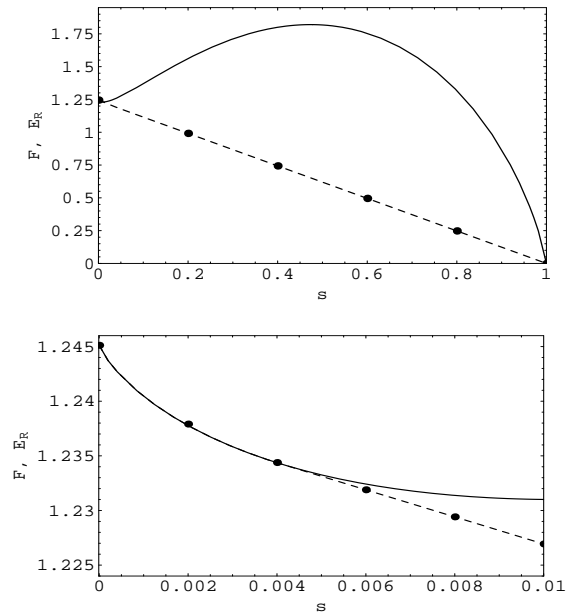


FIG. 3: Comparison of F , its convex hull, and the numerical value of E_R for the state $\rho_{4;0,3}(s)$. Upper panel shows the whole range $s \in [0, 1]$, whereas the lower panel shows a blow-up of the range $s \in [0, 0.01]$.

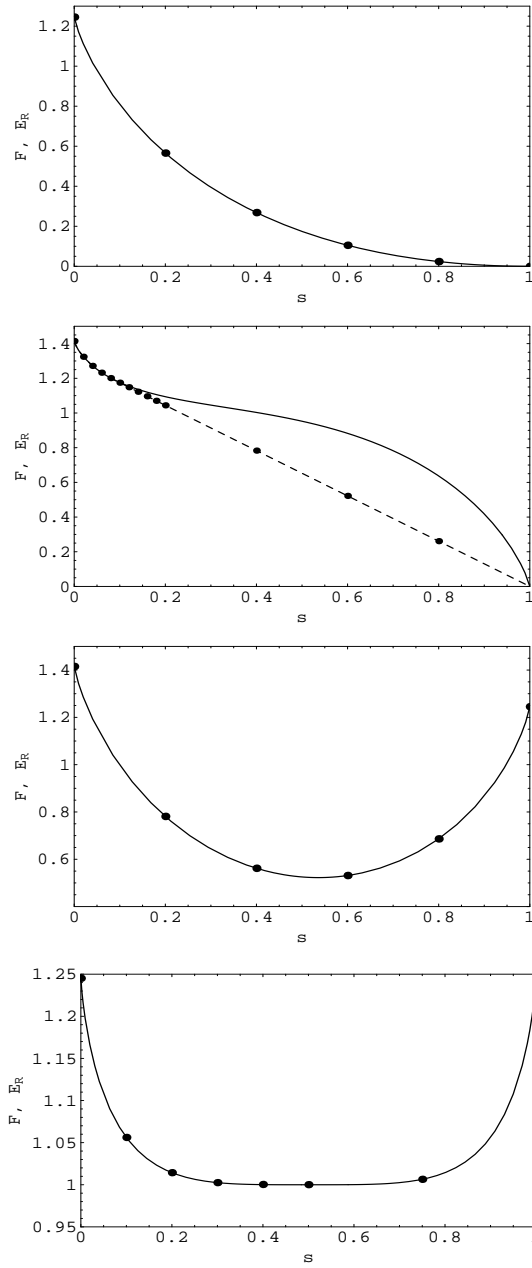


FIG. 4: Comparison of F , its convex hull, and the numerical value of E_R for the states $\rho_{4;0,1}(s)$, $\rho_{4;0,2}(s)$, $\rho_{4;1,2}(s)$, and $\rho_{4;1,3}(s)$ (from top to bottom).

For $\rho_{4;1,2}(s)$, it is

$$s \log_2 \frac{64s}{(2-s)(2+s)^3} + (1-s) \log_2 \frac{128(1-s)}{3(2-s)^2(2+s)^2}. \quad (52b)$$

For $\rho_{4;1,3}(s)$, it is

$$s \log_2 \frac{64s}{(3-2s)(1+2s)^3} + (1-s) \log_2 \frac{64(1-s)}{(3-2s)^3(1+2s)}. \quad (52c)$$

For states such as $\rho_{3;0,2}$, $\rho_{4;0,2}$, and $\rho_{4;0,3}$, convexifications (i.e. convex hull constructions) are needed; see Figs. 2, 3, and 4. In Fig. 5 we give an example of a seven-qubit state, viz., $\rho_{7;2,5}(s)$.

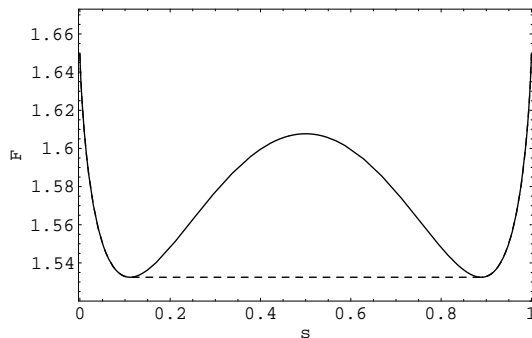


FIG. 5: The function F (solid curve) and its convex hull (dashed line indicates convexification) for the seven-qubit mixed state $\rho_{7;2,5}(s)$.

Although we have not been able to prove our conjecture, we have observed some supporting evidence, in addition to the numerical evidence presented above. We begin by noting that the states $\rho(\{p\})$ are invariant under the projection

$$P : \rho \rightarrow \int \frac{d\phi}{2\pi} U(\phi)^{\otimes n} \rho U(\phi)^{\dagger \otimes n} \quad (53)$$

with $U(\phi)\{|0\rangle, |1\rangle\} \rightarrow \{|0\rangle, e^{-i\phi}|1\rangle\}$. Vollbrecht and Werner [23] have shown that in order to find the closest separable mixed state for a state that is invariant under projections such as P , it is only necessary to search within the separable states that are also invariant under the projection. We can further reduce the set of separable states to be searched by invoking another symmetry property possessed by $\rho(\{p\})$: these states are also, by construction, invariant under permutations of all parties. Let us denote by Π_i one of the permutations of parties, and by $\Pi_i(\rho)$ the state obtained from ρ by permuting the parties under Π_i . We now show that the set of separable states to be searched can be reduced to the separable states that are invariant under the permutations. To see this, suppose that ρ is a mixed state in the family (37), and that σ^* is one of the closest separable states to ρ , i.e.,

$$E_R(\rho) \equiv \min_{\sigma \in \mathcal{D}} S(\rho||\sigma) = S(\rho||\sigma^*). \quad (54)$$

As ρ is invariant under all Π_i , we have

$$E_R(\rho) = \frac{1}{N_\Pi} \sum_i S(\rho||\Pi_i(\sigma^*)), \quad (55)$$

where N_Π is the number of permutations. By using the convexity of the relative entropy we have

$$E_R(\rho) \geq S\left(\rho||\left[\sum_i \Pi_i(\sigma^*)/N_\Pi\right]\right). \quad (56)$$

However, because of the extremal property, Eq. (54), the inequality must be saturated, as the left-hand side is already minimal. This shows that

$$\sigma^{**} \equiv \frac{1}{N_\Pi} \sum_i \Pi_i(\sigma^*) \quad (57)$$

also a closest separable mixed state to ρ , and is manifestly invariant under all permutations. Thus, we only need to search within this restricted family of separable states.

It is not difficult to see that the set \mathcal{D}_S of all separable mixed states that are diagonal in the basis of $\{|S(n, k)\rangle\}$ can be constructed from a convex mixture of separable states in Eq. (41). That is, for any $\sigma_s \in \mathcal{D}_S$ we have a decomposition

$$\sigma_s = \sum_i t_i \sigma(\theta_i), \quad (58)$$

where $t_i \geq 0$, $\sum_i t_i = 1$, and $\sigma(\theta_i)$ is of the form (41). This is because the separability of the states (37) implies that there exists a decomposition into pure states such that each pure state is a separable state. Furthermore, because

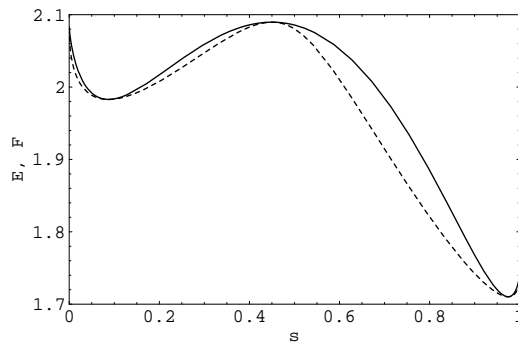


FIG. 6: Comparison of \mathcal{E} (dashed curve) and F (solid curve) for the eleven-qubit mixed state $\rho_{11;2,6}(s)$.

$\{|S(n, k)\rangle\}$ are eigenstates of $\rho(\{p\})$, the most general form of the pure state in its decomposition is

$$\sum_k \sqrt{q_k} e^{i\phi_k} |S(n, k)\rangle. \quad (59)$$

This pure state is separable if and only if it is of the form (42), up to an overall irrelevant phase. As $\rho(\{p\})$ is invariant under the projection P (53), a pure state in Eq. (42) will be projected to the mixed state in Eq. (41) under P . Thus, every separable state that is diagonal in $\{|S(n, k)\rangle\}$ basis can be expressed in the form (58).

Hence, our conjecture (46) ensures (via any necessary convexification) that it is at least the minimum (of the relative entropy) when the separable mixed states are restricted to \mathcal{D}_S . However, in order to prove the conjecture, one would still need to show that the expression is also the minimum when the restriction to \mathcal{D}_S is relaxed.

We remark that our conjecture is consistent with the results of Ishizaka [24], in that our conjectured σ^* satisfies the condition that $[\rho, \sigma^*] = 0$ and that σ^* has the same reduction as ρ for every party. Furthermore, suppose σ^* (diagonal in the basis $\{|S(n, k)\rangle\}$) represents the separable state that gives the conjectured value of RE:

$$\sigma^* = \sum_k r_k |S(n, k)\rangle \langle S(n, k)|, \quad (60)$$

where the r 's can be obtained by finding the convex hull of the function F in Eq. (47). Now consider any separable state τ in the Hilbert space *orthogonal* to the subspace spanned by $\{|S(n, k)\rangle\}$. We need to show that the separable state $\sigma(x) \equiv x\sigma^* + (1-x)\tau$, for any $x \in [0, 1]$, gives greater relative entropy with $\rho(\{p\})$ in Eq. (37) than σ^* does with $\rho(\{p\})$, i.e.,

$$S(\rho(\{p\})\|\sigma(x)) \geq S(\rho(\{p\})\|\sigma^*). \quad (61)$$

Writing out the expression explicitly, we have that

$$S(\rho(\{p\})\|\sigma(x)) = \sum_k p_k \log \frac{p_k}{x r_k} \geq \sum_k p_k \log \frac{p_k}{r_k} = S(\rho(\{p\})\|\sigma^*). \quad (62)$$

Note that τ gives no contribution in the relative entropy, as it is orthogonal to $\rho(\{p\})$, and that we have not used the fact that τ is separable. But to prove Conjecture 1 we need to show that Eq. (61) holds if separable τ is not orthogonal to the subspace spanned by $\{|S(n, k)\rangle\}$.

Recall that for pure states we found the inequality $E_{\log_2} \leq E_R$. Does this inequality hold for mixed states? We do not know the complete answer to this question, but for the mixed state $\rho(\{p\})$ we shall at least find that this inequality would hold if Conjecture 1 holds. To see this, we first establish that $\mathcal{E}(\{q\})$ is a lower bound on $F(\{q\})$; see the example in Fig. 6. The proof is as follows. Recall that

$$\mathcal{E}(\{p\}) = -2 \log_2 \left[\max_{\theta} \sum_k \sqrt{p_k} \sqrt{C_k^n} \cos^k \theta \sin^{n-k} \theta \right]. \quad (63a)$$

By the concavity of log, we then have

$$-2 \log_2 \left[\sum_k \sqrt{p_k} \sqrt{C_k^n} \cos^k \theta \sin^{n-k} \theta \right] \leq \sum_k p_k \log_2 \frac{p_k}{C_k^n \cos^{2k} \theta \sin^{2(n-k)} \theta}. \quad (63b)$$

Hence

$$\min_{\theta} -2 \log_2 \left[\sum_k \sqrt{p_k} \sqrt{C_k^n} \cos^k \theta \sin^{n-k} \theta \right] \leq \min_{\theta} \sum_k p_k \log_2 \frac{p_k}{C_k^n \cos^{2k} \theta \sin^{2(n-k)} \theta}, \quad (63c)$$

or equivalently

$$\mathcal{E}(\{p\}) \leq F(\{p\}). \quad (63d)$$

If Conjecture 1 is correct then by taking the convex hull of both sides of this inequality we would have

$$E_{\log_2} \leq E_R \quad (63e)$$

for the family of states (37). Notice that we have also shown that this relation holds for arbitrary pure states. It would be interesting to know whether it also holds for arbitrary mixed states.

IV. CONCLUDING REMARKS

We have provided a lower bound on the relative entropy of entanglement for arbitrary multipartite pure states in terms of their geometric measure of entanglement. For several families of pure states we have shown that the bound is in fact saturated, and thus provides the exact value of the relative entropy of entanglement. For mixtures of certain permutation-invariant states we have conjectured analytic expressions for the relative entropy of entanglement.

It is possible that our results on the relative entropy of entanglement might be applicable to the checking of the consistency of some equalities and inequalities [7, 25, 26] regarding minimal reversible entanglement generating sets (MREGSs). Consider, e.g., the particular family of n -qubit pure states $\{|S(n, k)\rangle\}$, the relative entropy of entanglement of which we have given in Eq. (28). Now, if we trace over one party we get a mixed $(n-1)$ -qubit state:

$$\text{Tr}_1 |S(n, k)\rangle\langle S(n, k)| = \frac{n-k}{n} |S(n-1, k)\rangle\langle S(n-1, k)| + \frac{k}{n} |S(n-1, k-1)\rangle\langle S(n-1, k-1)|. \quad (64)$$

We have also given a conjecture for the relative entropy of entanglement for this mixed state. If we trace over m parties, the reduced mixed state would be a mixture of $\{|S(n-m, q)\rangle\}$ [with $q \leq (n-m)$], and again we have given a conjecture for its relative entropy of entanglement. For example, if we start with $|S(4, 1)\rangle$, and trace over one party and then another, we get the sequence:

$$|S(4, 1)\rangle \rightarrow \rho_{3;0,1}(1/4) \rightarrow \rho_{2;0,1}(1/2), \quad (65)$$

for which we have given the corresponding relative entropies of entanglement in Eqs. (28), (51b) and (50). (To be precise, the second formula is a conjecture; the others are proven.) The afore-mentioned equalities and inequalities concerning MREGS usually involve only the von Neumann entropy and the regularized (i.e. asymptotic) relative entropy of entanglement of the pure state and its reduced density matrices. The regularized relative entropy of entanglement is defined as

$$E_R^\infty(\rho) \equiv \lim_{n \rightarrow \infty} \frac{1}{n} E_R(\rho^{\otimes n}). \quad (66)$$

The calculation of the regularized relative entropy of entanglement is, in general, much more difficult than for the non-regularized case, and the (in)equalities involving the regularized relative entropy of entanglement are thus difficult to check. Nevertheless, it is known that $E_R^\infty \leq E_R$, so we can check their weaker forms by replacing E_R^∞ by E_R , and the corresponding (in)equalities by weaker inequalities.

Plenio and Vedral [7] have derived a lower bound on the RE of a tripartite pure state $\rho_{ABC} = |\psi\rangle\langle\psi|$ in terms of the the entropies and RE's of the reduced states of two parties:

$$\max\{E_R(\rho_{AB}) + S(\rho_{AB}), E_R(\rho_{AC}) + S(\rho_{AC}), E_R(\rho_{BC}) + S(\rho_{BC})\} \leq E_R(\rho_{ABC}), \quad (67)$$

where $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$ (and similarly for ρ_{AC} and ρ_{BC}) and $S(\rho) \equiv -\text{Tr} \rho \log_2 \rho$ is the von Neumann entropy. They have further found that this lower bound is saturated by $|\text{GHZ}\rangle$ and $|\text{W}\rangle$. This raises an interesting question [27]: is the above lower bound (for n -partite pure states) saturated by the states that saturate the lower bound $E_{\log_2} = -2 \log_2 \Lambda_{\max}(\psi) \leq E_R(\psi)$? Numerical tests seem to suggest that the Plenio-Vedral bound is tighter than E_{\log_2} . If this

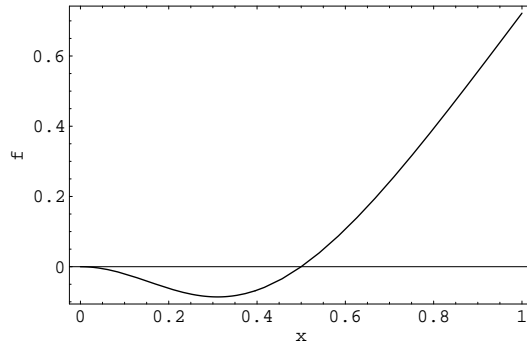


FIG. 7: The function $f(4, x)$. It shows the violation of monotone condition (74) when the function is negative.

is the case then all states that saturate the lower bound E_{\log_2} on E_R will saturate the Plenio-Vedral bound. Based on Conjecture 1, we can show that for $\rho_{12\dots n} = |S(n, k)\rangle\langle S(n, k)|$ the inequality

$$\max_i \{E_R(\rho_{12\dots\hat{i}\dots n}) + S(\rho_{12\dots\hat{i}\dots n})\} \leq E_R(\rho_{12\dots n}) \quad (68)$$

is saturated, where $\rho_{12\dots\hat{i}\dots n} \equiv \text{Tr}_i(\rho_{12\dots n})$ is the reduced density matrix obtained from $\rho_{12\dots n}$ by tracing out the i -th party. The proof is as follows. As $|S(n, k)\rangle$ is permutation-invariant, there is no need to maximize over all parties, and we can simply take $i = 1$, obtaining the reduced state $\rho_{n-1; k-1, k}(k/n)$ as in Eq. (64). As the corresponding function $F_{n-1; k-1, k}(s)$ of $\rho_{n-1; k-1, k}(s)$ is convex for $s \in [0, 1]$, we immediately obtain from Conjecture 1 that, for $\rho_{n-1; k-1, k}(k/n)$,

$$E_R(\rho_{n-1; k-1, k}(k/n)) = \log_2 \left[C_k^n \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \right] + \frac{k}{n} \log_2 \frac{k}{n} + \frac{n-k}{n} \log_2 \frac{n-k}{n} \quad (69a)$$

$$= E_R(|S(n, k)\rangle) - S(\rho_{n-1; k-1, k}(k/n)). \quad (69b)$$

Therefore, the bound in Eq. (68) is saturated for $\rho_{12\dots n} = |S(n, k)\rangle\langle S(n, k)|$.

A major challenge is to extend the ideas contained in the present Paper from the relative entropy of entanglement to its regularized version, the latter in fact being of wider interest than the former. The alternative way of defining the relative entropy via the optimization over PPT states may also be used, in view of the recent progress on the bipartite regularized relative entropy of entanglement [16].

We now explore the possibility that the geometric measures can provide lower bounds on yet another entanglement measure—the entanglement of formation. If the relationship $E_R \leq E_F$ between the two measures of entanglement—the relative entropy of entanglement E_R and the entanglement of formation E_F —should continue to hold for *multipartite* states (at least for pure states), and if E_F should remain a convex hull construction for mixed states, then we would be able to construct a lower bound on the entanglement of formation:

$$\begin{aligned} E_{\log_2}(\rho) &\equiv \min_{p_i, \psi_i} \sum_i p_i E_{\log_2}(|\psi_i\rangle) \leq \min_{p_i, \psi_i} \sum_i p_i E_R(|\psi_i\rangle) \\ &\leq \min_{p_i, \psi_i} \sum_i p_i E_F(|\psi_i\rangle) \equiv E_F(\rho), \end{aligned} \quad (70)$$

where $\{p_i\}$ and $\{\psi_i\}$ are such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Thus, $E_{\log_2}(\rho)$ is a lower bound on $E_F(\rho)$. By using the inequality $(1-x^2)\log_2 e \leq -2\log_2 x$ (for $0 \leq x \leq 1$), one further has that $(\log_2 e)E_{\sin^2}(\rho) \leq E_{\log_2}(\rho) \leq E_F(\rho)$.

We remark that E_{\sin^2} has been shown to be an entanglement monotone [12, 13], i.e., it is not increasing under local operations and classical communication (LOCC). However, E_{\log_2} is *not* a monotone, as the following example shows. Consider the bipartite pure state

$$|\psi\rangle \equiv \frac{1}{\sqrt{1+Nx^2}}|00\rangle + \frac{x}{\sqrt{1+Nx^2}}(|11\rangle + |22\rangle + \dots + |NN\rangle), \quad (71)$$

with $|x| \leq 1$, for which $E_{\log_2} = \log_2(1+Nx^2)$. Suppose that one party makes the following measurement:

$$\mathcal{M}_1 \equiv |0\rangle\langle 0|, \quad \mathcal{M}_2 \equiv |1\rangle\langle 1| + |2\rangle\langle 2| + \dots + |N\rangle\langle N|. \quad (72)$$

With probability $P_1 = 1/(1 + Nx^2)$ the output state becomes $|\psi_1\rangle = |00\rangle$; with probability $P_2 = Nx^2/(1 + Nx^2)$ the output state becomes $|\psi_2\rangle = (|11\rangle + |22\rangle + \dots + |NN\rangle)/\sqrt{N}$, for which $E_{\log_2} = \log_2 N$. For E_{\log_2} to be a monotone it would be necessary that

$$E_{\log_2}(\psi) \geq P_1 E_{\log_2}(\psi_1) + P_2 E_{\log_2}(\psi_2). \quad (73)$$

Putting in the corresponding values for the P 's and E_{\log_2} 's, we find that this inequality is equivalent to

$$f(N, x) \equiv \log_2(1 + Nx^2) - \frac{Nx^2}{1 + Nx^2} \log_2 N \geq 0. \quad (74)$$

As this is violated for certain values of x with $N > 2$, as exemplified in Fig. 7 for the plot of $f(4, x)$, we arrive at the conclusion that E_{\log_2} is, in general, not a monotone.

Note added. Certain results reported in the present Paper have recently been applied by Vedral [28] to the macroscopic entanglement of η -paired superconductivity.

Acknowledgments

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- [1] See, e.g., M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [2] For a review, see M. Horodecki, *Quantum Inf. Comput.* **1**, 3 (2001), and references therein.
- [3] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, *Phys. Rev. A* **63**, 012307 (2001).
- [4] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [5] C. H. Bennett, D. DiVincenzo, J. Smolin, and W. K. Wootters, *Phys. Rev. A* **53**, 3824 (1996).
- [6] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [7] M. B. Plenio and V. Vedral, *J. Phys. A* **34**, 6997 (2001).
- [8] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
- [9] A. Peres, *Phys. Lett. A* **202**, 16 (1995).
- [10] As we shall show later, this state has $E_R = \log_2(9/4)$, exactly the value cited in Ref. [7] from a numerical result.
- [11] A. Shimony, *Ann. NY. Acad. Sci.* **755**, 675 (1995).
- [12] H. Barnum and N. Linden, *J. Phys. A* **34**, 6787 (2001).
- [13] T.-C. Wei and P. M. Goldbart, *Phys. Rev. A* **68**, 042307 (2003).
- [14] T.-C. Wei, J. B. Altepeter, P. M. Goldbart, and W. J. Munro, to appear in *Phys. Rev. A* **70** (2004); quant-ph/0308031.
- [15] S. Ishizaka, *Phys. Rev. A* **67**, 060301 (2003).
- [16] K. Audenaert, J. Eisert, E. Jané, M. B. Plenio, S. Virmani, and B. De Moor, *Phys. Rev. Lett.* **87**, 217902 (2001); K. Audenaert, M. B. Plenio, and J. Eisert, *Phys. Rev. Lett.* **90**, 027901 (2003). We thank an anonymous referee for pointing out the alternative in defining the relative entropy of entanglement, as well as the above references.
- [17] S. Bravyi, *Phys. Rev. A* **67**, 012313 (2003).
- [18] A. Acín, G. Vidal, and J. I. Cirac, *Quantum Inf. and Comput.* **3**, 55 (2003). G. Vidal and J. I. Cirac, *Phys. Rev. Lett.* **86**, 5803 (2001). We thank an anonymous referee for pointing out these references.
- [19] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, *Phys. Rev. A* **67**, 022112 (2003).
- [20] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, *Phys. Rev. A* **64**, 030302 (2001).
- [21] T.-C. Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro, and F. Verstraete, *Phys. Rev. A* **67**, 022110 (2003).
- [22] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).
- [23] K. G. H. Vollbrecht and R. F. Werner, *Phys. Rev. A* **64**, 062307 (2001).
- [24] S. Ishizaka, *J. Phys. A: Math. Gen.*, **35**, 8075 (2002).
- [25] S. Wu and Y. Zhang, *Phys. Rev. A* **63**, 012308 (2000).
- [26] E. F. Galvao, M. B. Plenio, and S. Virmani, *J. Phys. A* **33**, 8809 (2000).
- [27] We thank an anonymous referee for raising this question.
- [28] V. Vedral, quant-ph/0405102.