# How entangled are bound entangled states?

Tzu-Chieh Wei[1], Joseph B. Altepeter[1], Paul M. Golbart[1],
William J. Munro
HP Laboratories Bristol
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[1] Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080 USA

# How entangled are bound entangled states?

Tzu-Chieh Wei,[1] Joseph B. Altepeter,[1] Paul M. Goldbart,[1] and William J. Munro[2]

[1]*Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080, U.S.A.*

[2]*Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol, BS34 SQ2, UK*

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Bound entangled states are states that are entangled but from which no entanglement can be distilled if all parties are allowed only local operations and classical communication. However, in creating these states one needs nonzero entanglement to start with. To date, no analytic results reveal the entanglement content of these strange states. Here, the entanglement of two distinct multipartite bound entangled states is determined analytically in terms of geometric measure of entanglement and a related quantity. The results are compared with those for the relative entropy of entanglement and the negativity, and plausible lower bounds on the entanglement of formation are given. Along the way, an intriguing example emerges, in which a bipartite mixed state, associated with Smolin's bound entangled state, can be reversibly converted into a bipartite Bell state, and vice versa. Furthermore, for any N-qubit state that is PPT for all bipartite partitionings, there is no violation of the two-setting, three-setting, and functional Bell inequalities.

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*Introduction*: We are motivated to study the quantification of entanglement for the basic reason that entanglement has been identified as a resource central to much of quantum information processing (see, e.g., Ref. [1]). To date, progress in the quantification of entanglement for mixed states has resided primarily in the domain of bipartite systems [2]. For multipartite systems in pure and mixed states, the characterization and quantification of entanglement presents even greater challenges.

As there are ambiguities in the generalizing of the entanglement of distillation ($E_{\rm D}$) and formation ($E_{\rm F}$) to multipartite systems [3], the study of other measures, such as the relative entropy of entanglement ($E_{\rm R}$), is thus relevant to understanding multipartite entanglement. Recently, a multipartite entanglement measure based on the geometry of Hilbert space has been proposed [4], and has been applied to several bipartite and multipartite cases. The merit of this measure is that it is suited for any-partite systems with any dimension, although determining it analytically for generic states remains a challenge.

One of the many test beds for entanglement measures includes a peculiar set of states: bound entangled states. These are states that are entangled, but from which no pure entangled state can be distilled if all parties are allowed only local operations and classical communication (LOCC). The entanglement of distillation (under LOCC) for these states is thus zero. Bound entangled states can be either bipartite or multipartite, the latter possibly exhibiting more features than the former. However, it does take nonzero amount of entanglement to *create* bound entangled states. It is thus desirable to see how much entanglement (e.g., $E_{\rm F}$) is needed to create them. So far, no analytic results on nontrivial entanglement measures for these states have been reported. Yet, as the two important measures—entanglement of distillation and of formation—have so far been limited to bipartite settings, in order to explore multipartite settings, it is, on the one hand, necessary to lay down bounds on the entanglement content for distillation and formation. On the other hand, applying other measures to multipartite states may prove helpful in understanding entanglement.

In this Letter, we study two distinct bound entangled states: Smolin's four-party unlockable bound entangled state [5, 6] and Dür's $N$-party Bell-inequality-violating bound entangled states [7]. For each, we determine analytically their geometric measure of entanglement, as well as a related quantity. These give lower bounds on their $E_{\rm F}$. In addition, we make conjectures on their relative entropies of entanglement. We have also found that when Smolin's state is partitioned 1:234, it has bipartite $E_{\rm F}=1$, equal to that of a Bell state. Moreover, this bipartite mixed state can be reversibly transformed into a bipartite Bell state, and vice versa, even though they have different system entropies.

*Geometric measure of entanglement*: We begin by briefly reviewing the formulation of this measure. Consider a general, $n$-partite, pure state (expanded in the local bases $\{|e^{(i)}_{p_i}\rangle\}$): $|\psi\rangle=\sum_{p_1\cdots p_n}\chi_{p_1p_2\cdots p_n}|e^{(1)}_{p_1}e^{(2)}_{p_2}\cdots e^{(n)}_{p_n}\rangle$. As shown in Ref. [4], the closest separable pure state,

$$|\phi\rangle\equiv\otimes_{i=1}^n|\phi^{(i)}\rangle=\otimes_{i=1}^n\Big(\sum\nolimits_{p_i}c^{(i)}_{p_i}\,|e^{(i)}_{p_i}\rangle\Big), \qquad (1)$$

satisfies the stationarity condition

$$\sum_{p_1\cdots\widehat{p_i}\cdots p_n}\chi^*_{p_1p_2\cdots p_n}c^{(1)}_{p_1}\cdots\widehat{c^{(i)}_{p_i}}\cdots c^{(n)}_{p_n}=\Lambda\,{c^{(i)}_{p_i}}^*, \qquad (2a)$$

(with $i=1,\ldots,n$) and its complex conjugate, where the eigenvalue $\Lambda\in[-1,1]$ is associated with the Lagrange multiplier enforcing $\langle\phi|\phi\rangle=1$, and the symbol $\widehat{\ }$ denotes exclusion. Moreover, the eigenvalue $\Lambda$ is the cosine of the angle between $|\psi\rangle$ and $|\phi\rangle$; the largest one, $\Lambda_{\max}$, which we call the *entanglement eigenvalue*, corresponds to the closest separable state, and is the maximal overlap:

$$\Lambda_{\max}(|\psi\rangle) = \max_{\phi} |\langle\phi|\psi\rangle|, \tag{3}$$

where $|\phi\rangle$ is separable but otherwise arbitrary pure state. In Ref. [4], $E_{\sin^2} \equiv 1 - \Lambda^2_{\max}(|\psi\rangle)$ was defined to be the geometric measure of entanglement for any pure state $|\psi\rangle$. In Ref. [8], $E_{\log}(|\psi\rangle) \equiv -2\log\Lambda_{\max}(|\psi\rangle)$ was shown to be a lower bound on $E_{\rm R}$ of $|\psi\rangle$.

The extension to mixed states can be built upon the pure-state theory, and is made via the use of the *convex hull* construction (indicated by "co"), as was done for $E_{\rm F}$ [2]. The essence is a minimization over all decompositions $\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|$ into pure states:

$$E_{\sin^2}(\rho) \equiv ({\rm co}E_{\sin^2})(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum\nolimits_i p_i\, E_{\sin^2}(|\psi_i\rangle), \tag{4a}$$

$$E_{\log}(\rho) \equiv ({\rm co}E_{\log})(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum\nolimits_i p_i\, E_{\log}(|\psi_i\rangle). \tag{4b}$$

This convex hull construction ensures that the measure gives zero for separable states; however, it also complicates the task of determining mixed-state entanglement. $E_{\sin^2}$ is an *entanglement monotone* [4] and although it remains to see whether $E_{\log}$ is also, it does, e.g., have the additive property:

$$E_{\log}\big(|\psi_{12}\rangle\otimes|\phi_{34}\rangle\big) = E_{\log}\big(|\psi_{12}\rangle\big) + E_{\log}\big(|\phi_{34}\rangle\big).$$

Moreover, if the relationship among the two measures—the relative entropy of entanglement ($E_{\rm R}$) and the entanglement of formation ($E_{\rm F}$)—still holds for any multipartite pure state: $E_{\rm R} \le E_{\rm F}$, and if $E_{\rm F}$ for mixed states is still constructed by the convex hull, then we have a lower bound on the entanglement of formation:

$$E_{\log}(\rho) \le E_{\rm F}(\rho). \tag{5}$$

Using the inequality $-2\log x \ge 1 - x^2$ (with $0 \le x \le 1$) for both base-2 and base-$e$, one further has

$$E_{\sin^2}(\rho) \le E_{\log}(\rho). \tag{6}$$

We shall calculate analytically $E_{\sin^2}$ and $E_{\log}$ for the two bound entangled states, Smolin's and Dür's.

*Smolin's four-party unlockable bound entangled state*: This is a four-qubit mixed state

$$\rho^{ABCD} \equiv \frac{1}{4}\sum_{i=0}^{3}\big(|\Psi_i\rangle\langle\Psi_i|\big)_{\rm AB}\otimes\big(|\Psi_i\rangle\langle\Psi_i|\big)_{\rm CD}, \tag{7}$$

where the $|\Psi\rangle$'s are the four Bell states: $(|00\rangle \pm |11\rangle)/\sqrt{2}$ and $(|01\rangle \pm |10\rangle)/\sqrt{2}$. Now, the state $\rho^{ABCD}$ can be conveniently rewritten as

$$\rho^{ABCD} = \frac{1}{4}\sum_{i=0}^{3}|X_i\rangle\langle X_i|, \tag{8}$$

where the $|X_i\rangle$'s are four orthogonal GHZ-like states:

$$|X_0\rangle \equiv \frac{1}{\sqrt{2}}\big(|0000\rangle + |1111\rangle\big),\ |X_1\rangle \equiv \frac{1}{\sqrt{2}}\big(|0011\rangle + |1100\rangle\big),$$
$$|X_2\rangle \equiv \frac{1}{\sqrt{2}}\big(|0101\rangle + |1010\rangle\big),\ |X_3\rangle \equiv \frac{1}{\sqrt{2}}\big(|0110\rangle + |1001\rangle\big).$$

The most general decomposition of an arbitrary mixed state $\rho$ into pure states can be expressed as

$$\rho = \sum_{k=1}^{\mathcal{M}}|\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|, \quad \text{with } |\tilde{\varphi}_k\rangle = \sum_{i=1}^{n}\mathcal{U}_{ki}\sqrt{\lambda_i}\,|\xi_i\rangle \tag{10}$$

where $\mathcal{M}$ is an integer not smaller than $n$, the number of orthonormal eigenvectors $\{|\xi_i\rangle\}$(with nonzero eigenvalues $\{\lambda_i\}$) of $\rho$, $|\tilde{\varphi}\rangle$'s are *un-normalized*, and $\mathcal{U}$ satisfies $\sum_{k=1}^{\mathcal{M}}\mathcal{U}_{ki}\,\mathcal{U}^*_{kj} = \delta_{ij}$. Thus, the most general (unnormalized) pure state that appears in the decomposition of Smolin's state is

$$|\tilde{\varphi}_k\rangle = \sum_{i=0}^{3}\frac{1}{2}\,\mathcal{U}_{ki}|X_i\rangle. \tag{11}$$

Our goal is to minimize $\sum_k p_k\, E_{\rm pure}\big(|\varphi_k\rangle\big)$ over all possible $\mathcal{U}$'s, where $E_{\rm pure}$ is some pure-state entanglement (either $E_{\sin^2}$ or $E_{\log}$ in our considerations), $p_k \equiv \langle\tilde{\varphi}_k|\tilde{\varphi}_k\rangle$, and $|\varphi_k\rangle$ is the normalized state $|\varphi_k\rangle \equiv |\tilde{\varphi}_k\rangle/\sqrt{p_k}$. Making a general minimization for an arbitrary mixed state is extremely difficult. However, for the mixed state $\rho^{ABCD}$ we shall show that the decomposition in Eq. (8) does indeed minimize the average entanglement over pure-state decompositions. As in Eq. (11), $|\varphi\rangle$ can be explicitly written as $|\varphi\rangle = \sum_{i=0}^{3}\sqrt{q_i}\,e^{i\,\phi_i}|X_i\rangle$, where the $q$'s are non-negative, satisfying $\sum_i q_i = 1$, and the $\phi$'s are phases. For fixed $q$'s, the state has a maximal entanglement eigenvalue when all phases are zero. We shall show shortly that its maximal entanglement eigenvalue is $1/\sqrt{2}$, which is achieved by the $|X\rangle$'s.

The entanglement eigenvalue of the state $|\varphi\rangle = \sum_{i=0}^{3}\sqrt{q_i}\,|X_i\rangle$ is the maximal overlap with the separable state $|\Phi\rangle = \otimes_{i=1}^{4}\big(c_i|0\rangle + s_i|1\rangle\big)$, where $c_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$ with $0 \le \theta_i \le \pi/2$. Thus

$$\begin{aligned}\langle\Phi|\varphi\rangle = {} & \sqrt{q_0/2}\,(c_1c_2c_3c_4 + s_1s_2s_3s_4) + \sqrt{q_1/2}\,(c_1c_2s_3s_4 \\ & + s_1s_2c_3c_4) + \sqrt{q_2/2}\,(c_1s_2c_3s_4 + s_1c_2s_3c_4) \\ & + \sqrt{q_3/2}\,(c_1s_2s_3c_4 + s_1c_2c_3s_4),\end{aligned}$$

which has maximum $1/\sqrt{2}$. To see this, use the Schwarz inequality, treating $\big\{\sqrt{q_0/2}, \sqrt{q_1/2}, \sqrt{q_2/2}, \sqrt{q_3/2}\big\}$ as one vector (whose modulus is $1/\sqrt{2}$), and the corresponding coefficients as another vector (whose modulus can be shown to be no greater than 1). The states $|X_i\rangle$ clearly saturate this bound; hence:

$$E_{\sin^2}(\rho^{ABCD}) = 1/2, \quad E_{\log}(\rho^{ABCD}) = \log 2. \tag{12}$$

We conjecture that the relative entropy of entanglement $E_{\rm R}$ for this state is $\log 2$ and that one of the closest separable mixed states is the state

$$\frac{1}{8}\big(|0000\rangle\langle 0000|+|1111\rangle\langle 1111|+|0011\rangle\langle 0011|+|1100\rangle\langle 1100| +|0101\rangle\langle 0101|+|1010\rangle\langle 1010|+|0110\rangle\langle 0110|+|1001\rangle\langle 1001|\big).$$

We remark that the negativity $\mathcal{N}$ (defined as twice the absolute sum of negative eigenvalues of the partial transpose (PT) of the density matrix with respect to some bipartite partitioning) is zero for any 2/2 partitioning, e.g., {AB : CD}, but nonzero for 1/3 partitioning, e.g.,{A:BCD}. Specifically, $\mathcal{N}_{\rm A:BCD}=1$ but $\mathcal{N}_{\rm AB:CD}=0$.

*Dür's N-party bound entangled state*: Dür [7] found that for $N\ge 4$ the following state has bound entanglement:

$$\rho_N\equiv\frac{1}{N+1}\left(|\Psi_G\rangle\langle\Psi_G|+\frac{1}{2}\sum_{k=1}^{N}\big(P_k+\bar{P}_k\big)\right), \qquad (13)$$

where $P_k\equiv|u_k\rangle\langle u_k|$ is a projector onto the state $|u_k\rangle\equiv|0\rangle_1|0\rangle_2\cdots|1\rangle_k\cdots|0\rangle_N$, $\bar{P}_k\equiv|v_k\rangle\langle v_k|$ projects onto $|v_k\rangle\equiv|1\rangle_1|1\rangle_2\cdots|0\rangle_k\cdots|1\rangle_N$, and

$$|\Psi_G\rangle\equiv\big(|0^{\otimes N}\rangle+e^{i\alpha_N}|1^{\otimes N}\rangle\big)/\sqrt{2}. \qquad (14)$$

For $N\ge 8$ this state violates the Mermin-Klyshko-Bell inequality for $N\ge 8$ [7]; violation was pushed down to $N\ge 7$ by Kaszlikowski et al. [9] for a three-setting Bell inequality; it was pushed further down to $N\ge 6$ by Sen *et al.* [10] for a functional Bell inequality. The phase $\alpha_N$ in $|\Psi_G\rangle$ can be eliminated by local unitary transformations, and hence we shall take $\alpha_N=0$ in the following discussion.

In fact, if we consider the family of states

$$\rho_N(x)\equiv x|\Psi_G\rangle\langle\Psi_G|+\frac{1-x}{2N}\sum_{k=1}^{N}\big(P_k+\bar{P}_k\big), \qquad (15)$$

we find that for $N\ge 4$ the state is bound entangled if $0<x\le 1/(N+1)$ and NPT entangled if $x>1/(N+1)$. This can be seen from the fact that the negativities of the state $\rho_N(x)$ with respect to the two different partitions $(1:2\cdots N)$ and $(12:3\cdots N)$ are

$$\mathcal{N}_{1:2\cdots N}\big(\rho_N(x)\big)=\max\{0,[(N+1)\,x-1]/N\}\,, \qquad (16a)$$
$$\mathcal{N}_{12:3\cdots N}\big(\rho_N(x)\big)=x. \qquad (16b)$$

By applying arguments similar to those used to calculate entanglement for Smolin's state, we have that the general pure state in the decomposition of $\rho_N(x)$ has the form

$$\sqrt{y}\,e^{i\phi_0}|\Psi_G\rangle+\sqrt{1-y}\sum_{k=1}^{N}\big(\sqrt{q_k}e^{i\phi_i}|u_i\rangle+\sqrt{r_k}e^{i\phi'_i}|v_i\rangle\big),$$

where $q$'s and $r$'s are non-negative and satisfy $\sum_k(q_k+r_k)=1$. In this family, the state with the least entanglement (or maximum $\Lambda_{\max}$) for fixed $\{y,q_k,r_k\}$ is the one with all phase factors zero:

$$|\Psi\big(y,\{q,r\}\big)\rangle\equiv\sqrt{y}|\Psi_G\rangle+\sqrt{1-y}\sum_{k=1}^{N}\big(\sqrt{q_k}|u_i\rangle+\sqrt{r_k}|v_i\rangle\big).$$

Next, we ask: For fixed $y$, what is the least entanglement that the above state can have? Take a separable state of the form $|\Phi\rangle=\otimes_{i=1}^{N}\big(c_i|0\rangle+s_i|1\rangle\big)$; its overlap with $|\Psi\big(y,\{q,r\}\big)\rangle$ is then

$$\langle\Psi|\Phi\rangle=\sqrt{y/2}\,(c_1\cdots c_N+s_1\cdots s_N) +\sqrt{1-y}\sum_{k=1}^{N}(\sqrt{q_k}\,c_1\cdots s_k\cdots c_N+\sqrt{r_k}\,s_1\cdots c_k\cdots s_N).$$

This can be shown to no greater than $\sqrt{(2-y)/2}$, again by a Schwarz inequality, taking

$$\Big\{\sqrt{y/2},\{\sqrt{(1-y)q_k}\},\{\sqrt{(1-y)r_k}\}\Big\}$$

as the first $(2N+1)$-component vector (with modulus $\sqrt{(2-y)/2}$) and the corresponding coefficients as the second one (the modulus of which can be shown to be no greater than 1 for $N\ge 4$). The bound can be saturated, e.g., by

$$|\psi_{\pm,u,k}(y)\rangle\equiv\sqrt{y}|\Psi_G\rangle\pm\sqrt{1-y}|u_k\rangle, \qquad (17a)$$
$$|\psi_{\pm,v,k}(y)\rangle\equiv\sqrt{y}|\Psi_G\rangle\pm\sqrt{1-y}|v_k\rangle, \qquad (17b)$$

for which $\Lambda_{\max}(y)=\sqrt{(2-y)/2}$ [11]. As $1-\Lambda^2_{\max}(y)$ is linear in $y$ and $-2\log\Lambda_{\max}(y)$ is convex in $y$, one immediately gets

$$E_{\sin^2}(\rho_N(x))=\frac{x}{2},\quad E_{\log}(\rho_N(x))=\log\frac{2}{2-x}, \qquad (18)$$

and one of the optimal decompositions is

$$\rho_N(x)=\frac{1}{4N}\sum_{k=1}^{N}\sum_{\alpha=\pm}\sum_{\beta=u,v}|\psi_{\alpha,\beta,k}(x)\rangle\langle\psi_{\alpha,\beta,k}(x)|. \qquad (19)$$

We conjecture that, for $N\ge 4$, $\rho_N(x)$ has $E_R(x)=x\log 2$, with one closest separable mixed state being

$$\frac{x}{2}\big(|0..0\rangle\langle 0..0|+|1..1\rangle\langle 1..1|\big)+\frac{1-x}{2N}\sum_{k=1}^{N}\big(P_k+\bar{P}_k\big).$$

The Mermin-Klyshko-Bell inequality, ${\rm Tr}\big(\mathcal{B}_N\rho\big)\le 1$ for local hidden variable theory, can be violated by $\rho_N(x)$ for certain values of $x$ and $N$ as ${\rm Tr}\big(\mathcal{B}_N\rho_N(x)\big)=2^{(N-1)/2}x$. Similar results can be obtained for two other inequalities [9, 10].

*Concluding remarks*: We have analytically determined the geometric measure of entanglement and a related logrithmic quantity for two bound entangled states. These two quantities provide lower bounds on entanglement of formation. We have also made conjecture on their relative entropies of entanglement (which are at least upper bounds). Assuming $E_{\rm D} \le E_{\rm R}$ holds, we then also have upper bounds on distillable entanglement for these states.

For Smolin's bound entangled state, we can establish its entanglement of formation for certain bipartite partitionings. For example, if we group the four parties ABCD in two, A:BCD, we can write the state as

$$\rho^{A:BCD} = \frac{1}{4}\sum_{i=0}^{3} |\bar{X}_i\rangle\langle\bar{X}_i|, \qquad (20)$$

with the 3-qubit states of BCD mapped on to the 8-level system ($000 \to \underline{0}, 001 \to \underline{1}, ..., 111 \to \underline{7}$) and arrive at the locally convertible states (by BCD)

$$|\bar{X}_0\rangle = \big(|0\underline{0}\rangle + |1\underline{7}\rangle\big)/\sqrt{2}, \quad |\bar{X}_1\rangle = \big(|0\underline{3}\rangle + |1\underline{4}\rangle\big)/\sqrt{2},$$
$$|\bar{X}_2\rangle = \big(|0\underline{5}\rangle + |1\underline{2}\rangle\big)/\sqrt{2}, \quad |\bar{X}_3\rangle = \big(|0\underline{6}\rangle + |1\underline{1}\rangle\big)/\sqrt{2}.$$

In order to find the entanglement for this bipartite state (of $C^2 \otimes C^8$), we need to consider the entanglement of the general pure state (properly normalized)

$$|\psi\rangle \equiv \sum\nolimits_i \sqrt{x_i}\, e^{i\phi_i} |\bar{X}_i\rangle$$

that appears in the pure-state decompositions. In fact, regardless of the values of the $x_i$'s, this pure state has a reduced density matrix (when tracing over BCD) of the form $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$. This shows that the mixed state $\rho^{A:BCD}$ has $E_{\rm F} = 1$, $E_{\sin^2} = 1/2$, and $E_{\log} = 1$. In fact, this is a general result for $C^2 \otimes C^{2m}$ states that are derived from mixing Bell-like states

$$|\Psi_k^{\pm}\rangle \equiv (|0, \underline{k}\rangle \pm |1, \underline{2m-k-1}\rangle)/\sqrt{2}, \qquad (22)$$

having *distinct* $k$'s, where $k = 0, 1, \cdots, m-1$.

The above considerations indicate that the bipartite state $\rho^{A:BCD}$ can be created by starting with an *ebit* shared between A and BCD. Suppose that they start in a bipartite state $|\bar{X}_0\rangle$. The party BCD sends the state $|\bar{X}_0\rangle$ into a local apparatus thatleaves the state unchanged (with probablity 1/4), or locally transforms $|\bar{X}_0\rangle$ to $|\bar{X}_1\rangle$ (via $\underline{0} \to \underline{3}$, $\underline{7} \to \underline{4}$, etc.), $|\bar{X}_2\rangle$, and $|\bar{X}_3\rangle$ (with probablities 1/4 each) respectively. This protocol achieves the mixed state $\rho^{A:BCD}$ from a bipartite ebit. On the other hand, the two parties can distill a bipartite Bell state using the protocol proposed by Smolin [5, 6], in which CD (as BCD are now in the same party) make a Bell measurement and pass the result to A and B such that they establish a Bell state. Thus we have found a bipartite mixed state that can be reversibly converted into a Bell state, even though their system entropies are different.

Although Dür's bound entangled state violates a Bell inequality, it has nonzero negativity under certain partitionings. One may raise the question: Does there exist a bound entangled state that is PPT under all partitionings but that still violates a Bell's inequality? For example, does a UPB bound entangled state [12] violate a Bell inequality? We shall see shortly that the answer is "No", at least for the three different Bell inequalities [7, 9, 10] mentioned earlier. Acín has shown [13] that if an $N$-qubit state violates a two-setting Bell inequality then it is distillable under certain bipartite partitioning. Using the results of Ref. [14] regarding distillability, one can repeat the same analysis for the other two Bell inequalities [9, 10] and obtain the same conclusion. This bipartite distillability then implies a negative partial transpose (NPT) under that bipartite partitioning according to Horodecki et al. [15] Hence, violating these Bell inequalities implies NPT under certain bipartite partitioning. Said equivalently, if a state has positive partial transpose (PPT) under all bipartite partitionings then the state never violates these Bell inequalities.

So far, the smallest $N$ for which Dür's bound entangled state violates a Bell inequality is $N = 6$. Can the number be further pushed down to 4 by some other forms of inequalities? If so, can Dür's $N = 4$ and $N = 5$ bound entangled states be distillable under some bipartite partitioning? These questions remain open.

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*Appendix A:* In this Appendix we sketch proofs of two useful inequalities and describe the deriviation of the entanglement eigenvalue for the states in Eqs. (17). We start with the first sought inequality:

$$(c_1c_2c_3c_4 + s_1s_2s_3s_4)^2 + (c_1c_2s_3s_4 + s_1s_2c_3c_4)^2$$
$$+(c_1s_2c_3s_4 + s_1c_2s_3s_4)^2 + (c_1s_2s_3c_4 + s_1c_2c_3s_4)^2 \le 1.$$

We have simplified the notation by using $c_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$. By subtracting the left-hand side from 1 and making some algebraic manipulation, we arrive at the non-negative expression (hence the sought result):

$$(c_1c_2c_3s_4 - s_1s_2s_3c_4)^2 + (c_1c_2s_3s_4 - s_1s_2c_3s_4)^2 +$$
$$(c_1s_2c_3c_4 - s_1c_2s_3s_4)^2 + (s_1c_2c_3c_4 - c_1s_2s_3s_4)^2.$$

The next sought inequality is (for $N \ge 4$):

$$f_N \equiv \big(c_1 \cdots c_N + s_1 \cdots s_N\big)^2 +$$
$$\sum_{k=1}^{N} \big\{(c_1 \cdots s_k \cdots c_N)^2 + (s_1 \cdots c_k \cdots s_N)^2\big\} \le 1.$$

First, making similar arguments, one can show that $f_4 \le 1$. One can also show that $f_{N+1} \le f_N$. Thus by induction, we have the sought result.

We now discuss why $\sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|u_k\rangle$ and $\sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|v_k\rangle$ have as their maximal entanglement eigenvalue $\Lambda_{\max}(y) = \sqrt{(2-y)/2}$. As one can make local relative phase shifts to transform $\sqrt{y}|\Psi_G\rangle + \sqrt{1-y}|u_k\rangle$ to $\sqrt{y}|\Psi_G\rangle - \sqrt{1-y}|u_k\rangle$, they have the same entanglement. The change from $\sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|u_k\rangle$ to $\sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|v_k\rangle$ is simply a flipping of 0 to 1, and vice versa. The mapping from $k$ to $k'$ is just a relabelling of parties. Thus, we need only consider the state

$$\sqrt{y/2}\,(|00\cdots 0\rangle + |11\cdots 1\rangle) + \sqrt{1-y}|10\cdots 0\rangle.$$

As this state is invariant under permutation of all parties except the first one, and as the coefficients are non-negative, in order to find the maximal overlap we can make the hypothesis that the closest separable state is of the form

$$\left(\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle\right) \otimes (\sqrt{q}|0\rangle + \sqrt{1-q}|1\rangle)^{\otimes N-1}.$$

We further see that in order for the overlap to be maximal, $q$ must be either 1 or 0. For the former case, we can further maximize the overlap to get $\sqrt{(2-y)/2}$. For the latter case, the maximum overlap is $\sqrt{y/2}$, which is less than $\sqrt{(2-y)/2}$ (as $0 \le y \le 1$). Hence, the state $\sqrt{y}|\Psi_G\rangle \pm \sqrt{1-y}|u_k\rangle$ has the entanglement eigenvalue $\sqrt{(2-y)/2}$.

*Appendix B:* In this Appendix we analyze the connection between violation of three Bell inequalities and bipartite distillability as was done in Ref. [13] for the two-setting inequality. It was shown by Dür and Cirac [16] that an arbitrary $N$-qubit state $\rho$ can be locally depolarized into the form

$$\rho_N = \lambda_0^+|\Psi_0^+\rangle\langle\Psi_0^+| + \lambda_0^-|\Psi_0^-\rangle\langle\Psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j\left(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|\right),$$

while preserving $\lambda_0^\pm = \langle\Psi_0^\pm|\rho|\Psi_0^\pm\rangle$ and $\lambda_j = \langle\Psi_j^+|\rho|\Psi_j^+\rangle + \langle\Psi_j^-|\rho|\Psi_j^-\rangle$, where $|\Psi_0^\pm\rangle \equiv (|0^{\otimes N}\rangle \pm |1^{\otimes N}\rangle)/\sqrt{2}$, and the $|\Psi_j^\pm\rangle$'s are GHZ-like states, i.e., the states in (22), unfolded into qubit notation. Normalization gives the condition

$$\lambda_0^+ + \lambda_0^- + 2\sum_j \lambda_j = 1.$$

Now define $\Delta \equiv \lambda_0^+ - \lambda_0^-$, which we assume to be non-negative (w.l.o.g). The condition that there is no bipartite distillability for some bipartite partitioning $P_j$ is [14]

$$2\lambda_j \ge \Delta.$$

Assuming non-distillability for *all* bipartite splittings, we have

$$2\sum_j \lambda_j = 1 - (\lambda_0^+ + \lambda_0^-) \ge (2^{N-1} - 1)\Delta.$$

As $\lambda_0^+ + \lambda_0^- \ge \Delta$, we have further that

$$1 - \Delta \ge (2^{N-1} - 1)\Delta. \tag{23}$$

For the two-setting Bell inequality considered by Acín [13], violation implies

$$\Delta > 1/2^{(N-1)/2}.$$

For the three-setting Bell inequality considered in [9], violation implies

$$\Delta > \sqrt{3}\,(2^N/3^N).$$

For the functional Bell inequality in [10], violation implies

$$\Delta > 2\,(2^N/\pi^N).$$

One can easily check that the three Bell inequalities considered are inconsistent with non-bipartite-distillability condition, Eq. (23). Hence, the violating of these three Bell inequalities implies the existence of some bipartite distillability.


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