Equilbrium in Copula Models of First Price Auctions

Kemal Guler
Intelligent Enterprise Technologies Laboratory
HP Laboratories Palo Alto
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We present a model of sealed-bid first-price auctions with affiliated private values (APV) where dependence of valuations is represented by a copula. We derive the equilibrium bidding strategies in a symmetric model with risk-neutral and risk-averse bidders.

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1 INTRODUCTION

We present a model of sealed-bid first-price auction with affiliated private values (APV) where dependence of valuations is represented by a copula. We derive the equilibrium bidding strategies in a symmetric model with risk neutral as well as risk-averse bidders. Copula-marginal representation of the joint distribution of a set of random variables has been used in a variety of application areas from decision and risk analysis to finance and insurance (see Clemen & Reilly 199, Embrechts et al 2001 for some recent examples). To the best of our knowledge, in auction models and econometrics of auction data copula-based formulations have not been used. In this paper we focus on characterization of Nash equilibrium bidding functions in copula-based models of symmetric first price auctions.

Characterization of bidding equilibrium is the first key step in game theoretic analysis of auctions. When closed form expressions are available for the equilibrium a number of analytical questions can be attacked relatively easily. Determination of equilibrium distribution seller's revenue and analysis of expected seller revenue for various changes in the auction environment (e.g., changes in the number of bidders, changes in the seller's reserve price, changes in strength of dependence in bidder valuations etc) are two broad applications that follow equilibrium characterization. Copula-marginal representation of the joint distribution of bidder valuations allows one to explore the effects of variation in the marginal distributions or variation in dependence independently of the other component. In future work, we intend to exploit the equilibrium characterization obtained in this paper to explore a number of such comparative statics problems.

Characterization of equilibrium is also important for empirical applications. In recent years, there has been growing interest in structural econometric models that employ game theoretic models for the statistical analysis auction data. In a companion paper

(Guler, Tang & Zhang []), we investigate applications of copula-based representation of joint distribution of bidders' private information in semiparametric structural econometrics of affiliated private value auctions.

Since economic theory never suggests parametric functional forms, almost all parametric models are driven by convenience and almost all parametric model specifications have some ad hoc flavor. Therefore, it is important that model specifications reduce parameterization to the minimum necessary for the purpose. Copula representation provides a convenient parameterization without having to assume anything about the marginal distributions. It allows one to reduce the dimensionality of models used for analytic and empirical purposes.

This paper proceeds as follows. Section 2 presents a number of preliminaries on copula representation of dependence of random variables. Section 3 describes a copula-based model of sealed-bid first price auction with affiliated private values. Section 4 presents the Bayesian-Nash equilibrium of the model. Section 5 contains some concluding remarks.

2 COPULA REPRESENTATION OF DEPENDENCE

The joint distribution of valuations is the object interest. The joint distribution is decomposed into two constituent elements each of which can be studied independently. The first element is the set of marginal distributions. The second is the structure of dependence, which is represented by a copula function.

We assume that the dependence between valuations is given by a copula. A copula is a multidimensional distribution function with uniform marginals. A copula joins univariate distribution functions to form multivariate distribution functions. By Sklar's theorem any multidimensional distribution may be represented by superposition of a copula and marginal distributions. Furthermore, if the marginal distributions are continuous, the copula is unique. By specifying a copula for the underlying joint distribution of bidder valuations one can identify the marginals and the full joint distribution. Copula-marginal representation of the joint distribution of a set of random variables has been used in a variety of application areas from decision and risk analysis to finance. See Embrechts, Lindskog and McNeil [], Clemen and Reilly [] for various applications of copulas. To the best of our knowledge, there has not been any use of copula representation in auction theory and auction econometrics.

We first present two fundamental results on copulas and a number of parametric copula families. The proofs are available in two recent excellent monographs by Joe (1997) and Nelsen (1999).

Theorem (Sklar): Let $F(x_1,...,x_n)$ be an n-dimensional joint distribution with continuous marginals $F_1(x_1),...,F_n(x_n)$. Then F has a unique copula representation

$$F(x_1,...,x_N) = C(F_1(x_1),...,F_n(x_n)) \qquad (1)$$

For any $(u_1, \ldots, u_N) \in [0,1]^N$, the copula representation can be stated alternatively as

$$F(F_1^{-1}(u_1), \ldots, F_N^{-1}(u_N)) = C(u_1, \ldots, u_N) \tag{2}$$

Conditional distributions:

Theorem: If $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ then the conditional distribution of $x_{-i} := x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ given x_i is

$$H(x_{-i} \mid x_i) = C_i(F_1(x_1), \ldots, F_i(x_i), \ldots, F_n(x_n)) \tag{3}$$

where $C_i(\cdot)$ is the partial derivative of the copula function $C(\cdot)$ with respect to the *i*th argument.

Examples

Independence copula:

When the random variables $u_1, ..., u_n$ are independent, the joint distribution is the product of marginal distributions:

$$C(u_1, ..., u_n) = u_1 u_2 \cdots u_n \tag{4}$$

Gaussian copula:

Let Φ denote the standard normal distribution and let Φ_R be the n-dimensional standard multivariate normal distribution with correlation R. The *Gaussian copula* is given by

$$C(u_1, ..., u_n) = \Phi_R(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)) \tag{5}$$

Gumbel copula:

$$C(u_1, ..., u_n) = exp(-((-\ln u_1)^q + ... + (-\ln u_n)^q)^{1/q}) \tag{6}$$

Archimedean copula:

$$C(u_1, ..., u_n) = (1 + [(u_1^{-1} - 1)^q + ... + (u_n^{-1} - 1)^q]^{1/q})^{-1} \tag{7}$$

Clayton Copula:

$$C(u_1, ..., u_n) = (1 - n + \sum_{i=1}^{n} u_i^{-q})^{-1/q} \tag{8}$$

3 A COPULA MODEL OF APV AUCTIONS

The model we employ is the standard model of sealed bid first auctions with affiliated valuation (Milgrom and Weber (1981), Riley and Samuelson (1981)). A single unit is to be sold using a sealed-bid first-price auction to one of n bidders. Bidders are symmetric. Bidders' valuations of the auctioned object are private and private information. These valuations, however, are affiliated across bidders. H denotes the joint distribution of bidder valuations. C and F denote the copula and the common marginal distributions, respectively.

A typical bidder's von Neumann-Morgenstern utility function is denoted by $u(\cdot)$. Assuming all rival bidders use the bidding strategy $B(v)$ with inverse $f(b)$, bidder i with valuation v and bid b expects to get:

$$p_i(v,b) = u(v-b)H(f(b),...,f(b)\,|\,v) = u(v-b)C_i(F(f(b)),...,F(v),...,F(f(b))) \quad (9)$$

The first order condition for bidder i's optimal bid is:

$$\begin{aligned} &-u'(v-b)C_i(F(f(b)),...,F(v),...,F(f(b))) + \\ &u(v-b)\sum_{j\neq i} C_{ij}(F(f(b)),...,F(v),...,F(f(b)))F'(f(b))f'(b) = 0, \end{aligned} \quad (10)$$

where $C_{ij}(\cdot)$ is the cross partial derivative of $C(\cdot)$ with respect to ith and jth arguments. Using symmetry and optimality of bidder i's bid, i.e. $v = f(b)$, we can write (9) as

$$u'(v-B(v))B'(v)C_i(F(v),...,F(v)) = u(v-B(v))F'(v)(n-1)C_{ij}(F(v),...,F(v)) \quad (11)$$

Alternatively,

$$B'(v) = (n-1)\frac{u(v-B(v))}{u'(v-B(v))}F'(v)\frac{C_{ij}(F(v),\ldots,F(v))}{C_i(F(v),\ldots,F(v))} \tag{12}$$

Together with the boundary condition $B(r) = r$, equation (11) or (12) characterizes the Nash equilibrium bidding strategy.

4 EQUILIBRIUM IN AN APV MODEL WITH GUMBEL COPULA

We present the Nash equilibrium bidding strategies in closed form for the parametric Gumbel family of copulae. We first present some preliminary results for the characterization of Nash equilibrium bidding strategies.

Figures 1 through 3 illustrate the density of two-dimensional Gumbel copula for various values of q. As is clear from the pictures the dependence under this copula is stronger at the extremes. For this copula, the ratio C_i / C_{ij} simplifies to get (see Appendix 2 for details)

$$\frac{C_i(u_1,...,u_n)}{C_{ij}(u_1,...,u_n)} = \frac{u_j[(-\ln u_1)^q + ... + (-\ln u_n)^q]}{(-\ln u_j)^{q-1}\{q-1+[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1/q}\}} \qquad (13)$$

Evaluated on the diagonal $(u_1,...,u_n) = (t,...,t)$ this ratio further simplifies to get

$$\frac{C_i(t,...,t)}{C_{ij}(t,...,t)} = \frac{nt(\ln 1/t)}{q-1+n^{1/q}(\ln 1/t)} \qquad (14)$$

Risk-Neutral Bidders:

If the bidders are risk-neutral, we have $u(x) = x$. It turns out that the initial value problem (12) can be solved explicitly to obtain the Nash equilibrium bid function.

Proposition 1: For the Gumbel model with symmetric risk-neutral bidders, Nash equilibrium bidding strategy is

$$B(v) = v - \int_r^v \left(Z(t;n,q,F(\cdot)) \big/ Z(v;n,q,F(\cdot)) \right)^{n-1} dt \qquad (15)$$

where

$$Z(t;n,q,F(\cdot)) = F(t)^{n^{\frac{1-q}{q}}}[\ln(1/F(t))]^{\frac{1-q}{n}} \tag{16}$$

Proof: See appendix.

Figures 4 through 6 illustrate the equilibrium bid function for various values of n and q for the case when the marginal distribution is uniform on the unit interval and reserve price is equal to .5.

Constant relative risk aversion (CRRA):

We next characterize the equilibrium when the bidders' von Neumann-Morgenstern utility function belongs to the constant relative risk aversion (CRRA) family

$$u(x) = x^{1-r}$$

where the parameter $r \in [0,1]$ measures the degree of relative risk aversion. For this class of utility functions the ratio u/u' takes the form

$$\frac{u(x)}{u'(x)} = \frac{x}{1-r}$$

so that the differential equation that characterizes the equilibrium becomes

$$n(\ln {}^{1}\!/_{F(v)})F(v)B'(v) = ({}^{1}\!/_{1-r})(n-1)(v-B(v))F'(v)\{q-1+n^{1/q}\ln {}^{1}\!/_{F(v)}\}.$$

Proposition 2: For the Gumbel model with symmetric bidders and constant relative risk aversion (CRRA), Nash equilibrium bidding strategy is

$$B(v) = v - \int_r^v \left(Z(t;n,q,F(\cdot)) \big/ Z(v;n,q,F(\cdot)) \right)^{(n-1)/(1-r)} dt \tag{17}$$

where $Z(t;n,q,F(\cdot))$ is defined in Proposition 1 above.

Proof: See appendix.

5 CONCLUDING REMARKS

We have characterized the Nash equilibrium bidding strategies when the joint distribution of bidder valuations is represented by the Gumbel copula. We are currently exploring extensions to other parametric copulae and to general copulae. In a companion paper (Guler, Tang, Zhang 2002) we explore the implications of copula-marginal representation for structural econometrics of affiliated value auctions.

We intend to apply the approach presented in this paper to study a number of comparative statics questions. Among such questions are the following:

- How is the degree of dependence related to aggressiveness of bidding? Do bidders behave more or less aggressively as valuation dependence increases?

- Do bidders necessarily bid more aggressively as the number of bidders increase?

- How is the seller's expected revenue related to the degree of dependence? To variations in the marginal distributions? What are the implications of variations in marginal distributions and dependence structures on the comparison of standard auction formats?

- How is the seller's optimal reserve price affected by bidder valuation dependence? Can one separate the effect of marginal distributions and the copulae on the seller's optimal reserve price?

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7 APPENDIX 1

Conditional distributions other derivations for the Gumbel model:

For the Gumbel copula,

$$C_i(u_1,...,u_n) = C(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1-q/q}$$

When evaluated at the diagonal of the hypercube this function simplifies to

$$C_i(u,...,u) = C(u,...,u)\frac{1}{u}n^{1-q/q}$$

The mixed partial derivative of the Gumbel copula is as follows:

$$\begin{aligned}
C_{ij}(u_1,...,u_n) &= C_j(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1-q/q} \\
&+ C(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}(1-q)(-1/u_j)(-\ln u_j)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1-2q/q} \\
&= C(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}(1/u_j)(-\ln u_j)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{2-2q/q} \\
&+ C(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}(q-1)(1/u_j)(-\ln u_j)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1-2q/q} \\
&= C(u_1,...,u_n)(1/u_i)(-\ln u_i)^{q-1}(1/u_j)(-\ln u_j)^{q-1}[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1-2q/q} \\
&\quad \{q-1+[(-\ln u_1)^q + ... + (-\ln u_n)^q]^{1/q}\}
\end{aligned}$$

Again, when we evaluate this expression at $(u,...,u)$ we get

$$C_{ij}(u,...,u) = C(u,...,u)\frac{1}{u^2}(-\ln u)^{-1}n^{1-2q/q}\{q-1-n^{1/q}\ln u\}$$

Proof of Proposition 1:

Tedious but straightforward manipulations give the following differential equation for the symmetric equilibrium when the joint distribution of bidder valuations is represented by the Gumbel copula:

$$nu'(v-B(v))\ln(1/F(v))F(v)B'(v)=(n-1)u(v-B(v))F'(v)\{q-1+n^{1/q}\ln 1/F(v)\}$$

If the bidders are risk neutral, this differential equation simplifies:

$$n\ln(1/F(v))F(v)B'(v)=(n-1)(v-B(v))F'(v)\{q-1+n^{1/q}\ln(1/F(v))\}$$

We introduce a change of variables and reparameterization:

$$t=1/q$$
$$L(v)=\ln F(v) \text{ so that}$$
$$q-1=1-t/t \text{ and}$$
$$L'(v)=F'(v)/F(v)$$

With this notation equilibrium differential equation becomes

$$B'(v)+B(v)\tfrac{n-1}{nt}[(1-t)\tfrac{L'(v)}{L(v)}-t\,n^t L'(v)]=v\,\tfrac{n-1}{nt}[(1-t)\tfrac{L'(v)}{L(v)}-t\,n^t L'(v)].$$

Multiplying both sides with the integrating factor

$$Y(v)=e^{-(n-1)[(1-t)\ln L(v)-t\,n^t L(v)]/(nt)}=\left(F(v)^{n^{t-1}}[\ln 1/F(v)]^{1-t/nt}\right)^{n-1},$$

we obtain an exact differential equation of the form

$$B'(v)Y(v)+B(v)Y'(v)=vY'(v).$$

Using the initial value $B(r)=r$, we get the equilibrium as

$$B(v)=v-\frac{\int_r^v Y(t)dt}{Y(v)}$$

Finally, substituting for $Y(t)$ and reverting back to the original parameters we get the equilibrium as stated in the proposition.

Proof of Proposition 2:

Following the same steps as in the proof of Proposition 1 and using the integrating factor

$$Y(v) = e^{-(n-1)[(1-t)\ln L(v) - t\,n^{t} L(v)]/(nt(1-r))} = \left(F(v)^{n^{t-1}} [\ln {}^{1}\!/\!_{F(v)}]^{{}^{1-t}\!/\!_{nt}} \right)^{{}^{n-1}\!/\!_{1-r}}$$

yields the solution claimed in the proposition.

8 APPENDIX 2

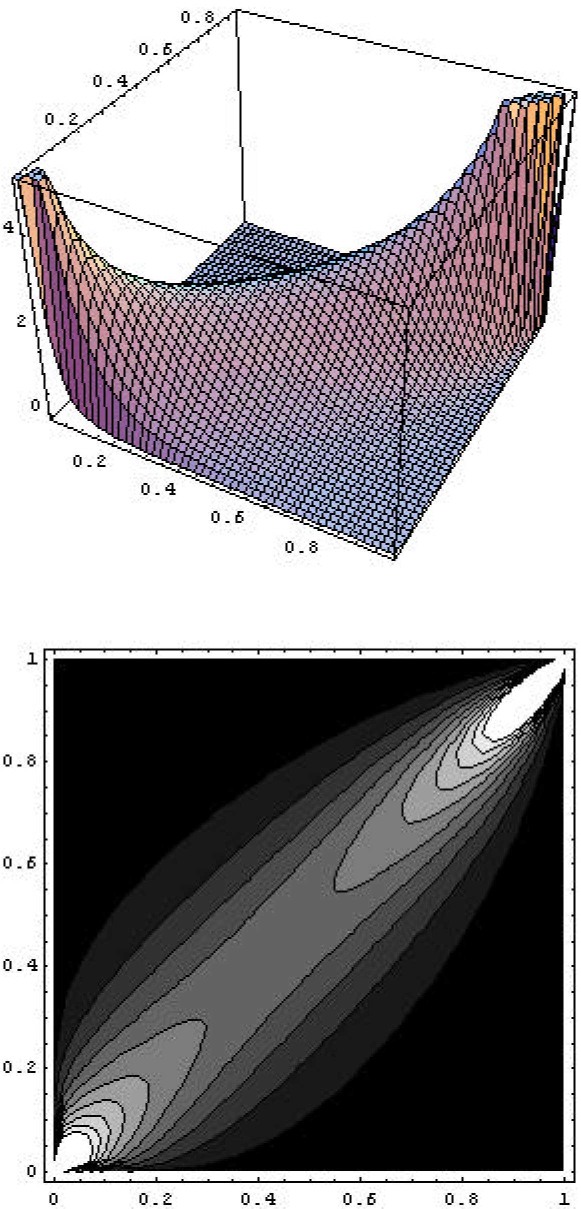

Figure 1: Gumbel copula density for $q = 3$.

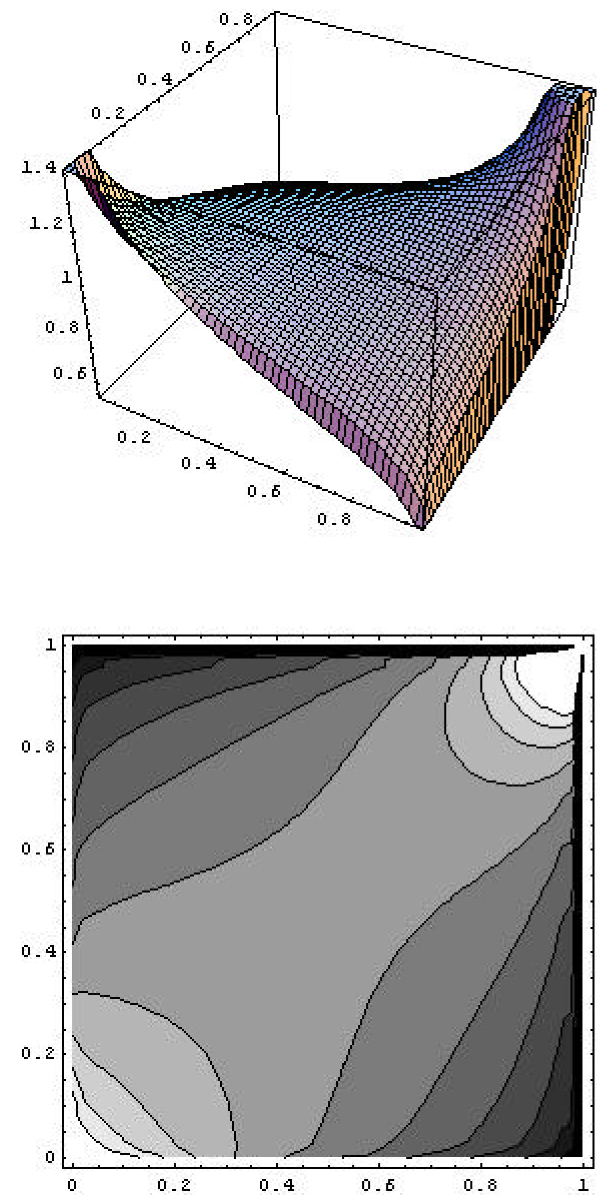

Figure 2: Gumbel copula density for $q = 1.1$.

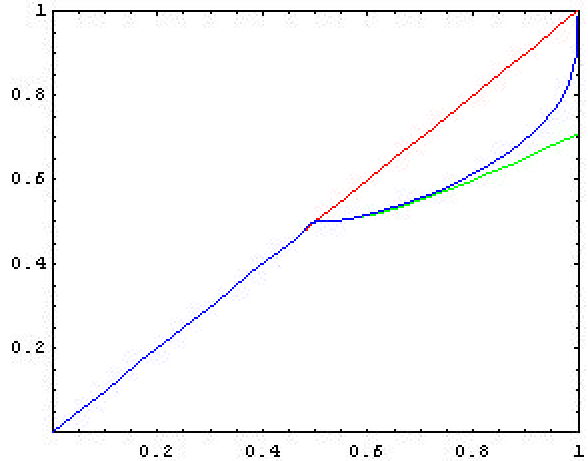

Figure 3: Equilibrium bid function for
$n = 3, r = .5$ and $F(v) = v, q \in \{1(green), 1.5(blue)\}$

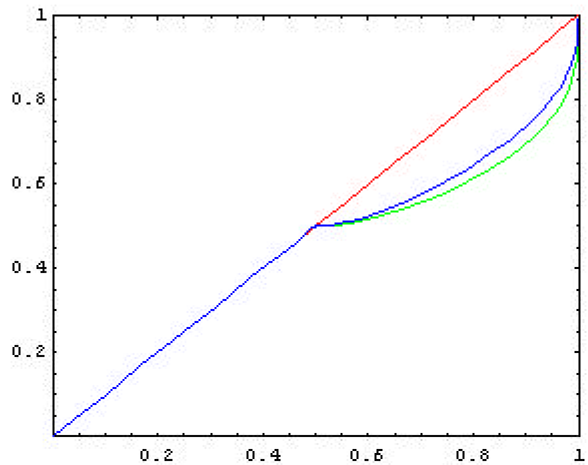

Figure 4: Equilibrium bid function for
$q = 1.5, r = .5$ and $F(v) = v, n \in \{3(green), 5(blue)\}$

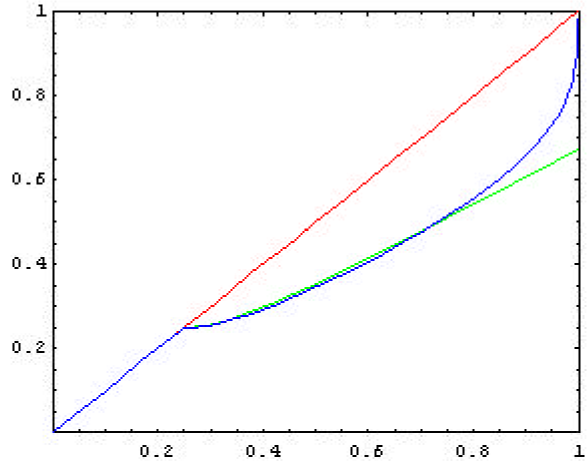

Figure 5: Equilibrium bid function for
$n = 3, r = .25$ and $F(v) = v, q \in \{1(green), 1.5(blue)\}$

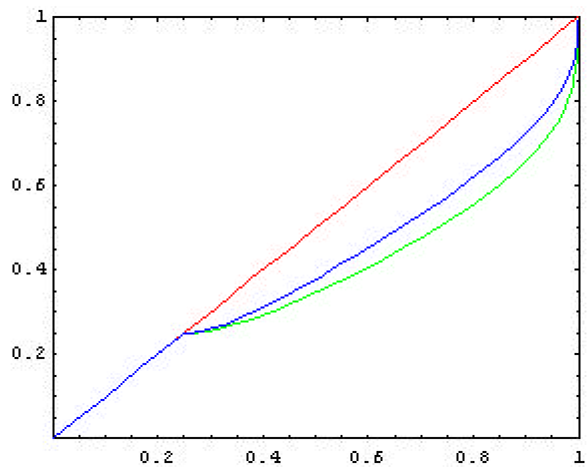

Figure 6: Equilibrium bid function for
$q = 1.5, r = .25$ and $F(v) = v, n \in \{3(green), 5(blue)\}$