Entangling gates using Josephson circuits

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A system consisting of two Josephson qubits coupled through a quantum monochromatic electromagnetic field mode is studied. It is shown that for certain values of the parameters, it can be used as an entangling gate, which entangles the two qubits whilst the electromagnetic field remains disentangled. The gate operates with decent fidelity to a $\sqrt{SWAP}$ gate and could form the basis for initial experimental investigations of coupled superconducting qubits.

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Abstract

A system consisting of two Josephson qubits coupled through a quantum monochromatic electromagnetic field mode is studied. It is shown that for certain values of the parameters, it can be used as an entangling gate, which entangles the two qubits whilst the electromagnetic field remains disentangled. The gate operates with decent fidelity to a $\sqrt{SWAP}$ gate and could form the basis for initial experimental investigations of coupled superconducting qubits.

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There is currently considerable interest in realisations of qubits for quantum information processing. Numerous candidates have been proposed and are under investigation. (e.g., [1, 2].) Fundamental qubits (ions, spins in molecules) currently lead the experimental race in terms of having demonstrated two qubit gates and simple algorithms, but in the long term condensed matter realisations could well be the best route to scalability in qubit number, provided that the problems of decoherence can be reduced. Josephson circuits exhibit many interesting quantum phenomena and form one of the leading condensed matter candidates for the implementation of quantum gates for quantum computing [3–6]. On paper these systems compare well against the Di Vincenzo checklist of criteria for quantum computing [7]. A number of recent experiments have indicated that single superconducting qubits can be constructed [8–11]. Preparation, evolution with good coherence and measurement have been shown to work well, but experimental demonstration of entanglement between two such qubits remains a goal.

Our work here addresses this specific point. In this paper we consider a system comprising two Josephson qubits interacting with a quantized monochromatic electromagnetic field. Earlier work on the interaction of Josephson devices with quantized electromagnetic fields has been presented in [12–15]. The two superconducting qubits (which we denote as A and B) are approximately described as two level systems when biased at a suitable external flux (magnetic or electric) value which separates the lowest two eigenstates from the rest of the spectrum. Proposals have been made [3–5] to couple superconducting qubits directly through a circuit element (capacitive or inductive), leading to direct interaction terms of the form $\sigma_x^A \sigma_x^B$ etc. in the Hamiltonian, using pseudospin notation for the approximate superconducting qubits. Clearly such terms are capable of producing maximal entanglement between the qubits. However, as practical circuits often have both inductive and capacitive components whether by design or not, in this work we consider coupling each of the qubits to an electromagnetic field mode which has its own dynamics. In practice this would be a mode of a resonant superconducting microwave structure. The interaction between the qubits is thus mediated through the electromagnetic field mode, and not through a direct circuit interaction, which is assumed small by comparison.

The model Hamiltonian we employ is [16] (in units $\hbar = c = k_B = 1$)

$$H = \frac{1}{2}\Delta\ \sigma_z^A + \frac{1}{2}\Delta\ \sigma_z^B + \omega_F(a^\dagger a + \frac{1}{2}) + \lambda_A\ \sigma_x^A(a + a^\dagger) + \lambda_B\ \sigma_x^B(a + a^\dagger) \qquad (1)$$

where σ are Pauli matrices; Δ is the energy separation between the ground state and first excited state (which is assumed to be the same for both qubits); ω_F is the frequency of the electromagnetic field; a and $a^\dagger$ annihilation and creation operators correspondingly, of the electromagnetic field; and λ_A, λ_B the coupling constants of the electromagnetic field with the qubits A and B, correspondingly. This Hamiltonian can describe both forms of superconducting qubit. In the case of charge qubits [3, 6] the σ_x eigenstates are the charge eigenstates and the qubits are assumed to be biased with gate voltages such that the energy (σ_z) eigenstates are the odd and even superpositions of adjacent charge states. In this case, the qubits are arranged so the charges couple to the electric field of the mode. In the case of flux or current qubits [4–6] the σ_x eigenstates represent current states of a superconducting ring, biased so that the energy eigenstates are superpositions of these. In this case the qubits are arranged so their currents couple to the magnetic field of the mode. Clearly, our analysis and simulations apply to any other qubit-field-qubit system which can be described by (1).

Here we consider the resonant case where $\omega_F = \Delta$. Such systems are generally operated at very low temperatures so that the thermal noise is smaller than the quantum noise ($\hbar\omega_F >> k_B T$). Our calculations assume that the effect of dissipation is negligible. The effects of dissipation, dephasing, finite temperatures and other couplings will be considered in future work.

The purpose of this paper is to show that for suitable values of the parameters the system generates entanglement between the qubits, and operates as an effective $\sqrt{SWAP}$ gate. This requires the demonstration that if at $t = 0$ the system is in one of the factorizable states $|0_A; 0_B; 0_F\rangle$, $|1_A; 0_B; 0_F\rangle$, $|0_A; 1_B; 0_F\rangle$, $|1_A; 1_B; 0_F\rangle$ then, at a later time t, it evolves as follows:

$$|0_A; 0_B; 0_F\rangle \rightarrow |0_A; 0_B; 0_F\rangle \tag{2}$$

$$|1_A; 1_B; 0_F\rangle \rightarrow |1_A; 1_B; 0_F\rangle \tag{3}$$

$$|0_A; 1_B; 0_F\rangle \rightarrow \frac{1}{1+i}(|0_A; 1_B\rangle + i|1_A; 0_B\rangle)|0_F\rangle \tag{4}$$

$$|1_A; 0_B; 0_F\rangle \rightarrow \frac{1}{1+i}(|1_A; 0_B\rangle + i|0_A; 1_B\rangle)|0_F\rangle \tag{5}$$

We have used the parameters $\omega_F = \Delta = 1.3{\cdot}10^{-4}$ and $\lambda_A = \lambda_B = 4.2{\cdot}10^{-5}$ (corresponding to $\omega_F = 2 \cdot 10^{11} rad \cdot s^{-1}$) and assuming that at $t = 0$ the system is in a state described by a density matrix $\rho(0)$ we calculated the density matrix of the system at a time t which is $\rho(t) = \exp(iHt)\rho(0)\exp(-iHt)$. For the numerical work we have truncated the infinite dimensional Hilbert space of the electromagnetic field to $N_{max} = 6$ and we have checked that greater values of $N_{max} = 6$ do not significantly increase the accuracy (the average number of photons in the system is less than 1 and $N_{max} = 6$ is sufficient). The $\exp(iHt)$ is therefore the exponential of a finite matrix which is calculated with MATLAB.

Partial tracing of the $\rho(t)$ with respect to some of the subsystems produces the density matrix of the remainder at time t. From these we can calculate various quantities. Assuming that at $t = 0$ the system is in the state $|0_A; 1_B; 0_F\rangle$ we calculated the average number of quanta $< N_A >$, $< N_B >$ and $< N_F >$ in the three modes as a function of time. We also calculated the entropic quantities $I_{ij}(t) = S(\rho_i) + S(\rho_j) - S(\rho_{ij})$ (where $i, j = A, B, F$) which describe correlations between the modes ij (non-zero value indicates the existence of classical correlations or entanglement). The results are shown in Fig.1 where it is seen that at $\omega_F t = 27.6$ we have $< N_A >= 0.5$, $< N_B >\approx 0.5$ and $< N_F >\approx 0$ $I_{AB} \approx 1.2$, $I_{AF} \approx 0.1$ $I_{BF} \approx 0.1$.

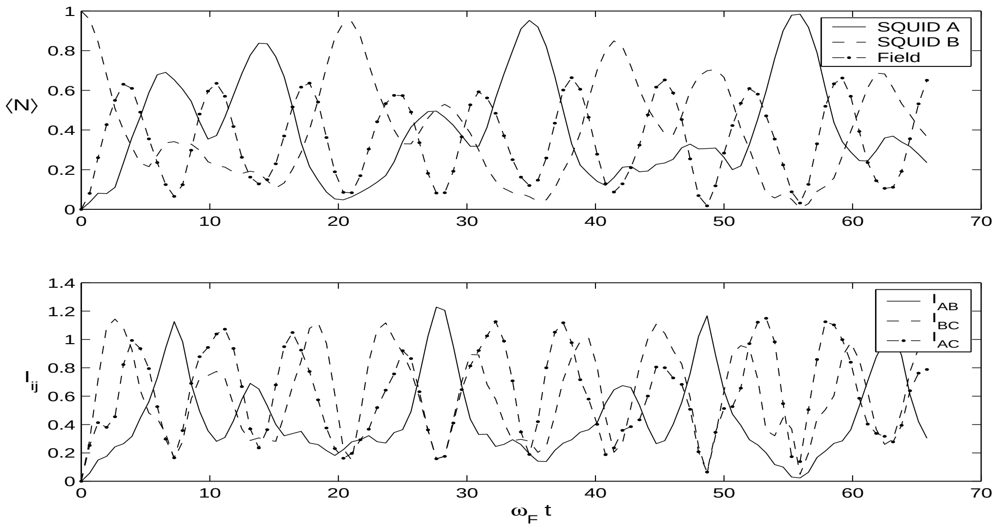

FIG. 1: Top: Average number of excitations $< N_A >$ (solid line), $< N_B >$ (dashed line) and $< N_F >$ (dashed-dot line) for the three subsystems with initial state $|0_A 1_B 0_F\rangle$. Bottom: The entropies I_{ij} in nats defined in the text.

This suggests that the field is close to a vacuum state and disentangled from the two qubits; and also that the two qubits are correlated. To demonstrate that they are close to maximally entangled and that the evolution has effected an approximate $\sqrt{SWAP}$ gate, we examined the density matrix of the system at $\omega_F t = 27.6$. First we checked that the electromagnetic field is disentangled from the qubits. The low values of the $I_{AF} \approx 0.1$ and $I_{BF} \approx 0.1$ indicate already that this is the case. We also calculated the matrix $\rho - (\rho_{AB} \otimes \rho_F)$ which in the case of exact factorization of the electromagnetic field with the two Josephson qubits, is equal to zero. As a measure of the deviation of this matrix from zero we calculated the sum of the absolute values of its elements and we normalized it by dividing over the total number of elements. We call this the *factorization coefficient* F and its value for our case is $F = 0.005$ which indicates that to a good approximation the matrix is of the form $\rho_{AB} \otimes \rho_F$. In table 1 we give the matrix elements of ρ_{AB} . It is seen that to a good approximation Eq. (4) is obeyed. Due to the symmetry between the two modes, this also means that Eq. (5) is obeyed.

	$\|1_A 1_B\rangle$	$\|0_A 1_B\rangle$	$\|1_A 0_B\rangle$	$\|0_A 0_B\rangle$
$\langle 1_A 1_B\|$	0.0123 + 0.0000i	0	0	0.0053 - 0.0060i
$\langle 0_A 1_B\|$	0	0.4962 + 0.0000i	-0.0102 - 0.4805i	0
$\langle 1_A 0_B\|$	0	-0.0102 +0.4805i	0.4834- 0.0000i	0
$\langle 0_A 0_B\|$	0.0053 + 0.0060i	0	0	0.0080 + 0.0000i

TABLE I: Matrix elements of ρ_{AB} at $\omega_F t = 27.6$ for a system in the state $|0_A; 1_B; 0_F\rangle$ at $t = 0$

We next repeat the same calculations for the initial states $|0_A; 0_B; 0_F\rangle$ and $|1_A; 1_B; 0_F\rangle$. For the initial state $|0_A; 0_B; 0_F\rangle$ we get $F = 0.004$ which indicates that the electromagnetic field is to a very good approximation disentangled; and the matrix ρ_{AB} is shown in table 2 where it is seen that Eq. (2) is obeyed.

	$\|1_A 1_B\rangle$	$\|0_A 1_B\rangle$	$\|1_A 0_B\rangle$	$\|0_A 0_B\rangle$
$\langle 1_A 1_B\|$	0.0131 + 0.0000i	0	0	0.0636- 0.0773i
$\langle 0_A 1_B\|$	0	0.0407+ 0.0000i	0.0407 + 0.0000i	0
$\langle 1_A 0_B\|$	0	0.0407 -0.0000i	0.0407 + 0.0000i	0
$\langle 0_A 0_B\|$	0.0636+ 0.0773i	0	0	0.9055 - 0.0000i

TABLE II: Matrix elements of ρ_{AB} at $\omega_F t = 27.6$ for a system in the state $|0_A; 0_B; 0_F\rangle$ at $t = 0$

For the initial state $|1_A;1_B;0_F\rangle$ we get $F = 0.006$ which again indicates that the electromagnetic field is to a very good approximation disentangled; and the matrix ρ_{AB} is shown in table 3 where it is seen that Eq. (3) is obeyed (although the approximation is not as good as before).

	$\|1_A1_B\rangle$	$\|0_A1_B\rangle$	$\|1_A0_B\rangle$	$\|0_A0_B\rangle$
$\langle 1_A1_B\|$	0.7225 - 0.0000i	0	0	0.1365 - 0.0252i
$\langle 0_A1_B\|$	0	0.0602 - 0.0000i	0.0602 - 0.0000i	0
$\langle 1_A0_B\|$	0	0.0602 + 0.0000i	0.0602 + 0.0000i	0
$\langle 0_A0_B\|$	0.1365 + 0.0252i	0	0	0.1450 - 0.0000i

TABLE III: Matrix elements of ρ_{AB} at $\omega_F t = 27.6$ for a system in the state $|1_A;1_B;0_F\rangle$ at $t = 0$

We note that a system which at time t operates as a *perfect* $\sqrt{SWAP}$ gate will at time $2t$ operate as a $SWAP$ gate. We have checked that this is indeed the case in our simulations, and at twice the time entanglement is generated the results show a $SWAP$. For example, for the case that the system is at $t = 0$ in the state $|0_A;1_B;0_F\rangle$ we present here the factorization coefficient $F = 0.005$ and the density matrix ρ_{AB} in table 4. It is seen that the $SWAP$ operation is performed with good accuracy, a fidelity [17] of 0.99.

	$\|1_A1_B\rangle$	$\|0_A1_B\rangle$	$\|1_A0_B\rangle$	$\|0_A0_B\rangle$
$\langle 1_A1_B\|$	0.0033 - 0.0000i	0	0	-0.0054 + 0.0014i
$\langle 0_A1_B\|$	0	0.0025 - 0.0000i	-0.0082 - 0.0413i	0
$\langle 1_A0_B\|$	0	-0.0082 + 0.0413i	0.9810 + 0.0000i	0
$\langle 0_A0_B\|$	-0.0054 - 0.0014i	0	0	0.0132 + 0.0000i

TABLE IV: Matrix elements of ρ_{AB} at $\omega_F t = 55.9$ for a system in the state $|0_A;1_B;0_F\rangle$ at $t = 0$

A better accuracy might be desirable in order to implement fault tolerant quantum computation with many superconducting qubits. However, this is not about to happen, as many experimental hurdles will have to be overcome to get to such quantum information technology. Next on the experimental agenda is the demonstration of a decent degree of entanglement between two condensed matter qubits. Our results are at the very least important for this stage. As we have shown, it is clear that two superconducting qubits can be entangled through their coupling to a microwave field mode. As well as being a next step towards quantum information technology, the experimental demonstration of entanglement between two superconducting qubits would be of great importance for the fundamentals of quantum mechanics, for in the case of flux qubits this would represent entanglement between two macroscopic spatially separated systems. It is also worth noting that for the form of evolution discussed here, the demonstration of entanglement would not require measurements on the two qubits in different bases (which is not likely to be easy for superconducting systems in the short term). Rather than through the violation of a Bell inequality, it is possible to infer entanglement through measurements at the $\sqrt{SWAP}$ and $SWAP$ points in the evolution in just a single basis [18]. Thus the qubit-field-qubit system discussed here is a very promising route for investigating the entanglement of superconducting circuits.

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