Quantum Teleportation of Optical Quantum Gates

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Quantum computation may allow certain problems to be solved more efficiently than is possible on any classical machine [1], and optical realizations of quantum computation are particularly important because of the negligible effects of decoherence on quantum optical states. However, many challenges still remain for fault-tolerant implementations of optical quantum computing. In proposals for quantum computation using optics [2–4], it is necessary to invoke some form of an optical nonlinear transformation, and unlike linear optics these nonlinear transformations either suffer from large losses (decoherence) or employ nondeterministic gates that fail a large fraction of the time.

The remarkable results by Gottesman and Chuang [5] show that quantum teleportation [6] can be used as a universal quantum primitive. In essense, quantum teleportation allows for the fault-tolerant implementation of "difficult" quantum gates that would otherwise corrupt the fragile information of a quantum state. The linear optics quantum computation (LOQC) scheme [2] relies on the quantum teleportation of nondeterministic nonlinear gates in the instances that they succeed in order to perform fault-tolerant quantum computation. However, the difficulty in performing Bell-state measurements in this optical encoding using only linear optics [7] requires innovative new schemes for near-deterministic quantum teleportation, and these schemes place severe demands on the photodetectors employed for the measurements. The requirements to perform fault-tolerant quantum teleportation in this scheme greatly exceed current technologies.

Fortunately, there exists an alternative approach for quantum teleportation in optical systems: continuous-variable (CV) quantum teleportation [8, 9]. The measurements involved in this scheme are the highly developed and efficient techniques of homodyne detection [10]. Also, experimental CV quantum teleportation can be performed unconditionally [9], and does not suffer from the difficulties associated with the Bell-state measurements of qubit quantum teleportation [11]. It is of great interest, then, to determine what optical quantum gates can be teleported using CV quantum teleportation, and if such gates can be used to implement fault-tolerant optical quantum computation.

In this letter, we show that CV quantum teleportation can easily teleport quantum gates in the CV Clifford group [12], defined below, which includes linear optics and squeezing operations. Also, by allowing squeezing operations conditioned on the results of measurements, one can teleport optical nonlinear gates generated by some Hamiltonians that are cubic polynomial in the canonical coordinates. We show that the gates which can be teleported in this fashion form a universal set of gates for quantum computation, and that this scheme allows for the fault-tolerant implementation of quantum computation in *any* optical scheme employing qubits, qudits or CV encodings, provided one can perform fault-tolerant linear optics and squeezing. We analyze "noisy" teleportation employing finitely-squeezed states and imperfect detectors, and the resulting effect on the teleported gates. We conclude by discussing the challenges involved in using CV quantum teleportation to teleport the nondeterministic gates in LOQC.

The transformations describing linear optics and squeezing possess a natural group structure which will be exploited for their quantum teleportation. Thus, we begin by reviewing the Pauli and Clifford groups for CV quantum computation [4, 12]. The standard Pauli group $\mathcal{C}_1$ on n coupled oscillator systems is the Heisenberg-Weyl group, which consists of phase-space displacement operators for the n oscillators. It is the group of "linear optics," which can be implemented by mixing states with strong coherent fields at a beamsplitter [13]. The algebra that generates this group is spanned by the $2n$ canonical operators $\hat{q}_i$, $\hat{p}_i$, $i = 1, \ldots, n$, along with the identity operator $\hat{I}$, satisfying the commutation relations $[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}\hat{I}$. (In the following, we set $\hbar = 1$.) For a single oscillator, the Pauli group consists of operators of the form $R(q,p) = \exp(-\mathrm{i}(q\hat{p} - p\hat{q}))$, with $q, p \in \mathbb{R}$. The Pauli operators for one system can be used to construct a set of Pauli operators $\{R_i(q_i, p_i); i = 1, \ldots, n\}$ for n systems (where each operator labeled by i acts as the identity on all other systems $j \neq i$). This set generates the Pauli group $\mathcal{C}_1$.

The Clifford group $\mathcal{C}_2$ is the group of transformations acting by conjugation that preserves the Pauli group $\mathcal{C}_1$; i.e., a gate U is in the Clifford group if $URU^{-1} \in \mathcal{C}_1$ for every $R \in \mathcal{C}_1$. The Clifford group $\mathcal{C}_2$ for continuous variables is easily shown [12] to be the (semidirect) product of the Pauli group and the linear symplectic group of all one-mode and two-mode squeezing transformations.

For illustrative purposes, we present a generating set of gates for the CV Clifford group. First, consider gates that act on a single oscillator mode. The *Fourier transform* F is the CV analog of the Hadamard transformation; it is defined as $F = \exp(\mathrm{i}\frac{\pi}{4}(\hat{q}^2 + \hat{p}^2))$, generated by the harmonic oscillator Hamiltonian. The *CV phase gate* $P(\eta)$ is a one-mode squeezing operation, defined by $P(\eta) = \exp(\frac{1}{2}\mathrm{i}\eta\hat{q}^2)$. The Fourier transform F, the CV phase gate $P(\eta)$, and the Pauli operators $R(q,p)$ generate the Clifford group for a single mode. In order to extend the Clifford group to multiple modes, we must define a basic two-mode gate that entangles the canonical operators for different modes. The *SUM gate* is the CV analog of the CNOT gate and provides the basic interaction gate for two oscillators i and j; it is defined as $\mathrm{SUM}_{ij} = \exp(-\mathrm{i}\hat{q}_i\hat{p}_j)$. This gate describes the unitary transformation used in a back-action evasion or quantum nondemolition process [13]. The n-mode Clifford group $\mathcal{C}_2$ can be generated using SUM gates, Fourier transforms, CV phase gates, and the n-mode Pauli group.

Transformations in the Clifford group do not form a universal set of gates for CV quantum computation. However, Clifford group transformations together with *any* higher-order nonlinear transformation acting on a single mode form a universal set of gates [14]. Because arbitrary nonlinear gates can be constructed with such a set, this result also applies to qubit-based optical realizations for any encoding of qubits [2–4]. In the following, we construct a scheme to use CV quantum teleportation to teleport such a universal set of gates.

Quantum teleportation, although initially proposed for remote parties, can also be used in a quantum circuit. Consider a three mode optical system (interferometer). Let mode 1 be in an arbitrary pure state $|\psi\rangle$ (although this circuit works equivalently well for mixed states), and modes 2 and 3 be in the maximally entangled Einstein-Podolsky-Rosen (EPR) state $|\mathrm{EPR}\rangle_{23} = \int |q\rangle_2 |q\rangle_3 \,\mathrm{d}q$, where $|q\rangle$ are position eigenstates defined by $\hat{q}|q\rangle = q|q\rangle$. Modes 1 and 2 are then subjected to joint projective measurements of the form

$$\Pi_{q,p} = R_1(q,p)|\mathrm{EPR}\rangle_{12}\langle\mathrm{EPR}|R_1^\dagger(q,p)\,, \qquad (1)$$

where $R_1(q,p)$ is the Pauli operator on mode 1. The measurement yields two classical numbers, q_0 and p_0,

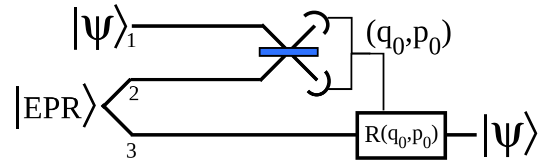

FIG. 1: Circuit diagram of CV quantum teleportation.

which are used to condition a Pauli operation $R_3(q_0, p_0)$ on mode 3. The result of this circuit is that mode 3 is left in the state $|\psi\rangle$, and we say that this state has been quantum teleported to mode 3. A circuit diagram for quantum teleportation is given in Fig. 1. Repeating this circuit on many modes allows for the quantum teleportation of multi-mode states, preserving entanglement between modes.

We now consider how CV quantum teleportation can be used to teleport a quantum gate. Consider an arbitrary transformation S in the single mode ($n = 1$) Clifford group, and suppose we wish to implement this gate on an arbitrary single mode state $|\psi\rangle$. In the following, we show that it is possible to implement S on one mode of an EPR state, and then employ a modified CV quantum teleportation circuit to "teleport" this gate onto the desired state $|\psi\rangle$.

We use the property that, if U is in the single mode Clifford group, then

$$UR(q_0,p_0) = (UR(q_0,p_0)U^{-1})U = R'(q_0,p_0)U\,,$$

and $R'(q_0,p_0) = UR(q_0,p_0)U^{-1}$ is an element of the Pauli group due to the definition of the Clifford group, determined by the gate U. Thus, the desired Clifford gate U can be quantum teleported onto the state $|\psi\rangle$ simply by implementing U on one mode of the EPR pair and by appropriately altering the conditional displacement R of the quantum teleportation. Using n single-mode quantum teleportation circuits, it is then possible to teleport any gate in the n-mode Clifford group. See Fig. 2 for an illustration of how this gate teleportation is performed.

For example, if one wished to teleport the Fourier transform F, they would employ the relation

$$FR(q_0,p_0) = (FR(q_0,p_0)F^{-1})F = R(-p_0,q_0)F\,,$$

and so, by acting with F on one half of the EPR pair, and by employing the conditional transformation $R(-p_0, q_0)$ rather than $R(q_0,p_0)$ in the quantum teleportation circuit, the result of teleporting an arbitrary state $|\psi\rangle$ will be the transformed state $F|\psi\rangle$.

In addition, this technique can be used to teleport more general quantum gates than those in the Clifford group. Consider the set of gates defined by $\mathcal{C}_3 = \{U | U\mathcal{C}_1U^{-1} \subseteq \mathcal{C}_2\}$. This set includes all gates that, when conjugating any Pauli group operation, yield a Clifford group operation. Note that $\mathcal{C}_3$ is not a group (it is not closed under composition) and contains nonlinear transformations.

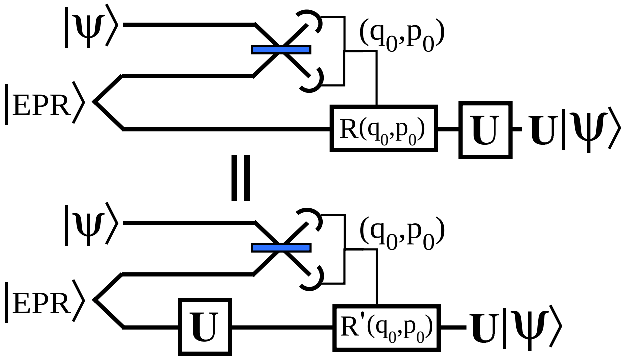

FIG. 2: Quantum teleportation of a gate U. The process of first teleporting the state then implementing U is equivalent to acting on one mode of the EPR state with U, followed by a modified quantum teleportation.

One example of a nonlinear gate in $\mathcal{C}_3$ is the *cubic phase gate* $V(\gamma) = \exp(\mathrm{i}\gamma\hat{q}^3)$, with $\gamma \in \mathbb{R}$, which Gottesman *et al* [4] have shown how to implement using Clifford group transformations and homodyne and photon counting measurements. (For details, see [15].) An example of a multimode gate in $\mathcal{C}_3$ is the *controlled phase gate* $CP_{ij} = \exp(\frac{1}{2}\mathrm{i}\hat{q}_i\hat{q}_j^2)$ acting on two modes i and j. In comparison to the CV phase gate, this gate performs a phase gate operation on mode j depending on the state of mode i. It is a controlled Clifford group transformation; others can be defined similarly.

We now define a quantum teleportation circuit to teleport a gate $U \in \mathcal{C}_3$. Commuting this gate through the conditional operations gives

$$UR(q_0,p_0) = (UR(q_0,p_0)U^{-1})U = R_2'(q_0,p_0)U\,,$$

where, by the definition of $\mathcal{C}_3$, $R_2'(q_0,p_0)$ is an element of the Clifford group. Thus, any gate in $\mathcal{C}_3$ can be quantum teleported using Clifford group transformations conditioned on measurement outcomes; i.e., conditional phase-space displacements and squeezing operations.

Consider, as an example, the CV quantum teleportation of the cubic phase gate $V(\gamma)$. Commuting this gate back through the Pauli operator gives

$$\begin{aligned}R_2'(q_0,p_0;\gamma) &= V(\gamma)R(q_0,p_0)V(\gamma)^{-1} \\ &= \exp\bigl(-\mathrm{i}\bigl(q_0\hat{p} - p_0\hat{q} + 3\gamma q_0\hat{q}^2\bigr)\bigr)\,.\end{aligned}$$

This operation, generated by an inhomogeneous quadratic polynomial in $\hat{q}$ and $\hat{p}$, lies in the Clifford group and can be implemented using a combination of conditional phase-space displacements and one-mode squeezing. Thus, to teleport $V(\gamma)$, this gate is performed on one mode of an EPR state, followed by a modified teleportation scheme with a conditional operation $R_2'(q_0,p_0;\gamma)$; the result is that a state $|\psi\rangle$ is quantum teleported into the transformed state $V(\gamma)|\psi\rangle$.

The cubic phase gate $V(\gamma)$, being a higher-order nonlinear gate on a single mode, can be combined with Clifford group gates of n modes to form a universal set of gates for quantum computation on n modes [14], and a scheme exists to implement this cubic phase gate [4]. Thus, with CV quantum teleportation it is possible to teleport a universal and realizable set of gates. If Clifford group transformations and CV quantum teleportation can be implemented fault-tolerantly, then it is possible to use this scheme to implement a fault-tolerant cubic phase gate (or other nonlinear gate in $\mathcal{C}_3$) using a gate that is not fault-tolerant. Thus, any nonlinear transformations can be moved "off-line" [4, 5]; although these transformations must still be performed in the quantum teleportation circuit, they can be made to act on EPR ancilla states non-deterministically rather than on the fragile encoded states.

We now consider the effects of realistic noise and errors that may occur in a teleportation scheme. We assume that Clifford group transformations can be performed fault-tolerantly, but must take into account finite squeezing and imperfect detection. In experiment [9], the EPR states are approximated by two-mode squeezed vacua [16]

$$|\eta\rangle_{23} = \sqrt{1-\eta^2}\sum_{n=0}^{\infty}\eta^n|n\rangle_2|n\rangle_3\,, \tag{2}$$

with $|n\rangle$ the n-boson Fock state and $0 < \eta \leq 1$. In the limit $\eta \to 1$, this state becomes the maximally entangled EPR state $|\mathrm{EPR}\rangle_{23}$. The projective measurements of Eq. (1) are implemented via homodyne detection, and the displacement operation $R_3(q_0,p_0)$ is performed by mixing the field with a local oscillator at a beamsplitter; see [8, 9] for details. Finite squeezing $\eta < 1$ and imperfect homodyne efficiency $\nu < 1$ can characterised by a Gaussian noise term with variance $\sigma = \exp(-2\tanh^{-1}\eta) + (1-\nu^2)/\nu^2$, defined such that $\sigma = 1/2$ is the level of vacuum noise [17]. For demonstrated CV quantum teleportation with fidelity $F > 0.5$, we require $\sigma < 1$; for fault-tolerant teleportation of quantum gates, σ must be significantly smaller unless appropriate quantum error correction can be applied (see below).

Imperfect quantum teleportation can then be described by a *transfer superoperator* as follows. If ρ is the density matrix for an initial state, the teleported state ρ' will be related to the input state ρ by

$$\rho' = \int\frac{\mathrm{d}q\,\mathrm{d}p}{\pi\sigma}\exp(-\frac{q^2+p^2}{\sigma})R(q,p)\rho R^\dagger(q,p)\,,$$

which defines the transfer superoperator $\mathcal{E}_\sigma$ for the imperfect quantum teleportation. Let $\mathcal{E}_U$ be the superoperator corresponding to a unitary transformation U, defined on a density matrix ρ to be $\mathcal{E}_U(\rho) = U\rho U^{-1}$. Using noisy quantum teleportation (with transfer superoperater $\mathcal{E}_\sigma$) to teleport the gate U, the resulting transformation is described by the superoperator $\mathcal{E}_U \circ \mathcal{E}_\sigma$.

Thus, noisy quantum teleportation of gates is described by a convolution of the unitary gate with a Gaussian noise process; the corresponding errors can be viewed as small ($\sim \sqrt{\sigma}$) random displacements in phase space. The effect of this Gaussian noise depends critically on the specific encoding used. For example, superpositions of coherent states [3] are extremely fragile to Gaussian noise, whereas the encoding of Gottesman *et al* [4] is specifically designed to protect against such Gaussian noise because small displacements in phase space can be detected and corrected. Thus, our proposal for the quantum teleportation of nonlinear gates can be made fault-tolerant if an encoding (such as [4]) is used that is protected against Gaussian noise and can be corrected using fault-tolerant Clifford group operations [18]. As Gaussian noise processes are indicative of optical systems, the investigation of "Gaussian protected" states that can be encoded using only Cliford group operations is essential for robust quantum information using optics.

Because most optical quantum information schemes employ the Kerr effect (generated by a Hamiltonian of the form $(\hat{a}^\dagger)^2\hat{a}^2$) as the nonlinear transformation outside of the Clifford group, it is of interest to consider how such a transformation can be implemented using the above universal set of gates. Using the relation $e^{iAt}e^{iBt}e^{-iAt}e^{-iBt} = e^{i[A,B]t^2} + \mathcal{O}(t^3)$, a combination of cubic phase gates and Clifford gates can be used to simulate the Kerr nonlinearity to any degree of accuracy. We also note that it is possible to iteratively define more higher-order sets of gates $\mathcal{C}_k = \{U | U\mathcal{C}_1U^{-1} \subseteq \mathcal{C}_{k-1}\}$ that can be implemented fault-tolerantly if gates in $\mathcal{C}_{k-1}$ can be implemented fault-tolerantly. The Kerr nonlinearity is related to transformations in the set $\mathcal{C}_4$.

Because CV quantum teleportation can teleport *any* state of an optical mode, we can consider the use of CV quantum teleportation in any optical quantum information process, even qubit-based schemes. One scheme [3] encoding qubits as coherent states could employ CV quantum teleportation to implement the "difficult" Hadamard transformations requiring a Kerr nonlinearity in a fault-tolerant way. In LOQC, nondeterministic gates are used to implement nonlinear transformations on arbitrary states with up to two photons. Qubit-based quantum teleportation is employed to implement these gates fault-tolerantly, but the near-deterministic quantum teleportation of LOQC places strong demands on photodetectors and requires highly entangled multimode states. To instead employ CV quantum teleportation, the problem arises that any gate to be teleported must be made to act on one mode of an EPR state. The probability of a successful gate operation may be affected by the photon number cutoff used for these states (the nonlinear sign gate being one such example). For high-fidelity quantum teleportation, one must employ highly squeezed states that have a corresponding high photon number cutoff (see Eq. (2)); thus, any effective teleportable gate must be designed to operate on such high-photon states. However, the challenges to design such a non-deterministic gate that can be teleported using this scheme may be less significant than the difficulties involved in the near-deterministic teleportation of LOQC.

In conclusion, we have shown that fault-tolerant implementations of Clifford group transformations and CV quantum teleportation is sufficient to perform fault-tolerant quantum computation in any optical scheme: qubits, qudits or CV. A nonlinear transformation such as the cubic phase gate (possible using Clifford group transformations, homodyne detection, and photon counting) can be performed "off-line" on EPR states and used as a resource to perform universal quantum computation. Such gates would allow the implementation of difficult but critical nonlinear transformations such as the Kerr effect in a fault-tolerant way. The definition of an infinite series of gates $\mathcal{C}_k$ highlights which gates can be teleported in a straightforward manner. In particular, the cubic phase gate (rather than a Kerr nonlinearity) is identified as the most direct nonlinear operation that would achieve a universal set of gates. The simplicity and power of this gate clearly motivates its experimental demonstration and its use in future quantum information processing tasks. The use of CV teleportation in LOQC relies on the design of new nondeterministic nonlinear gates that can be teleported using this scheme. Finally, the Gaussian noise introduced through realistic CV quantum teleportation can be corrected with suitable "Gaussian protected" encodings and Clifford group transformations, and the results presested here highlight the need for new, realistic quantum error correction schemes of this form.

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