



Quantum nonlocality, Bell inequalities and the memory loophole⁺

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In the analysis of experiments designed to reveal violation of Bell-type inequalities, it is usually assumed that any hidden variables associated with the n th particle pair would be independent of measurement choices and outcomes for the first $(n-1)$ pairs. Models which violate this assumption exploit what we call the *memory loophole*. We focus on the strongest type of violation, which uses the *2-sided* memory loophole, in which the hidden variables for pair n can depend on the previous measurement choices and outcomes in both wings of the experiment. We show that the 2-sided memory loophole allows a systematic violation of the CHSH inequality when the data are analysed in the standard way, but cannot produce a violation if a CHSH expression depending linearly on the data is used. In the first case, the maximal CHSH violation becomes small as the number of particle pairs tested becomes large. Hence, although in principle the memory loophole implies a slight flaw in existing analyses of Bell experiments, the data still strongly confirm quantum mechanics against local hidden variables. We consider also a related loophole, the *simultaneous measurement loophole*, which applies if all measurements on each side are carried out simultaneously. We show that this can increase the probability of violating the linearised CHSH inequality as well as other Bell-type inequalities.

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I. THE MEMORY LOOPHOLE

Bell's work [1] in the early 1960s made precise the sense in which classical intuitions based on the principles of special relativity conflict with quantum theory. Theoretical and experimental investigations have continued ever since, leading, inter alia, to the understanding of entanglement as a quantifiable resource of fundamental importance for quantum cryptography, communication and computation.

The experiment analyzed by Bell is the following [1]. A source prepares a pair of particles in some entangled state. One particle is sent to Alice and one to Bob, Alice and Bob being situated far from each other. When the particle arrives at Alice, Alice subjects it to a measurement X , chosen by her at random amongst many possible measurements A_1, A_2, \dots . Similarly, Bob subjects his particle to a measurement Y selected by him at random amongst many possible measurements B_1, B_2, \dots . The experiment is repeated many times. Everything is arranged such that each pair of measurements performed by Alice and Bob is space-like separated. After the experiment ends, Alice and Bob come together and compare their results.

Bell asked whether the correlations between the results of the measurements predicted by quantum mechanics can be explained by any classical model. More precisely, he formulated a model, known as a *local hidden variable model*, which is supposed to describe *all possible ways* in which classical systems can generate correlated answers in an experiment as above. He then went on to prove that quantum mechanical correlations cannot be obtained from such a model.

The key words above are “all possible ways.” To guarantee that one has found all possible ways in which a system may behave is a problematic, and formally not well-defined statement. Nevertheless, Bell's model, which we describe in detail below, is very powerful, and it has been generally accepted that it covers indeed all possibilities. Here however we argue that there are possibilities that have not been accounted for in Bell's model, which rely in one way or another on what, for reasons which will be obvious, we call the *memory loophole*.

The rest of our paper is organized as follows. In this section we describe Bell's original hidden variables model and present the memory loophole. In section II, we introduce an inequality that is equivalent to the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] and define some terms. In section III, we summarize the results of the paper. In sections IV to VI, we analyze the inequality from the point of view of Bell's original model and various different versions of the memory loophole. We show that the probability of violating a standard CHSH inequality is affected by the loophole, but that the effect is not significant for a large sample. Finally, in section VII, we consider a related

loophole which arises in experiments in which all N measurements on each side are made simultaneously. Section VIII concludes.

The model proposed by Bell is the following. When two particles are prepared at the source in some (entangled) state Ψ , they both receive an index λ which is called a *local hidden variable*. This index is chosen at random according to some distribution $\rho(\lambda)$. The hidden variable essentially prescribes, in a *local* way, how the particles behave when subjected to different measurements. That is, when Alice subjects her particle to a measurement A , the particle gives an outcome a according to some probability distribution $P(a; A, \lambda)$ which depends on the measurement A and on the hidden variable λ but not on the measurement B performed by Bob on his particle or on the result b of this measurement. Similarly, Bob's particle yields an outcome b according to the probability distribution $P(b; B, \lambda)$ which depends on the measurement B to which it is subjected and on the hidden variable λ but not on A or a . The joint probability $P(a, b; A, B)$ that the particles yield the outcomes a and b when subjected to the measurement of A and B respectively is then given by

$$P(a, b; A, B) = \int d\lambda \rho(\lambda) P(a; A, \lambda) P(b; B, \lambda) \quad (1)$$

The above model has been hitherto considered to describe all possible ways in which classical particles can yield long distance correlations while respecting the relativistic constraint of no superluminal signaling, which prevents Alice's particle from modifying its behavior according to what Bob does if there is not enough time for a light signal to arrive from Bob to Alice, and vice versa. Bell showed that quantum mechanics predicts correlations which cannot be obtained from such a model. The inconsistency of quantum theory with the hypothesis of local hidden variables is often — slightly confusingly — referred to as quantum nonlocality.

A way of testing whether or not some given correlations can be obtained from a local hidden variables (LHV) model is to test some signatures of such models, called Bell inequalities. The best known Bell-type inequality is the CHSH inequality. Suppose that Alice and Bob chose at random between two measurements A_1 or A_2 and B_1 or B_2 respectively. Suppose furthermore that each of these measurements has only two possible outcomes, $+1$ and -1 . Then as CHSH have shown [2], if the particles behave according to any LHV model,

$$E(A_1 B_1) + E(A_1 B_2) + E(A_2 B_1) - E(A_2 B_2) \leq 2, \quad (2)$$

where $E(AB)$ denotes the expectation value of the product of the outcomes of the measurements A and B . On the other hand, one can find quantum mechanical states $|\Psi\rangle$ (for example, any entangled pure state [3,4]) and appropriate measurements so that the CHSH inequality is violated. For example, if the state $|\Psi\rangle$ is the singlet state of two spin $1/2$ particles,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+1\rangle|-1\rangle - |-1\rangle|+1\rangle) \quad (3)$$

where $|+1\rangle$ and $|-1\rangle$ represent spin polarised “up” and “down” along the z axis, one can find appropriate spin measurements which yield

$$\langle\Psi|A_1 B_1|\Psi\rangle + \langle\Psi|A_2 B_1|\Psi\rangle + \langle\Psi|A_1 B_2|\Psi\rangle - \langle\Psi|A_2 B_2|\Psi\rangle = 2\sqrt{2}, \quad (4)$$

violating the LHV limit (2). (Here we have used the quantum mechanical formula for the expectation value $E(AB) = \langle\Psi|AB|\Psi\rangle$.)

Bell's LHV model has generally been thought to cover all possible ways in which classical particles can behave. We show now, however, that this is not the case.

In order to determine correlations one has to perform measurements not on a single pair of particles but on many such pairs, and gather a large number of outcomes which will determine the statistics. Now, according to the LHV model above (1), all the pairs in the ensemble are *uncorrelated*. This assumption appears natural from the perspective of quantum mechanics. In quantum theory, when we have a number of pairs, each pair being described by the same wave-function, the pairs are uncorrelated. However, we can imagine the following scenario. A first pair of particles is emitted by the source. One of the particles arrives at Alice and it is subjected to a measurement and gives an outcome according to the LHV model (1). However, it also leaves in the environment information indicating to what measurement it was subjected and what outcome it yielded. Now, when a particle in the second pair arrives at Alice, it will read this message and it will give an outcome which depends not only on the measurement it is subjected to, but also on the message left by the first particle, i.e. on what has happened to the first particle. Particles on Bob's side behave in a similar way. The consequence is that the original LHV model (1) is now replaced by

$$P(a^{(n)}, b^{(n)} | A^{(n)}, B^{(n)}) = \int d\lambda \rho(\lambda) P(a^{(n)} | A^{(n)}, M, \lambda) P(b^{(n)} | B^{(n)}, M, \lambda), \quad (5)$$

where

$$P(a^{(n)} | A^{(n)}, M, \lambda) = P(a^{(n)} | A^{(n)}, A^{(1)}, \dots, A^{(n-1)}, a^{(1)}, \dots, a^{(n-1)}, \lambda) \quad (6)$$

and

$$P(b^{(n)} | B^{(n)}, M, \lambda) = P(b^{(n)} | B^{(n)}, B^{(1)}, \dots, B^{(n-1)}, b^{(1)}, \dots, b^{(n-1)}, \lambda). \quad (7)$$

Here M stands for the local record, or *memory*, of the previous measurements. We call this a local hidden variable model with *1-sided memory*.

There is a further interesting variation of Bell's original model. Suppose that the source emits pairs of correlated particles one by one. Suppose too that on each pair Alice and Bob perform their measurements space-like separated from one another, so while Alice is performing her measurement no signal can arrive from Bob's measurement. However, the time between the measurements on the different pairs is long enough, so that by the time Alice measures her n -th particle, the particle could have received information about what has happened in Bob's measurements on all previous particles $(1, \dots, n-1)$, and similarly for Bob. One could imagine local hidden variable models in which this information is indeed communicated and used, in which case the probability in (6) is replaced by

$$P(a^{(n)} | A^{(n)}, M, \lambda) = P(a^{(n)} | A^{(n)}, A^{(1)}, \dots, A^{(n-1)}, a^{(1)}, \dots, a^{(n-1)}, B^{(1)}, \dots, B^{(n-1)}, b^{(1)}, \dots, b^{(n-1)}, \lambda) \quad (8)$$

and similarly for the probability on Bob's side. This is a local hidden variable model with *2-sided memory*.

In principle, Bell's original argument can be extended to render both types of memory loophole irrelevant. We could require that separated apparatuses are used for each particle pair, and that *every* measurement is space-like separated from every other — but it seems unlikely that such an experiment will be done any time soon with a large enough sample of particles to demonstrate statistically significant violations of Bell inequalities. Even the much weaker constraint that all of Alice's measurements are space-like separated from all of Bob's — which would exclude the 2-sided but not the 1-sided loophole — has not been satisfied in any experiment to date. (See, e.g., [9] and references therein).

It is worth emphasizing that the memory loopholes described above have a different status from that of other loopholes such as the well-known detection loophole [5] or the recently discussed collapse locality loophole [6]. The detection loophole does not identify a problem with Bell's local hidden variables model *per se*, but only states that technological limitations have to be taken into account — which can be done in the framework of the original model. Similarly, given a precise theory of state reduction (which is required to characterise the loophole precisely in the first place), the collapse locality loophole could be closed by carrying out standard Bell experiments using sufficiently advanced technology. [6] On the other hand, although, as we have noted, the memory loophole could be eliminated by new types of Bell experiments, it does highlight an intrinsic limitation of Bell's model as applied to standard Bell experiments. This is not to say that local hidden variable theories exploiting the memory loophole are necessarily more plausible than theories exploiting other loopholes (indeed, a contrary view can be argued [6]). It does, though, mean there is need for a reanalysis of the power of general local hidden variable theories in standard Bell experiments.

Another interesting theoretical question arises if Alice decides to measure all her particles simultaneously, and Bob does likewise. Since the measurements take some finite time, all the particles on Alice's side could conceivably communicate to each other. Hence the outcome given any particle n may depend on what happened with all the other particles (i.e. to what measurements they are subjected and what outcomes are they yielding). We call the resulting loophole the *simultaneous measurement loophole* and LHV models which exploit it *collective* LHV models.

One might wonder what the point of considering all these loopholes is. Each seems to involve more conspiracy on Nature's part than the last, and none of them appears to lead to plausible physical models. Given the importance of the Bell-type experiments, however, and their consequences for our world view, we feel that it is important to analyze the experiments as rigorously as possible and in particular to distinguish between logical impossibility and physical implausibility of the models.

There is another more practical motivation [7,8]. It is well known that quantum key distribution schemes which use entanglement have significant security advantages over other schemes; they can also be extended by the use of quantum repeaters to allow secure key distribution over arbitrary distances. The security of these schemes relies crucially on the fact that the states created and measured are genuinely entangled. The most obvious and seemingly reliable way to verify this is to use Bell-type tests as security checks within the protocols. However, any such tests need to be interpreted with care. If a quantum cryptosystem is acquired from a not necessarily reliable source, or possibly exposed to sabotage, then a cautious user must consider the possibility that devices have been installed which use classical communication to simulate, as far as possible, the behaviour of quantum states, while allowing third parties to extract illicit information about the key. Such devices effectively define a local hidden variable model, and the usual criterion of physical plausibility no longer applies. A saboteur could set up communication and computing devices

that use any information available anywhere in the cryptosystem. In particular, saboteurs might well try to exploit memory loopholes, as well as other Bell experiment loopholes, if they could gain a significant advantage by so doing.

Having established, therefore, that the original version of the local hidden variables model as proposed by Bell has to be modified, we now examine the consequences.

II. CHSH-TYPE INEQUALITIES. GENERAL CONSIDERATIONS.

We first revisit the usual Bell inequalities experiments, and emphasize in more detail the statistical aspects of the measurements.

The standard CHSH inequality is described in (2) and it is claimed that every ordinary (i.e. as originally constructed by Bell) local hidden variables model must obey the inequality.

Of course, even in an ideal experiment, an ordinary local hidden variables model can violate the CHSH bound. The quantities which figure in the CHSH expression are theoretical expectation values, which are abstract concepts. In reality each expectation value is determined by repeating a measurement a large number of times and estimating the probabilities (and hence the expectation values) as frequencies of events. These measured expectation values are subject to statistical fluctuations, which can yield violations of the CHSH bound. Our first task is to examine the problem in detail, defining precisely the operational meaning of the different quantities, and get an accurate understanding of what exactly is the meaning of violation of Bell's inequalities. Only after all these are clarified will we be able to see the effect of the various memory loopholes. In particular, we will see that memory can allow particles to take advantage of statistical fluctuations and build them up into a systematic bias. We will also see, however, that, if the CHSH expressions are defined in the usual way, the biases that can thus be obtained tend to zero as the number of pairs tested increases. Moreover, we will see that a simpler linearised form of the CHSH expressions is "memory-proof", in the sense that the probability of a given level of violation is no greater for memory-dependent local hidden variable models than for optimally chosen memoryless models.

We use the CHSH inequality in the form [10]

$$\begin{aligned} P_{CHSH} &= P_c(A_1, B_1) + P_c(A_1, B_2) + P_c(A_2, B_1) + P_a(A_2, B_2) \\ &\leq 3 \end{aligned} \tag{9}$$

for local hidden variable theories, where $P_c(A, B)$ is the probability that A and B have the same outcome (are correlated), and $P_a(A, B)$ is the probability that A and B have different outcomes (are anti-correlated). A_1, A_2, B_1, B_2 are chosen so that quantum mechanics predicts the maximal value, $P_{CHSH} = 2 + \sqrt{2}$.

What we actually mean by (9) in an experimental context is the following. We suppose that Alice and Bob perform measurements on N pairs of particles. For each of their particles Alice and Bob choose at random what measurement to perform, A or A' for Alice and B or B' for Bob. We define $\#(A, B)$ to be the number of pairs on which operators A and B were measured, $\#_c(A, B)$ and $\#_a(A, B)$ to be the number of times the outcomes were correlated and anti-correlated in these measurements. Note that Alice and Bob should not pre-arrange the sequence of their measurements - this would introduce well-known loopholes; the entire experiments of Alice and Bob, including the decision of what to measure on each particle have to be space-like separated from each other. Consequently Alice and Bob do not have total control on how many times a specific pair of measurements, say A, B is performed, but this number, $\#(A, B)$ is a random variable.

We define

$$X_N = \frac{\#_c(A_1, B_1)}{\#(A_1, B_1)} + \frac{\#_c(A_1, B_2)}{\#(A_1, B_2)} + \frac{\#_c(A_2, B_1)}{\#(A_2, B_1)} + \frac{\#_a(A_2, B_2)}{\#(A_2, B_2)}, \tag{10}$$

$$Y_N = \frac{4}{N}(\#_c(A_1, B_1) + \#_c(A_1, B_2) + \#_c(A_2, B_1) + \#_a(A_2, B_2)). \tag{11}$$

X_N is the experimental meaning of the CHSH inequality (9); the index N denotes that the experiment has been performed on N pairs. Indeed, the expression $\frac{\#_c(A_1, B_1)}{\#(A_1, B_1)}$ is the frequency of correlations between the outcomes of A_1 and B_1 , and it is therefore the experimental definition of the correlation probability $P_c(A_1, B_1)$ and so on.

Note that our definition of X_N assumes that $\#(A, B) > 0$ for all pairs of operators A, B . If not, X_N is undefined. Strictly speaking, our expressions for the expectation and other functions of X_N should thus all be conditioned on the event that X_N is defined. We will neglect this below, assuming that N is large enough that the probability of X_N being undefined is negligible. One could, alternatively, use an experimental protocol which ensures that X_N is defined. For instance, one could require that, if $\#(A, B) = 0$ for any A, B after N pairs have been tested, the

experiment continues on further pairs until $\#(A, B) > 0$ for all A, B , and then terminates. Our analyses would need to be modified slightly to apply to such a protocol, but the results would be essentially the same.

Y_N is another experimental quantity closely related to X_N . The two quantities are equal if the expressions $\#(A_i, B_j)$ are equal for all i, j . For large N , the $\#(A_i, B_j)$ are almost always nearly equal, and so the same is true of X_N and Y_N . Although it is traditional to use X_N in analyzing Bell experiments, Y_N is in fact much better behaved and easier to analyze, since it is a linear expression.

III. CHSH-TYPE INEQUALITIES. EXPECTATION VALUES AND FLUCTUATIONS.

X_N and Y_N represent quantities determined by making measurements on a batch of N pairs of particles. We do not assume the pairs behave independently: they may be influenced by memory, and we will analyze the different types of memories. We are interested in the maximum possible expectation value of X_N and Y_N , and the maximum probability of X_N or Y_N taking a value much larger than the expectation.

Obviously, the expectation and fluctuations of X_N and Y_N could be experimentally estimated only by repeating the whole series of N experiments a large number of times, and then only under the assumption that different batches of N pairs behave independently. Without some restriction on the scope of the memory loophole, we would need to allow for the possibility that any experiment we perform in the future could in principle be influenced by the results obtained in all experiments to date.

Those who require probabilities to have a frequency interpretation in order to be meaningful may thus have some difficulty interpreting the results of memory loophole analyses. The only certain way to circumvent this difficulty would be to set up many spacelike separated experiments. On the other hand, if probability is viewed simply a measure of the plausibility of a theory, there is no interpretational difficulty. As we will see, it can be shown that the probability of obtaining experimental data consistent with quantum theory, given a local hidden variable theory using a memory loophole, for a large sample, is extremely small. Since the cumulative data in Bell experiments are indeed consistent with quantum theory, we conclude that they effectively refute the hypothesis of memory-dependent local hidden variables — so long, of course, as these hidden variables are assumed not also to exploit other well-known loopholes such as the detector efficiency loophole.

The results for which we have complete proofs can be summarized in the following table:

LHV Model	$E(X_N)$	$P(\hat{X}_N > 5\delta)$	$E(Y_N)$	$P(\hat{Y}_N > \delta)$
Memoryless	≤ 3	$< 5f_N^\delta$	≤ 3	$< f_N^\delta$
1-sided Memory	$< 3 + o(N^{-1/2+\epsilon})$	$< 5f_N^\delta$	≤ 3	$< f_N^\delta$
Collective	?	?	≤ 3	?
2-sided Memory	$< 3 + o(N^{-1/2+\epsilon})$	$< 5f_N^\delta$	≤ 3	$< f_N^\delta$

Here $\hat{X}_N = X_N - 3$, $\hat{Y}_N = Y_N - 3$, and we have simplified the presentation by taking δ to be small enough that $(3 + \delta) < (3 + 5\delta)(1 - \delta)$. The expression $o(N^{-1/2+\epsilon})$ denotes a term that asymptotically tends to zero faster than $N^{-1/2+\epsilon}$ for any $\epsilon > 0$.

$$f_N^\delta = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{\delta\sqrt{N}} \exp\left(-\frac{1}{6}\delta^2 N\right). \quad (12)$$

The proofs are given in the following sections.

The significance of these results is as follows. The memoryless case represents the results for standard local hidden variables which behave independently for each pair. The result $E(X_N) \leq 3$ is the standard expression of the CHSH inequality. Although values of X_N larger than 3 can be experimentally obtained from a local hidden variables model, the probability of obtaining $3 + \delta$ decreases exponentially as $5f_N^\delta$. Hence, for a given δ and sufficiently large N , observing $3 + \delta$ when performing N experiments can be taken as a very good confirmation of the fact that it is not due to an LHV model. In the memoryless case, $E(Y_N) \leq 3$ and the fluctuations also decrease exponentially.

In the 2-sided memory case, the expectation value of Y_N again satisfies $E(Y_N) \leq 3$. Hence the existence of memory makes no difference here. Memory also makes no difference to the fluctuations: they still decrease exponentially. On the other hand, the expectation value of X_N can be larger than in the standard memoryless case. Hypothetically, if Bell experiments are analysed by using X_N and the effect of the memory loophole is neglected, a 2-sided memory LHV model could mistakenly be interpreted as exhibiting non-locality. Fortunately, we can put an upper bound of

$$E(X_N) \leq 3 + 5N^{-1/2+\epsilon} + 5\sqrt{3/2\pi}N^{-\epsilon} \exp(-N^{2\epsilon}/6), \quad (13)$$

for any small $\epsilon > 0$. Thus, for large enough N , X_N is almost as good as Y_N at distinguishing quantum theory from local hidden variable models.

In the 1-sided memory case, we can use the 2-sided memory results to show that Y_N is unaffected by the presence of memory, and X_N is affected in a negligible way for sufficiently large N . Actually, we have not succeeded in finding a 1-sided memory model for which $E(X_N)$ or $P(\hat{X}_N > \delta)$ are larger than the maximal values attainable by memoryless models, for any N . We thus cannot exclude the possibility that 1-sided memory is of no use at all in helping LHV models come closer to reproducing quantum mechanics.

In the collective case, $E(Y_N) \leq 3$. However, we present a collective LHV model which has bigger fluctuations than are possible without memory. To have a reliable test of non-locality, we need the fluctuations to become small as N gets large. We conjecture that this is indeed the case: however, the question marks in the table reflect the fact that we have no rigorous proof.

IV. CHSH-TYPE INEQUALITIES IN BELL'S NO MEMORY MODEL

We first revisit the derivation of the CHSH inequality in Bell's model, using techniques which will be useful for analyzing the different memory models.

We first recall how these quantities are interpreted in standard analyses, when the Bell pairs are measured sequentially and the memory loophole is neglected. Let Z_N be a binomially distributed variable with N trials, each of which has the two possible outcomes 0 and 1, with probability $p \neq 0, 1$ of outcome 1 for each: The normal approximation to the binomial distribution gives us that

$$P(Z_N > pN + z\sqrt{Np(1-p)}) \rightarrow 1 - \mathcal{N}(z) \quad (14)$$

as $N \rightarrow \infty$, where

$$\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(-\frac{1}{2}y^2\right) dy \quad (15)$$

is the normal distribution function, which obeys

$$1 - \mathcal{N}(z) \approx \frac{1}{\sqrt{2\pi}} z^{-1} \exp\left(-\frac{1}{2}z^2\right). \quad (16)$$

For large N , and for z large compared to 1 and small compared to $N^{1/2}$, the errors in these approximations are small and can be rigorously bounded [11]. Below we consider N and z in these ranges and neglect the error terms, which make no essential difference to the discussion.

Now

$$Y_N = \frac{4}{N} \sum_{n=1}^N Y_N^n, \quad (17)$$

where

$$Y_N^n = \delta_c^n(A_1, B_1) + \delta_c^n(A_1, B_2) + \delta_c^n(A_2, B_1) + \delta_a^n(A_2, B_2). \quad (18)$$

Here $\delta_c^n(A, B)$ is 1 if A and B are measured at the n^{th} round and found to be the same, and 0 otherwise, and $\delta_a^n(A, B)$ is 1 if A and B are measured at the n^{th} round and found to be different, and 0 otherwise.

In a memoryless local hidden variable theory, the Y_N^n are independent random variables taking values 0 or 1. We have that

$$E(\delta_c^n(A_1, B_1)) = \frac{1}{4} p_c^n(A_1, B_1), \quad (19)$$

where $p_c^n(A_1, B_1)$ is the probability that $A_1 = B_1$ if (A_1, B_1) is measured at the n^{th} trial, and similarly for the other three terms in (18). So, from (2) we have that

$$y_n = E(Y_N^n) = \frac{P_{CHSH}}{4} \leq \frac{3}{4}. \quad (20)$$

Clearly, for any N and any $\delta > 0$, the probability $P(Y_N > 3 + \delta)$ is maximised when the Y_N^n are identically distributed, with $y_n = 3/4$ for all n . For small δ we have that

$$\begin{aligned} P(Y_N > 3 + \delta) &= P(NY_N/4 > 3N/4 + \delta N/4) \\ &\approx 1 - \mathcal{N}(\delta\sqrt{N}/\sqrt{3}) \\ &\approx \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{\delta\sqrt{N}} \exp\left(-\frac{1}{6}\delta^2 N\right), \end{aligned} \quad (21)$$

for large N , which tends to zero fast as $N \rightarrow \infty$. A similar argument shows that quantum mechanics predicts that $P(Y_N < 2 + \sqrt{2} - \delta)$ tends to zero fast. A long run of experiments can thus distinguish quantum mechanics and memoryless local hidden variables with near certainty.

Although the analysis of Y_N is simpler and arguably more natural, Bell experiments are traditionally interpreted via the quantity X_N . Since

$$\begin{aligned} E\left(\frac{\#_c(A, B)}{\#(A, B)}\right) &= \sum_{n=1}^N p(\#(A, B) = n) \frac{E(\#_c(A, B) | \#(A, B) = n)}{n} \\ &= \sum_{n=1}^N p(\#(A, B) = n) \frac{nP_c(A, B)}{n} \\ &= P_c(A, B), \end{aligned} \quad (22)$$

and similarly $E(\frac{\#_a(A, B)}{\#(A, B)}) = P_a(A, B)$, equations (2) and (10) imply that $E(X_N) \leq 3$. (Recall that we assume the $n = 0$ terms in these sums have negligible probability.)

Moreover, since

$$P(\#(A, B) < N/4(1 - \delta)) \approx \frac{\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right), \quad (23)$$

we have that

$$P\left(X_N > \frac{1}{1 - \delta} Y_N\right) \lesssim \frac{4\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right) \quad (24)$$

and

$$P\left(X_N > \frac{3 + \delta}{1 - \delta}\right) \lesssim \frac{5\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right). \quad (25)$$

Similarly, quantum mechanics predicts that $P(X_N < 2 + \sqrt{2} - \delta)$ tends to zero fast. Thus, for large N , X_N distinguishes the predictions of quantum mechanics and memoryless local hidden variables almost as well as Y_N does.

V. THE TWO-SIDED MEMORY LOOPHOLE

Now we consider the case where the LHV model for N trials is allowed to exploit the memory loophole, predicting results at each round of measurement which may depend upon the previous measurements and outcomes on both sides.

Since equations (19) and (20) still hold, we have that

$$E(Y_N) = \frac{4}{N} \sum_{n=1}^N E(Y_N^n) \leq \frac{4}{N} \sum_{n=1}^N \frac{3}{4} = 3. \quad (26)$$

Thus memory does not help increase $E(Y_N)$. We shall now show that it does not help the probability of a large fluctuation in Y_N . First, we note that Y_N is just (a constant times) the sum of Y_N^n , where Y_N^n is a random variable at the n^{th} trial. Now, Y_N^n can only take values of 0 or 1. To maximize the probability of a large Y_N , we should try to maximize the probability of each Y_N^n being 1. This at first appears complicated, since with memory LHV models

there will be correlations between $P(Y_N^n = 1)$ for different n . The key is to note that, regardless of what happens in later rounds, for all LHV memory models,

$$P(Y_N^n = 1 | \text{events in trials } 1 \dots n-1) \leq 3/4. \quad (27)$$

This is because, for any fixed set of events in the earlier rounds, the model in round n is just an LHV model, whose probabilities have been chosen with no prior knowledge of the measurements which will be performed in round n , and must therefore satisfy the CHSH inequality.

It follows that, for any N and any $\delta > 0$, the probability $P(Y_N > 3 + \delta)$ is maximised when $P(Y_N^n = 1) = 3/4$ for all n . But an LHV model can maximize the probability that $Y_N^n = 1$, for any n , by a strategy independent of the outcomes of the previous measurements, for instance by predicting the outcome 1 for any measurement on either side. Since $Y_N^n = 0$ or 1, any such strategy maximizes the probability $P(Y_N > 3 + \delta)$, and so equation (21) still holds even when the memory loophole is taken into account. The memory loophole does not alter the distinguishability of the predictions of quantum mechanics and local hidden variables, if Y_N is used as the correlation measure, since neither the maximal expectation nor the maximal variance of Y_N are increased by memory-dependent strategies.

Now let us turn to X_N . We know that if the particles are described by identical LHV models, then $E(X_N) \leq 3$. Also, even when the particles have memory, equations (23-25) hold. Suppose we take $\delta = N^{-1/2+\epsilon}$, for some small $\epsilon > 0$, and let N be large enough that $\frac{3+\delta}{1-\delta} < 3 + 5\delta$. Then from (25), since X_N is always bounded by 4, we have that

$$\begin{aligned} E(X_N) &\leq 4P(X_N > 3 + 5\delta) + (3 + 5\delta)(1 - P(X_N > 3 + 5\delta)) \\ &\lesssim 3 + 5N^{-1/2+\epsilon} + 5\sqrt{3/2\pi}N^{-\epsilon} \exp(-N^{2\epsilon}/6), \end{aligned} \quad (28)$$

so that $(E(X_N) - 3)$ is bounded by a term that decays faster than $N^{-1/2+\epsilon}$, for any $\epsilon > 0$. This means that no LHV model can produce $E(X_N)$ much above 3 for large N ; it also means that the X_N remain efficient discriminators of quantum mechanics and local hidden variable theories even when the memory loophole is taken into account.

So far we have shown that the memory loophole makes no essential difference to Bell inequalities, so long as we use a large number of particles. We shall now show that if we only use a small number of particles, the 2-sided memory loophole does indeed make a difference. We shall give a memory-dependent LHV model with $E(X_N) > 3$. To construct a simple example, we take a model which gives $X_N = 3$ with certainty, and modify it a little so that the expectation increases above 3. We set $N = 101$. We can get $X_{101} = 3$, with certainty, simply by outputting +1 regardless of the observables measured. Our new model is identical to this one, except for the case when, after 100 measurements, we have measured (A_1, B_1) , (A_1, B_2) and (A_2, B_1) 33 times each, and (A_2, B_2) once. Our new model is allowed memory, so it can count how many times the various observables are measured, and thus tell when this is the case. In this (rather unlikely) case, the new model will output +1 on side A regardless of the measurement, and $B_1 = +1$, if measured, while $B_2 = -1$, if measured.

The two models will give identical values for X_{101} unless the above unusual state of affairs occurs after 100 rounds. Conditioned upon this event occurring, the old model still has an expectation of X_{101} equal to 3, whereas the new model has slightly more, almost 25/8. Since the expectation of the new model is 3 in all other cases, this increases the unconditional expectation of the new model to very slightly greater than 3.

The intuition behind the modification is that if one term in X_N (e.g. $\frac{\#_c(A_1, B_1)}{\#(A_1, B_1)}$) has a small denominator compared to another term, then we will gain more by increasing the numerator in the term with the small denominator than in the term with the big denominator.

Now that we have this model with $E(X_N) > 3$, it is easy to see how to modify it to make a model which does better. The idea is to start trying to increase the numerator in the best places from the start. In each round, there are 4 possible pairs of observables which could be measured ((A_1, B_1) , (A_1, B_2) , etc.). We can send a list which is guaranteed to give the correct sort of correlation or anticorrelation to at most 3 of the possible pairs, where we can choose which ones. So at each stage our model must choose one pair which, if measured, will give the wrong sort of correlation. After all the measurements are finished, the model would like to give the "incorrect" correlation to the pair of observables which has been measured most (since this term has the biggest denominator). There is no way for it to be sure of doing this, since it does not know at the start which pair will be measured most. So, our new model simply guesses.

More precisely, the improved model is as follows. In the first round of measurements it gives outcome +1, whatever is measured. From the second round it looks to see which pair, eg. (A_1, B_2) , has been measured most, and arranges that if that pair is measured in the next round, the correlations will be "incorrect", whereas if any other pair is measured in the next round the correlations will be "correct". It is easy to see this model produces $E(X_N) > 3$ for all N large enough that there is a negligible probability of one of the four observable pairs not being measured. Of course, our earlier bounds imply that $E(X_N) \rightarrow 3$ as $N \rightarrow \infty$. We conjecture that the model produces the maximum value of $E(X_N)$ attainable by a local hidden variable theory with 2-sided memory.

We comment briefly on the case of the 1-sided memory loophole, represented by a model of the form (5). We do not know whether such models can increase the value of $E(X_N)$ above 3, or come any closer to simulating quantum theory than memoryless LHV models. Note, however, that 1-sided memory models are a restricted class of the 2-sided memory models, and thus all the upper bounds proven for 2-sided models still apply. In particular, $E(Y_N) \leq 3$, and equation (21) still holds, ie. $P(Y_N > 3 + \delta) \approx \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{\delta\sqrt{N}} \exp(-\frac{1}{6}\delta^2 N)$. These are in fact tight bounds, since they can be obtained without any memory.

The two sided bounds also apply for X_N . However, we do not know whether they are tight: it may be that 1-sided memory LHV models are no more powerful than memoryless LHV models.

VII. SIMULTANEOUS MEASUREMENT LOOPHOLE

Although we have seen that the memory loophole gives LHV models some small wiggle room, it makes little essential difference. Both Y_N and X_N remain efficient discriminators between the predictions of quantum mechanics and of LHV models in the presence of the memory loophole. One might conjecture that this is also true of the simultaneous measurement loophole, and that this can be shown by essentially the same argument.

However, things are more complicated. It is true that $E(Y_N) \leq 3$, since equation (26) still holds, following the same reasoning as before. However, the fluctuations cannot be dealt with so easily. Our arguments to date have relied on the fact that $P(Y_N^n = 1) \leq 3/4$ holds true in any LHV model, not just a priori, but after conditioning on events up to round $(n - 1)$, even when the memory loophole allows the behaviour of the round n LHVs to depend on the earlier results. In particular we have that

$$P(Y_N^n = 1 \mid Y_N^{n-1} = i_{n-1}, \dots, Y_1^N = i_1) \leq 3/4, \quad (29)$$

for any values of i_1, \dots, i_{n-1} , in any memory-dependent LHV model.

However, the derivation of equation (29) relies on the fact that results from earlier rounds are necessarily uncorrelated with measurement choices from later rounds. This need not be true when the simultaneous measurement loophole can be exploited, as the following simple example illustrates.

Take $N = 2$, and consider simultaneously measured local hidden variables with the following outcome rules: on side A , the outcomes are $(1, 1)$ unless the operators measured are (A^1, A^2) , when the outcomes are $(1, 0)$; on side B , the outcomes are $(1, 1)$ unless the operators measured are (B^2, B^1) , when the outcomes are $(0, 1)$.

Here the outcomes and operators are ordered so that, for example, an outcome (i, j) means that i was obtained on the relevant particle from the first pair, and j was obtained on the relevant particle from the second pair. The pairs themselves are ordered by some convention: it does not matter which, so long as the ordering is consistent on each sides.

It is easy to verify that, in this model,

$$P(Y_2^1 = 1 \text{ and } Y_2^2 = 1) = 10/16, \quad (30)$$

whereas equation (29) would imply

$$P(Y_2^1 = 1 \text{ and } Y_2^2 = 1) \leq 9/16. \quad (31)$$

In other words, the simultaneous memory loophole allows an LHV model to increase the probability of getting a larger than expected value for Y_2 , beyond that attainable by any model in which the Y_N^n are independent random variables. The arguments of the preceding sections thus no longer apply.

We conjecture, nonetheless, that the predictions of quantum mechanics and of local hidden variables using the simultaneous measurement loophole can be discriminated by Y_N for large N . If so, then in theory the detector efficiency loophole could be countered by setting up an experiment in which a single pair of photons “simulates” N spin singlet states: i.e., many degrees of freedom of a single pair of photons are entangled, so that the joint state is isomorphic to the state of N singlets. One could then choose random measurements on each photon which simulate independent measurements on individual photons in the N singlets. Ignoring (admittedly somewhat unrealistically) losses in the beam-splitters used to set up the measurements, this means that results for all N simulated singlets are obtained whenever the detectors on both sides fire. If both detectors are of efficiency f , this will happen with probability f^2 — a gain of f^{2N-2} over the probability of obtaining a full set of results if N pairs of photons were separately measured. Choosing small ϵ , and taking N such that $P(Y_N > 3 + \epsilon) \ll f^2$ for any collective local hidden

variables model, would allow the hypothesis of collective local hidden variables to be refuted in a single successful experiment (which will take approximately f^{-2} attempts).

A strategy for combatting detector efficiency which uses the same basic idea of working with a highly entangled state of two photons, but is conceptually rather different, has been proposed by Massar [12].

VIII. CONCLUSION

We have seen that in the analysis of Bell-type experiments, one ought to allow for the possibility that the particles have memory, in the sense that outcomes of measurements on the n th pair of particles depend on both measurement choices and outcomes for the 1st, \dots , $(n - 1)$ th pairs. The standard form for local hidden variable models, originally due to Bell and summarized in equation (1), does not allow for this possibility, so a new analysis is needed. We have distinguished 1-sided and 2-sided versions of this loophole and shown that in the 2-sided case, a systematic violation of a Bell-type inequality can be obtained. In the case of the CHSH inequality, however, we have derived an upper bound on the probability of large deviations and thereby shown that the expected violation tends to zero as the number of particle pairs tested becomes large. Thus the CHSH inequality is robust against the memory loophole and the corresponding experimental tests remain good discriminators between quantum mechanics and local hidden variables — there is no need to design improved experiments in which more (or even all) measurements are space-like separated from one another.

We have also shown that if the analysis is performed in terms of the quantities Y_N , rather than X_N , then the memory models give no advantage over standard, memoryless, local hidden variables. Finally, we have considered a related loophole, the simultaneous measurement loophole, which would arise if Alice and Bob each performed all his measurements simultaneously, thus allowing for collective local hidden variables. We have seen that in this case, the probability of a significant deviation above the CHSH bound can be larger than would be allowed for a standard local hidden variable model. However, we suspect that this extra freedom is small, in the sense that the predictions of collective local hidden variables can be distinguished from those of quantum mechanics for a large enough number of particle pairs.

Note added

After this work was completed, we became aware of independent work by Gill [13], in which similar bounds on the probabilities of simulating quantum mechanical results via memory loophole local hidden variable models are presented. The existence of the memory loophole was independently noticed by Accardi and Regoli, [14] whose speculation that it might allow local hidden variables to simulate quantum mechanics is refuted by Gill's (and our) analyses.

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