Nonlinearity without Superluminality

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Quantum theory is compatible with special relativity. In particular, though measurements on entangled systems are correlated in a way that cannot be reproduced by local hidden variables, they cannot be used for superluminal signalling. As Czachor, Gisin, and Polchinski pointed out, this is not generally true of general nonlinear modifications of the Schrodinger equation. Excluding superluminal signalling has thus been taken to rule out most nonlinear versions of quantum theory. The no superluminal signaling constraint has also been used for alternative derivations of the optimal fidelities attainable for imperfect quantum cloning and other operations. These results apply to theories satisfying the rule that their predictions for widely separated and slowly moving entangled systems can be approximated by non-relativistic equations of motion with respect to a preferred time coordinate. This paper describes a natural way in which this rule might fail to hold. In particular, it is shown that quantum readout devices which display the values of localised pure states need not allow superluminal signalling, provided that the devices display the values of the states of entangled subsystems as defined in a non-standard, although natural, way. It follows that any locally defined nonlinear evolution of pure states can be made consistent with Minkowski causality.

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I. MOTIVATIONS

There are at least three good reasons to look for alternatives to quantum theory: the measurement problem, the difficulty in reconciling quantum theory with general relativity, and the desirability of finding new classes of theories against which certain quantum principles, such as linearity, can be tested. Yet it has proved rather difficult to find alternatives to quantum theory which respect the relativity principle and do not allow some form of superluminal signalling. For this and other reasons, the subtle relationship between quantum theory and special relativity is a source of continuing fascination.

Special relativity is not necessarily sacrosanct, of course, and moreover superluminal signalling need not be inconsistent with the relativity principle [1]. But the motivations just given suggest that alternatives to quantum theory which respect the relativity principle and do not allow superluminal signalling may be especially interesting and valuable. If the aim is to unify quantum theory and general relativity, abandoning the relativity principle or Minkowski causality seems an unpromising start. Also, one would prefer to test principles such as linearity by varying as little else as possible. For instance, if a test confirms a theory which respects linearity and relativity against a theory which respects neither, it is not so clear whether to interpret this as a confirmation of linearity or of relativity. And then, the very fact that respecting relativity and Minkowski causality seems to be difficult could be a hint that it is necessary. Constraints which are difficult (but not impossible) to satisfy are particularly interesting, since it would be nice to believe that the fundamental theory of nature is defined by a few compelling principles, rather than chosen arbitrarily from a large class of equally plausible possibilities.

All these points suggest reconsidering the relation between quantum theory and relativity.

II. ZWEISTEINE'S STATE READOUT MACHINE

Your colleague Zweisteine has long been a zealous admirer of special and general relativity but robustly sceptical about quantum theory. He reserves a special venom for the treatment of measurement within quantum theory. Naturally, the imprecision of the notion of measurement has not escaped his attention, and he believes that quantum theory needs to be augmented by a precise theory of state reduction. But he maintains also a less widely held view. He feels it is inconceivable that nature can have created objects so subtly intricate as quantum states, in such a form that we can access them only by the brutally destructive process encapsulated in the projection postulate. Positive operator valued measurements make him no happier: he sees them merely as projections applied to a larger

Hilbert space, bringing essentially the same unsatisfactory tradeoff between limited information gain and significant disturbance.

It must, he believes, be possible to access the information encoded in a state more directly and less destructively. Accordingly, he has for some years been working on a quantum state readout machine. This is supposed to accept a qubit — Mark I will be restricted to two dimensional systems — which it returns unaltered after printing out a high precision description.

Quite some time ago, you drew his attention to the no-cloning theorem [2] and related work [3]. He replied that these results illuminate very elegantly the limitations of quantum theory, and more generally the poverty of a universe limited to unitary or linear evolution laws. Fortunately, he added with an admonitory wag of the finger, we know from general relativity that nature is essentially non-linear.

More recently, after a particularly fraught departmental meeting, you were tactless enough to mention various papers that discuss the relation of quantum nonlinearity to superluminal signalling [4–7] and even query whether a natural construction of nonlinear theories is possible [8]. These cumulatively cast him into a state of great gloom, from which even the visit of an eminent Everettian, with whom he would normally have delighted in fencing, failed to rouse him.

Yet today, the spring is again in his step, a gleam of triumph in his eye. He has seen a way around the no-superluminal signalling constraint, he announces, and his state readout machine is complete. What can you do but indulge him? You prepare a qubit in state $|\psi\rangle = a|0\rangle + b|1\rangle$ in your lab, a being positive real and b complex, each specified to several decimal places. You bring it across, feed it into the machine. The printout reads $a|0\rangle + b|1\rangle$. You test the returned qubit, measuring P_ψ, and get the answer 1. A lucky guess, perhaps. After several similar experiments, though, another explanation seems required.

Whatever trickery is afoot, you know how to expose it. Your old colleague Bella, now based on Callisto, is happy to assist. This evening, she prepares a pair of particles in a singlet state,

$$(1/\sqrt{2})(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

and sends you the second particle. At noon tomorrow, universal time, she will carry out a projective measurement, in a basis of her choice, on the first particle. If she then reported the basis and result immediately by radio, the signal would reach you at 1pm. Guided by some faint premonition, though, you ask her to delay sending the signal for half an hour.

The next morning, you feed the entangled qubit into Zweisteine's machine. It whirrs, while you watch in amusement, and then prints out something surprising:

$$1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| \,.$$

Taking the returned qubit, you wait till 12.01, for the crucial test, and resubmit the qubit. The machine's opinion is unaltered:

$$1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| \,.$$

Aha! The machine's failed, as expected. The qubit is now in a pure state, not a mixture. You explain this, and your arrangement with Bella, to Zweisteine, who listens intently, and asks you nonetheless to continue.

So, at 12.59pm, you feed the qubit in again, and again read

$$1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| \,.$$

At 1.01pm you try once more, and for the second time that day are surprised by the printout:

$$c|0\rangle + d|1\rangle \,,$$

an opinion which the machine maintains as you desultorily resubmit the qubit over the next half hour. When Bella's radio message arrives at 1.30, you find she measured in the basis $c|0\rangle + d|1\rangle, \bar{d}|0\rangle - \bar{c}|1\rangle$, and obtained the second state.

This can't be fraud. Bella and Zweisteine have never met, and anyway she is entirely trustworthy. You remind yourself that, for all his eccentricities, and despite his scandalous neglect of the quant-ph arxiv, Zweisteine is a dedicated scientist, and a good one. He has been exploring unfamiliar physics, ranging from quantum effects in neurophysiology and consciousness to strong-field gravity, and not without success. In fact, some recent effects he's discovered are said by experts to be inexplicable by conventional theory. And his lab has, come to think of it, lately taken delivery of some specially bioengineered neural circuits and premium grade black holes.

You begin to reconsider...

III. WHAT COULD A PURE STATE READOUT DEVICE DESCRIBE?

Zweisteine's machine appears to be functioning as a genuine quantum state readout machine for pure states. When presented with a state of an entangled subsystem, it appears to recognise that it is entangled. However, it is apparently unaware of distant measurements that disentangle the state, until the point when information about those measurements could have reached it by light speed communication. What principles could it be following, consistent with quantum theory and relativity?

To simplify the notation, consider distinguishable pointlike particles located at fixed points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ in some inertial coordinate system $(\underline{x}, t)$, and that the particles' spatial wave function spread is negligible throughout the following discussion. The particles have some internal degrees of freedom, and their joint state is, we'll assume, entangled at $t = 0$:

$$|\psi(0)\rangle = \sum_{i_1 \ldots i_N} a_{i_1 \ldots i_N} |i_1\rangle_1 \ldots |i_N\rangle_N \, .$$

Suppose also that the particles have no mutual interactions and have been undisturbed, prior to $t = 0$, for a time long compared to their spatial separation, and remain so up to time $t_1 > 0$:

$$|\psi(t)\rangle = |\psi(0)\rangle \text{ for } -T < t < t_1 \, ,$$

where $T \gg \max_{i,j}(\|\underline{x}_i - \underline{x}_j\|)$.

What is the state of particle 1 at $t = 0$? The standard textbook answer is that it has no pure state, but is in an (improper) mixed state:

$$\rho_1(0) = \mathrm{Tr}_{2,\ldots,N}(|\psi(0)\rangle\langle\psi(0)|) \, .$$

We now want to consider how measurements affect the state. It will be assumed that measurement is an objectively definable process, and that a genuine state vector reduction takes place during measurement. That is, the quantum state of the measured system alters to one of the possible measurement outcomes; it does not enter into an entangled superposition with the apparatus which includes all the possible results. Of course, this is not everyone's favoured approach to the measurement problem. But it is one of the standard options. The aim here is to explore the scope for hypothetical readout devices and nonlinear theories under the assumption that it is correct.

Suppose now that a projective measurement is carried out on particle 2 at time $t_1 > 0$, and it is found to be in state $|j\rangle_2$. Write $P^j = |j\rangle\langle j|$ and $P_2^j = I \otimes P_j \otimes I \otimes \ldots \otimes I$. The textbook version of the projection postulate, the state of the full system is now, up to a normalisation factor,

$$|\psi(t_1)\rangle = P_2^j |\psi(0)\rangle \, ,$$

and the state of particle 1 is now, again up to normalisation,

$$\rho_1(t_1) = \mathrm{Tr}_{2,\ldots,N}(|\psi(t_1)\rangle\langle\psi(t_1)|) \, .$$

Generally, $\rho_1(t_1)$ and $\rho_1(0)$ will be different. On this account, the state of particle 1 has instantaneously changed as a result of a distant measurement on particle 2.

Of course, had we used a different reference frame, we would have found the state of particle 1 changing instantaneously at a different point on its worldline. Hence, famously, we cannot consistently maintain both the relativity principle and that the state of particle 1 — as defined by these calculations — represents an objective physical fact about the particle. In particular, if we go further and postulate a hypothetical device that reads out the value of the state as we have defined it, we need to assume the device functions with respect to some preferred reference frame, and it then allows instantaneous signalling in that frame over arbitrary distances.

The dilemma pointed out by EPR, of course, is that there is a plausible-seeming reason to think that the physical state of particle 1 really might be objectively defined by $\rho_1(t_1)$, not $\rho_1(t_0)$ after the measurement (and so one might think any sensible hypothetical state readout device *should* output $\rho_1(t_1)$ after the measurement). Namely, measurements on particle 1 after time t_1 have outcome probabilities in accordance with $\rho_1(t_1)$, not $\rho_1(t_0)$, and so the state of the particle, which is supposed to be the best available physical description, should be $\rho_1(t_1)$. But then the relativity principle suggests the state of particle 1 should have been $\rho_1(t_1)$ *before* time t_1. This leads us to introduce a local hidden variables hypothesis, and then Bell's theorem seems to refute this whole line of thought.

Could there, though, be a genuinely objective description of each of the particles that is weaker — in the sense that it is not always sufficient to reproduce all the predictions of quantum theory — but *is* consistent with relativity? Yes: in fact, there is a natural candidate, defined as follows. (See Figure 1.)

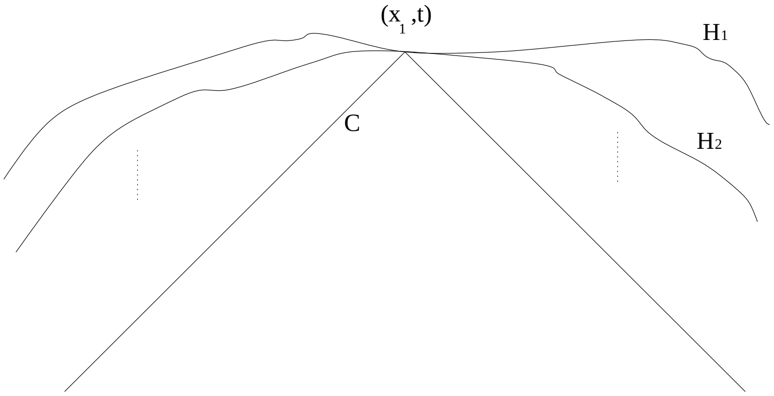

FIG. 1. Spacelike hypersurfaces tending to the past light cone

Consider C, the surface of the past light cone of particle 1 at time t. Take a family $\{H_n : n = 1, 2, \ldots\}$ of spacelike hypersurfaces which go through $(\underline{x_1}, t)$ and which asymptotically tend to C. Let $|\psi^n\rangle$ be the state vector of the joint system on H_n, and define

$$\rho^n = \mathrm{Tr}_{2,\ldots,N}(|\psi^n\rangle\langle\psi^n|) \, .$$

Finally, define the *local state* of particle 1 at time t to be

$$\rho_1^{loc} = \lim_{n\to\infty} \rho^n \, .$$

In words: the particle's local state is given by taking the joint wave function of the complete system, defined by allowing for only those projective measurements in the past light cone of the particle, and then tracing out the rest of the system.

Clearly, ρ_1^{loc} is Lorentz invariant. It also has a natural physical interpretation: $\rho_1^{loc}(\underline{x_1}, t)$ is the best possible description of the state obtainable by an observer located at $(\underline{x_1}, t)$. Such an observer can obtain ρ_1^{loc} by knowing the initial state and having arranged for radio signals of all measurement outcomes to be sent to him as soon as the measurements take place: he will thus have details of all measurements within the past light line cone of $(\underline{x_1}, t)$.

IV. LOCAL STATE READOUT DOES NOT ALLOW SUPERLUMINAL SIGNALLING

The above construction of ρ^{loc} has another significant implication. Assuming that standard quantum theory is correct, that we know the initial state of a system, and that we can identify all measurement events on that system and obtain their results, we could in principle construct a local state readout machine emulator for the system — that is, a device that will have the same operational action as a local state readout device for local subsytems.

To do this would require complete information about the system's hamiltonian and ideal technology — communication devices set up everywhere that broadcast signals reporting the results of measurements at light speed in all directions, and computers set up everywhere that carry out arbitrarily fast calculations. Given these things, and the value of the initial state, we can program the computers to take account of all measurement results as soon as the signal reporting them arrives, and use these together with knowledge of the hamiltonian evolution in the past light cone to calculate the local state and print it out. All of this can be done classically: the computers do not need to carry out any additional measurements on the system or disturb its quantum state in any way. Hence they emulate the state readout device, as required, by producing the state's value while leaving it undisturbed.

Obviously, these assumptions are unrealistic. We do not know the initial state of the universe, nor can we identify all measurements in our past light cone, nor can we construct the ideal technology required. But none of these assumptions contradicts standard quantum theory: each of them can consistently be added to it without changing the underlying theory. Hence, since standard quantum theory does not allow superluminal signalling, nor does quantum theory augmented by devices which emulate local state readout machines. And since there is no operational distinction between a local state readout machine emulator and a local state readout machine, quantum theory augmented by genuine local readout readout machines does not allow superluminal signalling either (happily for Zweisteine).

V. EMULATING NONLINEAR THEORIES

Given the hypothesis of local state readout machines, we can go further and devise experiments in which the hamiltonian acts on the quantum state as usual defined, but is defined in terms of fields which depend locally on the local quantum state. To construct such experiments, we would simply need to connect the readout to another device which controls an applied field. For instance, given a system of separated qubits and with some fixed basis, we could arrange for the hamiltonian to include a term $\pi/4\langle 0|\rho_{\rm loc}|0\rangle\sigma_z$. More generally, we could implement any locally varying nonlinear evolution laws of our choice *provided that the nonlinearity arises through dependence on the local state.*

Now, we have already seen that a device operationally indistinguishable from a local state readout device could be constructed within standard quantum theory, given sufficient knowledge and computational power, and hence that such a device does not allow superluminal signalling. It follows that superluminal signalling cannot be possible in any experiment of this type. But these experiments emulate a situation in which nature (through presently unknown physics) uses locally varying nonlinear evolutions that depend on the local state. Hence no theory of this type can allow superluminal signalling either.

VI. CONCLUSIONS

Could Zweisteine be right? Might unknown physics give a direct way of carrying out local state readout, or at least some partial information about the local state, on general quantum systems? Despite all the lessons of Bell's theorem, there is still some attraction in the idea that there is *something* objectively "there" in a localised part of an entangled quantum system. If not the local state, what?

Suppose, for instance, that, as has sometimes been speculated, that the gravitational field is actually fundamentally classical, while matter is quantum. The gravitational field then has to couple to some object defined by the quantum realm, and the local state seems a plausible candidate. One might also wonder whether a theory of consciousness, which (according to one line of thought) has to attach consciousness to some definite physical quantity, might possibly use local quantum states.

The problem, of course, in taking these thoughts beyond coffee table speculation into specific detail is that infinitely many local state dependent evolution laws could be written down. One of the initial hopes — that requiring consistency with special relativity might reduce the number of nonlinear theories to a few candidates — has not been fulfilled. Perhaps it might be possible to identify a restricted class of sensible ansätze for coupling the local state to gravity, though.

These speculations aside, the fact remains that a theory which implies nonlinear evolution of pure quantum states need not allow superluminal signalling, or otherwise violate relativity. With this concern lifted, testing quantum linearity seems a more respectable enterprise than it has lately been painted.

VII. ACKNOWLEDGEMENTS

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