Inferring superposition and entanglement from measurements in a single basis

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We discuss what can be inferred from measurements on one and two qubit systems using a single measurement basis at various times. We show that, given reasonable physical assumptions, carrying out such measurements at quarter-period intervals is enough to demonstrate coherent oscillations of one or two qubits between the relevant measurement basis states. One can thus infer from such measurements alone that an approximately equal superposition of two measurement basis states has been created in a coherent oscillation experiment. Similarly, one can infer that a near maximally entangled state of two qubits has been created in an experiment involving a putative SWAP gate. These results apply even if the relevant quantum systems are only approximate qubits. We discuss applications to fundamental quantum physics experiments and quantum information processing investigations.

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INTRODUCTION

It is well known from tomography that given many copies of a quantum state, measuring enough observables allows reconstruction of the state [1] However, the relevant quantum tomographic schemes generally require measurements in more than one basis. In many systems this doesn't present any difficulties, for example polarisation measurements on photons are easily carried out in any basis. However for many condensed matter qubit systems, there is usually just one natural measurement basis, often defined to be the computational basis or, in spin-1/2 notation, the z-basis: $|0\rangle = |\uparrow_z\rangle$ and $|1\rangle = |\downarrow_z\rangle$. This is the motivation for this paper. We would like to know what, if anything, can be inferred about the evolution of a one or two qubit system if measurements on each qubit are restricted to a single basis, but can be made at different times. Specifically, we are interested in the problems of demonstrating coherent oscillations between $|0\rangle$ and $|1\rangle$ in an experimental arrangement that is meant to function as a quantum NOT gate, and between $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ in an arrangement that is meant to function as a quantum SWAP gate.

Consider the first example. The standard coherent oscillation scenario is ideally realised by arranging for the spin qubit Hamiltonian to be of the form $H = -\frac{\Gamma}{2}\sigma_x$, with $\hbar$ set to 1. Inputting either computational basis (σ_z) eigenstate, the probabilities of finding the system in these basis states oscillate coherently with period $2\pi\Gamma^{-1}$. With a dimensionless time $T \equiv \Gamma t$, a single oscillation is complete at $T = 2\pi$. A NOT gate is effected by the action of H for a time $T = \pi$ and a root NOT by acting for $T = \pi/2$.

The usual approach to inferring that a superposition $|0\rangle + |1\rangle$ really was created at $T = \pi/2$ in an experiment is to plot out the sinusoidal oscillations of the computational basis measurement probabilities in detail. This is done by preparing, evolving for a small fraction of $T = 2\pi$, measuring in the computational basis, repeating many times to build up statistics of $p(0)$ and $p(1)$ for this value of T, then incrementing T and repeating the whole procedure until one reaches $T = 2\pi$.

Coherent oscillation of the quantum state appears much the simplest and likeliest explanation of a sinusoidal oscillation in the measurement probabilities. One might worry, though, that, some other evolutions could also be consistent with the data. For example, measurement probabilities $p(0) = p(1) = 1/2$ at $\pi/2$ would also be obtained if the system generated a mixed state $|0\rangle\langle 0| + |1\rangle\langle 1|$, rather than the desired pure state $|0\rangle + |1\rangle$, at this point. Conversely, one might wonder whether carrying out measurements at many evolution times is necessary or even helpful for inferring the state at $\frac{\pi}{2}$.

The tomographer's task is further complicated by the fact that real world plots will at best only be approximately consistent with ideal coherent oscillation: for example, the experimental oscillation amplitudes will decay over time. Moreover, the relevant quantum system may only be an approximate qubit: it may have a small but nonzero probability of being found in states other than $|0\rangle$ or $|1\rangle$ during the experiment. One would ideally like to be able to estimate how close the state generated under these conditions is to the superposition obtained by an ideal evolution.

In this paper we show that, in fact, the existence of coherent superpositions can be inferred simply by carrying out computational basis measurements at the relevant times, without a full sinusoidal oscillation plot. This is true for both the NOT and SWAP gate evolutions, and even for approximate qubits and imperfect evolutions. In all cases a reasonably simple bound on the fidelity to an ideal superposition state can be obtained. Our arguments require fairly minimal and physically reasonable assumptions (which are necessary to infer coherent superposition even if measurements are made at many different evolution times). Our procedure thus simplifies the experimenter's task considerably, without weakening the evidence for the results inferred.

Textbook discussions of quantum computing often assume the qubits are exact, in that they correspond to two-dimensional quantum systems such as the polarisation states of a single photon. However, as we have already noted, real world qubits may be approximate, corresponding to systems which live in a larger Hilbert space but have parameters set so that two of the states have very small transition probabilities to any of the rest. The computational basis states of an approximate qubit are often the two lowest-lying energy eigenstates of the system (or orthogonal superpositions). Two interesting condensed matter approximate qubits are based on superconductivity. The computational basis states can either be two charge states of a microscopic superconducting island (or box) differing by a single Cooper pair [2], or two flux states of a closed superconducting ring [3, 4].

The former is interesting because coherent oscillations have already been seen in this approximate charge qubit [2], so this system is one of a number of good candidates for condensed matter qubits for quantum computing. The latter flux system may also be a useful qubit. In addition, it is very interesting from the fundamental quantum physics viewpoint. Two flux states of a superconducting ring, differing by the flux quantum $\Phi_0 = h/2e$, correspond to macroscopically distinct circulating currents in the ring, so a superposition of two such states would be a real analogue of Schrödinger's cat. Recent flux experiments [3, 4] have not demonstrated coherent flux oscillations, but have provided evidence for flux superpositions in the frequency domain, in effect through spectroscopy. Experiments to demonstrate coherent flux oscillations are underway. If successful, these will be recognised as clear evidence for Schrödinger cat superpositions (in the same way that charge oscillations [2] have been recognised as showing clear evidence for charge superpositions). The single approximate qubit results we present here are particularly relevant for the ongoing flux oscillation experiments.

For two qubit systems there are a number of potential gates that can be investigated. Here however we restrict our attention to the SWAP and root SWAP gates: as is well known, universal quantum computations can be carried out by combining root SWAP with single qubit gates. For a system of two qubits, a and b, these gates can be realised with an interaction Hamiltonian of the Heisenberg form $H = \frac{J}{4}\underline{\sigma}_a.\underline{\sigma}_b$. This interaction is relevant for electron spins in coupled quantum dots [5] and other spin realisations of qubits in condensed matter systems. The computational basis states $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ swap after a dimensionless time $Jt \equiv T = \pi$. After a time $T = \pi/2$ a root SWAP gate is effected [5]. Approximate implementations of root SWAP have also been proposed for other condensed matter qubit systems, electron on helium [6] and Coulomb coupled quantum dots [7] where the approximate qubits are fictitious spins.

PHYSICAL ASSUMPTIONS

Our fundamental assumption is that, in any experiment we consider, we can characterise the Hilbert space corresponding to the quantum system(s) of interest, and distinguish system degrees of freedom from those corresponding to the rest of the experimental apparatus, thermal radiation, and the rest of the outside world, all of which we collectively refer to as the environment. So the total Hilbert space $\mathcal{H}_{\text{total}}$ can be factored as

$$\mathcal{H}_{\text{total}} = \mathcal{H}_S \otimes \mathcal{H}_E \text{ or } \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \otimes \mathcal{H}_E \,,$$

for a one or two qubit system respectively. Exact qubits correspond to two dimensional spaces $\mathcal{H}_S$ and $\mathcal{H}_{S_i}$ with an orthonormal basis $|0\rangle, |1\rangle$. For approximate qubits we can without loss of generality consider countably infinite dimensional spaces with a basis $|0\rangle, |1\rangle, |2\rangle, \ldots$, with the two distinguished states $|0\rangle$ and $|1\rangle$ defining the qubit computational basis.

We assume that the system-environment interactions are such that we can initialise the system in any computational basis state without materially affecting the environment and can treat the environment as effectively constant during subsequent evolutions. More precisely:

(i) the state of the experiment at any time can be characterised by a density matrix ρ_S or $\rho_{S_1 \otimes S_2}$ describing (only) the system state.

(ii) the evolution of the system between any times t and $t' > t$ is described by a quantum operation $\mathcal{E}_{t,t'}$ which is trace-preserving, convex linear on density matrices and completely positive.

(iii) the operations $\mathcal{E}_{t,t'}$ is independent of the initial state of the system and is the same in each experimental run.

We turn now to the detailed analyses of the various qubit evolutions and what may be inferred about superposition and entanglement from measurements.

AN EXACT QUBIT

There is a particularly simple strategy for inferring superpositions in the case of an experiment testing a prototype NOT gate by attempting to produce coherent oscillations of a single exact qubit. We run the experiment setting the initial system state to be either of the two computational basis states $\rho_0 = |0\rangle\langle 0|$ and $\rho_1 = |1\rangle\langle 1|$. Assume that we have identified, either theoretically or empirically, a time t such that $\mathcal{E}_t \equiv \mathcal{E}_{0,t}$ implements an approximate NOT operation. We can quantify the degree of approximation by carrying out a series of experiments in which computational basis measurements are performed at time t and estimating the probabilities of obtaining 0 and 1 from the states evolved from ρ_1 and ρ_0. We write

$$p_i^j \;=\; \langle i|\mathcal{E}_t(\rho_j)|i\rangle \,. \tag{1}$$

To proceed we need the following definitions and results[8]. The *trace distance* between states ρ and ρ' is defined as $D(\rho,\rho')=\frac{1}{2}\mathrm{Tr}|\rho-\rho'|$, where $|A|=(A^\dagger A)^{\frac{1}{2}}$ is the positive square root of $A^\dagger A$. It has the following properties. First, D is a metric: $D(\rho,\rho')=0$ if and only if $\rho=\rho'$, $D(\rho,\rho')=D(\rho',\rho)$ and $D(\rho,\rho'')\leq D(\rho,\rho')+D(\rho',\rho'')$. Second, D is non-increasing under trace-preserving quantum operations: $D(\mathcal{E}(\rho),\mathcal{E}(\rho'))\leq D(\rho,\rho')$. Third, the *fidelity* $F(\rho,\rho')=\mathrm{Tr}(\rho^{\frac{1}{2}}\rho'\rho^{\frac{1}{2}})^{\frac{1}{2}}$ and D obey

$$1-F(\rho,\rho')\leq D(\rho,\rho')\leq(1-F(\rho,\rho')^2)^{\frac{1}{2}}\,. \tag{2}$$

If our system really is an exact qubit, we find that $p_0^1+p_1^1=1=p_0^0+p_1^0$. Geometrically, $\mathcal{E}_t(\rho_1)$ lies in the intersection of the Bloch sphere with a plane at height $2p_0^1-1$ above the equator and parallel to it, while $\mathcal{E}_t(\rho_0)$ lies in the intersection of the sphere with a parallel plane $2p_1^0-1$ below the equator. Since the trace distance is half the Euclidean distance within the Bloch sphere, we have that

$$D(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_0))\geq(p_0^1+p_1^0-1)\,.$$

As the trace distance is non-increasing during quantum evolution, the two qubits evolved from ρ_0 and ρ_1 must always have been separated by at least this distance before time t. We can picture the qubit evolution as defining that of a perhaps contracting rod (whose endpoints are the qubits) moving inside the Bloch sphere. It is then easy to see that, for them to have swapped hemispheres by time t and maintained at least the above separation throughout, at least one of them must have had fidelity no less than $(\frac{p_0^1+p_1^0}{2})^{\frac{1}{2}}$ to a maximally superposed state of the form $|0\rangle+e^{i\phi}|1\rangle$ at some time before t. Hence we can infer the creation of a near-maximally superposed state during the experiment — though note that this argument does not identify which superposed state was approximated, or when.

AN APPROXIMATE QUBIT

Now suppose our system is only approximately characterised by a qubit. Clearly, any argument for coherent oscillations will need some measure of how closely the evolved state resembles a qubit. For any given state ρ that can be repeatedly prepared, we can obtain a good measure as follows. Define P to be the projection onto the two-dimensional qubit space, and note that if p_0^ρ and p_1^ρ are the probabilities of getting outcomes 0 and 1 when measuring in the computational basis, we can obtain $\mathrm{Tr}(P\rho)=p_0^\rho+p_1^\rho$. Let $\rho_P=P\rho P/\mathrm{Tr}(\rho P)$ be the normalised density matrix given by projecting ρ into the qubit space. Then $F(\rho_P,\rho)=(\mathrm{Tr}(P\rho))^{\frac{1}{2}}=(p_0^\rho+p_1^\rho)^{\frac{1}{2}}$ and so we have

$$D(\rho_P,\rho)\leq(1-p_0^\rho-p_1^\rho)^{\frac{1}{2}}\,.$$

We can now proceed as previously and obtain estimates for the probabilities (1) associated with an approximate NOT evolution, giving us that

$$\begin{aligned} D(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_0)) \;&\geq\; D(\mathcal{E}_t(\rho_1)_P,\mathcal{E}_t(\rho_0)_P)-D(\mathcal{E}_t(\rho_0),\mathcal{E}_t(\rho_0)_P)-D(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_1)_P) \\ &\geq\; (q_0^1+q_1^0-1)-(1-p_0^0-p_1^0)^{\frac{1}{2}}-(1-p_0^1-p_1^1)^{\frac{1}{2}}\,, \end{aligned} \tag{3}$$

where $q_0^1=p_0^1/(p_0^1+p_1^1)$ and $q_1^0=p_1^0/(p_0^0+p_1^0)$.

To complete the argument, we need some way of bounding the possible deviation from an exact qubit at a point when we expect a near-maximally superposed state to have been created. A general bound of the form $\mathrm{Tr}(P\mathcal{E}_{t'}(\rho_i))\geq 1-\delta$ for all times $t'<t$ and for $i=0,1$ would suffice. However, getting good empirical evidence for such a bound would require carrying out a series of measurements at many times between 0 and t, which would require the same amount of experimental labour as plotting sinusoidal oscillations, and would still leave the worry that the deviation might have been greater at some unmeasured intervening point.

A more watertight procedure is to identify empirically a time $t_{1/2}<t$ (for coherent oscillations we expect $t_{1/2}\approx t/2$) such that, if we write $r_i^j=\langle i|\mathcal{E}_{t_{1/2}}(\rho_j)|i\rangle$, we have $r_0^0\,,r_1^0\,,r_0^1\,,r_1^1\approx 1/2$. We can then directly argue that the evolution came close to a maximally superposed state at $t_{1/2}$ as follows.

Writing $\sigma_i=\mathcal{E}_{t_{1/2}}(\rho_i)$ and $\sigma_i^P=P\sigma_iP/\mathrm{Tr}(\sigma_iP)$ for $i=0,1$ we have $F(\sigma_i^P,\sigma_i)=(\mathrm{Tr}(P\sigma_i))^{\frac{1}{2}}=(r_0^i+r_1^i)^{\frac{1}{2}}$ and

$$D(\sigma_0^P,\sigma_1^P) \ \geq \ D(\sigma_0,\sigma_1) - D(\sigma_0,\sigma_0^P) - D(\sigma_1,\sigma_1^P) \qquad (4)$$
$$\geq \ (q_0^1+q_1^0-1) - (1-p_0^0-p_1^0)^{\frac{1}{2}} - (1-p_0^1-p_1^1)^{\frac{1}{2}} - (1-r_0^0-r_1^0)^{\frac{1}{2}} - (1-r_0^1-r_1^1)^{\frac{1}{2}} \, .$$

We will write the right hand side of this last inequality as $1-\epsilon'$ and let $s_j^i = r_j^i/(r_0^i+r_1^i)$, with $\Delta s_j^i = s_j^i - 1/2$. A little Bloch sphere trigonometry shows that at least one of the states σ_i^P must lie in a cap, centred on a maximally superposed state, whose Euclidean height is no more than

$$\delta = 1 - \sqrt{(1-\epsilon')^2 - (\Delta s_0^1 + \Delta s_1^1)^2} \, .$$

The relevant σ_i^P is separated from the maximally superposed state $|\psi_M\rangle$ by trace distance no more than $(\frac{\delta^2}{4} + (\Delta s_0^i)^2)^{\frac{1}{2}}$, and thus we have

$$\begin{aligned} D(\sigma_i,|\psi_M\rangle) \ &\leq \ D(\sigma_i,\sigma_i^P) + D(\sigma_i^P,|\psi_M\rangle) \qquad (5) \\ &\leq \ (1-r_0^i-r_1^i)^{\frac{1}{2}} + (\frac{\delta^2}{4} + (\Delta s_0^i)^2)^{\frac{1}{2}} \\ &\leq \ \max_j\{(1-r_0^j-r_1^j)^{\frac{1}{2}} + (\frac{\delta^2}{4} + (\Delta s_0^j)^2)^{\frac{1}{2}}\} \, . \end{aligned}$$

This last expression is experimentally measurable and bounds the separation between one of the evolved states and a maximally superposed state, although the argument does not identify either state.

ENTANGLEMENT GENERATION VIA CANDIDATE SWAP GATES

Consider now an experiment implementing a candidate SWAP gate on a system of two exact or approximate qubits. If we initialise the system in the state $|0\rangle|1\rangle$, an exact exchange interaction would evolve the system to $|1\rangle|0\rangle$ via a maximally entangled state of the form $|\psi_M\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + e^{i\phi}|1\rangle|0\rangle)$. We can consider the two qubits as defining a single effective qubit in the two dimensional space with computational basis $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$. If the system state is initially in this space, it will, to good approximation, remain there provided that the evolution is well approximated by an exchange interaction. Our results for the NOT evolution of approximate qubits thus apply directly.

Moreover, by carrying out appropriate measurements, we can verify that an entangled state is generated during the candidate SWAP evolution and give figures of merit for the entanglement. Define the projection Q to be on the four dimensional space defined by the two qubits. A measurement of Q on the state σ_i would produce the exact qubit state $\sigma_i^Q = Q\sigma_i Q/\mathrm{Tr}(Q\sigma_i)$ with probability $p_Q = \sum_{l=0,1;m=0,1} p_{lm}^i$, where $p_{lm}^i = \langle l|\langle m|\sigma_i|l\rangle|m\rangle$. For any maximally entangled state $|\psi_M\rangle$ we have

$$\begin{aligned} \max_i \langle\psi_M|\sigma_i^Q|\psi_M\rangle \ &\geq \ \min_j\{\frac{p_{01}^j + p_{10}^j}{p_{00}^j + p_{10}^j + p_{01}^j + p_{11}^j}\} \times \\ &\qquad \max_k\{\langle\psi_M|\sigma_k^P|\psi_M\rangle\} \, . \qquad (6) \end{aligned}$$

As we can calculate both expressions on the right hand side, we have a lower bound on the fidelity of a σ_i^Q to a maximally entangled state. If this is greater than one half, the relevant σ_i^Q is entangled, and as σ_i^Q can be obtained from σ_i by local measurements, σ_i is also entangled. The advantage of this procedure over more efficient ways of measuring entanglement[9] is that it requires no gates other than the one being tested.

DISCUSSION

We have shown that, if a candidate NOT gate acts on an exact qubit, the qubit must pass close to an equal weight superposition of $|0\rangle$ and $|1\rangle$ if the NOT operation is demonstrated to work accurately on both basis states as inputs. For an approximate qubit measurements are also needed half way through the NOT operation, to check that the system has not wandered far out of the qubit subspace. However, in both cases superposition can be inferred without plotting oscillations in detail, given reasonable physical assumptions.

These physical assumptions should hold true in the types of experiment we consider. However, it should be stressed that they could in principle be violated given sufficient pathologies. To take an extreme example, one could imagine a single qubit experiment in which, unknown to the experimenter, the environment happened to contain a second qubit, coupled to the first by a fast but only infrequently acting swap gate. The initially prepared computational basis state in the first qubit could then be exchanged with a maximally mixed state in the second, rotated via a fortuitous NOT operation in the second qubit, and then exchanged back into the first. Applying our inequalities would lead to the incorrect conclusion that the first qubit evolves through a near-maximal superposition state. In fact, in this case, such a state is indeed created, but in the second qubit lurking in the environment. Of course, it is very unlikely indeed that any experiment will accidentally incorporate a hidden qubit or anything similar.

The arguments used to infer a superposition in a single qubit from computational basis measurements can also be used to infer entanglement in certain multi-qubit systems. For a two qubit system arranged to effect a SWAP operation, demonstration that this works well for the two inputs $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ and measurements at the half way point (root SWAP) showing that the system hasn't wandered far out of the subspace spanned by these states are sufficient to infer that the system passed close to a maximally entangled state. Fidelities can be bounded and the existence of entanglement established, even for states well away from being maximally entangled. Once again, all this works whether the qubits are exact or approximate.

Clearly, for candidate condensed matter systems to be feasible qubits for quantum computation, reliable single qubit gates will eventually need to be constructed. This would remove the need to restrict to measurements in the computational basis, and would make tomography much simpler. However, state of the art is currently well short of this goal. At present, experimenters are trying to realise approximate one and two condensed matter qubit gates and test them. In particular, generating and demonstrating two qubit entanglement remains a goal. We believe that the results presented here should be extremely useful during the present investigative period. Even in the longer term, our results should be relevant for fundamental investigations such as creating multi-qubit GHZ states or entangling two Schrödinger cats, which may well be carried out on systems where our measurement restrictions apply.

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