



## Efficient Quantum Algorithms for Estimating Gauss Sums

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# Efficient Quantum Algorithms for Estimating Gauss Sums

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## Abstract

We present an efficient quantum algorithm for estimating Gauss sums over finite fields and finite rings. This is a natural problem as the description of a Gauss sum can be done without reference to a black box function. With a reduction from the discrete logarithm problem to Gauss sum estimation we also give evidence that this problem is hard for classical algorithms. The workings of the quantum algorithm rely on the interaction between the additive characters of the Fourier transform and the multiplicative characters of the Gauss sum.

## 1 Introduction

Let  $\chi : R \rightarrow \mathbb{C}$  be a multiplicative character and  $e : R \rightarrow \mathbb{C}$  an additive character over a finite ring  $R$ . The *Gauss sum*  $G$  of this triplet  $(R, \chi, e)$  is the inner product between  $\chi$  and  $e$ , that is:  $G(R, \chi, e) := \sum_{x \in R} \chi(x)e(x)$ . Gauss sums are useful on many fronts for the analysis of finite fields  $R = \mathbb{F}_p$  and rings  $R = \mathbb{Z}/n\mathbb{Z}$ . In combination with the closely related *Jacobi sums*, they have been used to prove theorems about Diophantine equations, difference sets, primality testing, et cetera. One can view Gauss sums as the finite versions of the gamma function  $\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx$ . See Brendt et al. [1] for a book entirely devoted to these topics.

The theory of quantum computation investigates if the laws of quantum physics allow us to process information in a more efficient way than is possible by the classical, Turing machine model of computation. Strong support for the claim that quantum computers are indeed more powerful than classical ones was given in 1994 by Peter Shor who proved the existence of efficient quantum algorithms for factoring and the discrete logarithm problem [12]. More recently, Hallgren showed that also Pell's equation can be solved in polynomial time on a quantum computer [8]. The common ingredient of these (and other) quantum algorithms is the use of quantum Fourier transform to extract the periodicity of an unknown function in time logarithmic in the size of the domain. See the book by Nielsen and Chuang for a thorough introduction to this field [11].

In this article we describe a quantum algorithm that, given the specification of the characters  $\chi$  and  $e$ , efficiently approximates the corresponding Gauss sum, for  $R$  a finite field  $\mathbb{F}_p$  or a 'mod  $n$ ' ring  $\mathbb{Z}/n\mathbb{Z}$ . Because determining the norm  $|G|$  of a Gauss sum is relatively straightforward, our algorithm focuses on estimating the angle  $\gamma \bmod 2\pi$  in the equation  $G = |G| \cdot e^{i\gamma}$ . We describe a quantum transform that induces this angle as a relative phase by a mapping  $|0\rangle + |1\rangle \mapsto |0\rangle + e^{i\gamma}|1\rangle$ . Because this transformation can be implemented efficiently, we can sample the output state  $O(\frac{1}{\varepsilon})$  times to get an estimation  $\tilde{\gamma}$  of the angle  $\gamma$  with expected error  $\varepsilon$ . The time complexity of this algorithm is  $O(\frac{1}{\varepsilon} \cdot \text{polylog}|R|)$ . Using a reduction from the discrete log problem to the approximation of Gauss sums, we provide evidence that this is a hard task on a classical computer. A discussion on the merits of Gauss sum estimation is included at the end of the article.

Section 2 gives the definitions and known results that we will use for the estimation of Gauss sums over finite fields. The basic quantum procedures that we use for our algorithm are defined in Section 3; the algorithm itself is described in Section 4. Next, we discuss the possibility of estimating Gauss sums with classical algorithms. The relationship between this problem and the discrete logarithm problem, Galois

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automorphisms and random walks is explained in Section 5. Section 6 gives some background that is necessary to define the Gauss sum problem for finite rings. A quantum algorithm for this problem is given in Section 7. The final Section 8 discusses the connection between the presented algorithm for Gauss sum estimation and other quantum algorithms. Also the relative hardness of the problem with respect to other known problems is addressed. Throughout the article, results that are already known are indicated as ‘facts’.

## 2 Gauss Sums over Finite Fields

### 2.1 Definitions and Notation: Gauss Sums over Finite Fields

Let  $\zeta_p$  denote the  $p$ th root of unity:  $\zeta_p := e^{2\pi i/p}$ . The *trace* of an element  $x$  of the finite field  $\mathbb{F}_{p^r}$  over  $\mathbb{F}_p$  is  $\text{Tr}(x) := \sum_{j=0}^{r-1} x^{p^j}$ . It can be shown that for every  $x \in \mathbb{F}_{p^r}$ , its trace is an element of the base-field:  $\text{Tr}(x) \in \mathbb{F}_p$ . For any  $\beta \in \mathbb{F}_{p^r}$  we also have the related functions  $x \mapsto \text{Tr}(\beta x)$ . These trace functions are all the linear functions  $\mathbb{F}_{p^r} \rightarrow \mathbb{F}_p$  (note that  $\beta = 0$  gives the trivial function 0). When we write  $\zeta_p^{\text{Tr}(x)}$  we interpret the value  $\text{Tr}(x)$  as an element of the set  $\{0, 1, \dots, p-1\} \subset \mathbb{Z}$ . For  $\beta \in \mathbb{F}_{p^r}$ , the functions  $e_\beta(x) := \zeta_p^{\text{Tr}(\beta x)}$  describe all possible additive characters  $\mathbb{F}_{p^r} \rightarrow \mathbb{C}$ .

Let  $g$  be a primitive element of  $\mathbb{F}_{p^r}$ , i.e. the multiplicative group  $\langle g \rangle$  generated by  $g$  equals  $\mathbb{F}_{p^r}^*$ . For each  $0 \leq \alpha \leq p^r - 2$ , the function  $\chi(g^j) := \zeta_{p^r-1}^{\alpha j}$  (complemented with  $\chi(0) := 0$ ) is a multiplicative character  $\mathbb{F}_{p^r} \rightarrow \mathbb{C}$ . Also, every multiplicative character can be written as such a function. For a non-zero  $x \in \mathbb{F}_{p^r}^*$ , the discrete logarithm with respect to  $g$  is defined by  $\log_g(g^j) := j \bmod p^r - 1$ . Hence, every multiplicative character can be expressed by  $\chi(x) := \zeta_{p^r-1}^{\alpha \log_g(x)}$  for  $x \neq 0$  and  $\chi(0) := 0$ . The trivial multiplicative character is denoted by  $\chi^0$  and is defined by  $\chi^0(0) = 0$  and  $\chi^0(x) = 1$  for all  $x \neq 0$ . Using the equality  $\zeta_{p^r-1}^{\alpha \log_g(x)} \zeta_{p^r-1}^{\beta \log_g(x)} = \zeta_{p^r-1}^{(\alpha+\beta) \log_g(x)}$ , it is easy to see that the pointwise multiplication between two characters establishes the isomorphism  $\hat{\mathbb{F}}_{p^r}^* \simeq \mathbb{Z}/(p^r - 1)\mathbb{Z}$ .

**Definition 1 (Gauss sums over Finite Fields)** *For the finite field  $\mathbb{F}_{p^r}$ , the multiplicative character  $\chi$ , and the additive character  $e_\beta$ , we define the Gauss sum  $G$  by*

$$G(\mathbb{F}_{p^r}, \chi, \beta) := \sum_{x \in \mathbb{F}_{p^r}} \chi(x) \zeta_p^{\text{Tr}(\beta x)}. \quad (1)$$

**Example 1** *Let  $\chi : \mathbb{F}_5 \rightarrow \{0, 1, -1, i, -i\}$  be the multiplicative character defined by:  $\chi(0) = 0$ ,  $\chi(1) = 1$ ,  $\chi(2) = i$ ,  $\chi(3) = -i$  and  $\chi(4) = -1$ . We see that  $G(\mathbb{F}_5, \chi, 1) = \zeta_5 + i\zeta_5^2 - i\zeta_5^3 - \zeta_5^4 = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}(1 - \sqrt{5} - 2i) = \sqrt{5} \cdot e^{2\pi i \cdot 0.338\dots}$ .*

Obviously,  $G(\mathbb{F}_{p^r}, \chi^0, 0) = p^r - 1$ ,  $G(\mathbb{F}_{p^r}, \chi^0, \neq 0) = -1$ , and  $G(\mathbb{F}_{p^r}, \chi, 0) = 0$  for  $\chi \neq \chi^0$ . In general, for  $\beta \neq 0$  we have the following fact.

**Fact 1** *For  $\beta \neq 0$  it holds that  $G(\mathbb{F}_{p^r}, \chi, \beta\delta) = \chi(\beta^{-1})G(\mathbb{F}_{p^r}, \chi, \delta)$ .*

**Proof:** For  $G(\mathbb{F}_{p^r}, \chi, \beta\delta)$  we have  $\sum_{x \in \mathbb{F}_{p^r}} \chi(x) \zeta_p^{\text{Tr}(\beta\delta x)} = \chi(\beta^{-1}) \sum_{z \in \mathbb{F}_{p^r}} \chi(z) \zeta_p^{\text{Tr}(\delta z)}$ , where we used the substitution  $x \leftarrow z\beta^{-1}$  and the multiplicativity of  $\chi$ .  $\blacksquare$

From now on we will assume that the Gauss sum concerns a nontrivial character  $\chi$  and  $\beta \neq 0$ . The inverse of a character  $\chi$  is defined by  $\chi^{-1}(x) := \overline{\chi(x)}$  for all  $x \neq 0$  and  $\chi^{-1}(0) := 0$  (where  $\overline{z}$  is the complex conjugate of  $z$ ). It is known that the norm of a Gauss sum obeys  $|G(\mathbb{F}_{p^r}, \chi, \beta)| = \sqrt{p^r}$ , and more specifically  $G(\mathbb{F}_{p^r}, \chi, \beta)G(\mathbb{F}_{p^r}, \chi^{-1}, \beta) = \chi(-1)p^r$ .

### 2.2 The Approximate Gauss Sum Problem

If we want to define the problem of estimating Gauss sums as a computational task, we have to make clear what the length of the input is. As stated above, any multiplicative character  $\chi : \mathbb{F}_{p^r} \rightarrow \mathbb{C}$  can be described by a triplet  $(p^r, g, \alpha)$ , where  $g \in \mathbb{F}_{p^r}^*$  is a generator of  $\mathbb{F}_{p^r}^*$  and  $\alpha \in \mathbb{F}_{p^r}$  the index of  $\chi$ . As a result, the

specification of the problem “What is  $G(\mathbb{F}_{p^r}, \chi, \beta)$ ?” as defined below, requires no more than  $O(r \log p)$  bits of information.

**Definition 2 (Gauss Sum Problem for Finite Fields)** *Let  $\mathbb{F}_{p^r}$  be a finite field,  $\chi$  a nontrivial character over  $\mathbb{F}_{p^r}$  and  $\beta \in \mathbb{F}_{p^r}^*$ . What is (approximately) the angle  $\gamma \bmod 2\pi$  in the Gauss sum equation  $G(\mathbb{F}_{p^r}, \chi, \beta) = \sqrt{p^r} \cdot e^{i\gamma}$ ?*

A *quadratic character* is a nontrivial  $\chi$  such that  $\chi(x) \in \{0, 1, -1\}$  for all  $x$ . By the isomorphism  $\hat{\mathbb{F}}_{p^r}^* \simeq \mathbb{Z}/(p^r - 1)\mathbb{Z}$  one sees that such a character is only possible if  $p$  is odd and where  $\chi$  is defined by  $\chi(g^j) = (-1)^j$ . Unlike the case of general characters, the Gauss sums of such quadratic characters are known completely:  $G(\mathbb{F}_{p^r}, \chi, 1) = -(-1)^r \sqrt{p^r}$  if  $p \equiv 1 \pmod{4}$ , and  $G(\mathbb{F}_{p^r}, \chi, 1) = -(-i)^r \sqrt{p^r}$  if  $p \equiv 3 \pmod{4}$ . (See Theorem 11.5.4 in [1] for a proof.)

### 3 Quantum Computing

In this section we give a brief overview of the known results on quantum computation that are relevant for the rest of this article. For more information, we refer the reader to [11].

#### 3.1 Efficient Quantum Procedures

**Fact 2 (Quantum Phase Estimation)** *Let  $\gamma$  be an unknown phase  $\bmod 2\pi$  of the qubit state  $|x_\gamma\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\gamma}|1\rangle)$ . If we measure this qubit in the orthogonal basis  $|m_\phi\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$  and  $|m_\phi^\perp\rangle := \frac{1}{\sqrt{2}}(|0\rangle - e^{i\phi}|1\rangle)$ , then the respective outcome probabilities are  $\text{Prob}(m_\phi|x_\gamma) = \frac{1}{2} + \frac{1}{2} \cos(\gamma - \phi)$  and  $\text{Prob}(m_\phi^\perp|x_\gamma) = \frac{1}{2} - \frac{1}{2} \cos(\gamma - \phi)$ . Hence, if we can sample  $t$  copies of  $|x_\gamma\rangle$  (with various different angles  $\phi$ ), then we can obtain an estimate  $\tilde{\gamma}$  of the unknown  $\gamma$  within an expected error of  $O(\frac{1}{t})$ .*

Shor’s famous article [12] implies the following result.

**Fact 3 (Efficient Quantum Algorithm for the Discrete Logarithm)** *There exists a quantum algorithm that, given a base  $g \in (\mathbb{Z}/n\mathbb{Z})^*$  and an element  $x = g^j \bmod n$ , determines the discrete logarithm  $\log_g(x) := j$  in time  $\text{polylog}(n)$ .*

**Fact 4 (Efficient Quantum Fourier Transform)** *Let  $\beta \in \mathbb{F}_{p^r}^*$ . The quantum Fourier transform  $\mathcal{F}_\beta$  over the finite field  $\mathbb{F}_{p^r}$ , which is defined as the unitary mapping*

$$\mathcal{F}_\beta : |x\rangle \mapsto \frac{1}{\sqrt{p^r}} \sum_{y \in \mathbb{F}_{p^r}} \zeta_p^{\text{Tr}(\beta xy)} |y\rangle \quad (2)$$

*for every  $x \in \mathbb{F}_{p^r}$ , can be implemented efficiently on a quantum computer. Similarly, we can also perform the Fourier transform over the group  $\mathbb{Z}/n\mathbb{Z}$  in an efficient way.*

Sometimes we use the hat notation in  $\mathcal{F} : |\psi\rangle \mapsto |\hat{\psi}\rangle$  to denote the Fourier transform of a state.

#### 3.2 Quantum State Preparation

For every function  $f : S \rightarrow \mathbb{C}$ , we define the state

$$|f\rangle := \frac{1}{\|f\|_2} \sum_{x \in S} f(x) |x\rangle \quad \text{with the } \ell_2 \text{ norm } \|f\|_2 := \sqrt{\sum_{x \in S} |f(x)|^2}. \quad (3)$$

We also allow ourselves to use the shorthand  $|S\rangle := \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$ , for any set  $S$ .

In this article we are mostly concerned with the preparation of states  $|\chi\rangle$  that refer to a multiplicative character  $\chi : R \rightarrow \mathbb{C}$ , which is zero for those values that are not in the multiplicative subgroup  $R^*$  and that are powers of  $\zeta_{|R^*|}$  for the elements that are in  $R^*$ .

**Fact 5 (Phase Kickback Trick [3])** *If the computation  $|x\rangle \mapsto |x\rangle|f(x)\rangle$  with  $f(x) \in \mathbb{Z}/n\mathbb{Z}$  can be performed efficiently, then the phase changing transformation  $|x\rangle \mapsto \zeta_n^{f(x)}|x\rangle$  can be performed exactly and coherently in time  $\text{polylog}(n)$  as well.*

**Proof:** First, create the state  $|x\rangle|\hat{1}\rangle := |x\rangle \otimes \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \zeta_n^j |j\rangle$ , by applying the Fourier transform over  $(\mathbb{Z}/n\mathbb{Z})$  to the rightmost part of the initial state  $|x\rangle|1\rangle$ . Next, consider the evolution that is established by subtracting  $f(x) \bmod n$  to that same rightmost register:

$$|x\rangle \otimes \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \zeta_n^j |j\rangle \mapsto |x\rangle \otimes \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \zeta_n^j |j - f(x)\rangle = \zeta_n^{f(x)} |x\rangle \otimes \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \zeta_n^k |k\rangle, \quad (4)$$

where we used the substitution  $j \leftarrow k + f(x)$  and the additivity  $\zeta_n^{k+f(x)} = \zeta_n^{f(x)} \cdot \zeta_n^k$ . Clearly, the overall phase change of this transformation is the one desired.  $\blacksquare$

The phase kickback trick enables us to induce the character values  $\chi(x)$  as phases in a quantum state. For those  $x$  that have  $\chi(x) = 0$  we will use the amplitude amplification process of Grover's search algorithm to change the amplitudes of the states  $|x\rangle$ .

**Fact 6 (Quantum Amplitude Amplification)** *Let  $f : S \rightarrow \{0, 1\}$  be function of which we know the total 'weight'  $\|f\|_1 := \sum_{x \in S} f(x)$ , but not the specific positions for which  $f(x) = 1$ . The corresponding state  $|f\rangle$  can be efficiently and exactly prepared on a quantum computer with  $O(\sqrt{|S|/\|f\|_1})$  queries to the function  $f$ .*

**Proof:** See the standard literature ([2, 6, 7] for example).  $\blacksquare$

The Facts 5 and 6 show that it is easy to create the state  $|\chi\rangle := \frac{1}{\sqrt{p^r-1}} \sum_{x \in \mathbb{F}_{p^r}} \chi(x)|x\rangle$  with a constant number of queries to the function  $\chi$ . Furthermore, we know that  $\chi$ , specified by the triplet  $(p^r, g, \alpha)$ , is defined by  $\chi(x) = \zeta_{p^r-1}^{\alpha \log_g(x)}$  for  $x \in \mathbb{F}_{p^r}$  and  $\chi(0) = 0$ . Using Shor's discrete logarithm algorithm (Fact 3), we can calculate this discrete log, from which it follows that given  $(p^r, g, \alpha)$ , we can create the state  $|\chi\rangle$  efficiently in the following way.

**Lemma 1 (Efficient  $\chi$  state preparation)** *For a finite field  $\mathbb{F}_{p^r}$  and  $(p^r, g, \alpha)$  the specification of a multiplicative character  $\chi$ , the state*

$$|\chi\rangle := \frac{1}{\sqrt{p^r-1}} \sum_{x \in \mathbb{F}_{p^r}} \chi(x)|x\rangle, \quad (5)$$

*and its Fourier transform  $|\hat{\chi}\rangle$  can be created in  $\text{polylog}(p^r)$  time steps on a quantum computer.*

**Proof:** First, use the amplitude amplification process on the set  $\mathbb{F}_{p^r}$  and the Fourier transform over  $\mathbb{Z}/(p^r-1)\mathbb{Z}$  to create the initial state

$$|\mathbb{F}_{p^r}^*\rangle|\hat{1}\rangle := \frac{1}{\sqrt{p^r(p^r-1)}} \sum_{x \in \mathbb{F}_{p^r}^*} |x\rangle \sum_{j=0}^{p^r-2} \zeta_{p^r-1}^j |j\rangle. \quad (6)$$

Next, in superposition over all  $x \in \mathbb{F}_{p^r}^*$  states, calculate the discrete logarithm values  $\log_g(x)$  and subtract  $\alpha \log_g(x) \bmod (p^r-1)$  to the state in the rightmost register. By the phase kickback trick of Fact 5 we thus obtain the desired state:

$$|\mathbb{F}_{p^r}^*\rangle|\hat{1}\rangle \mapsto \frac{1}{\sqrt{p^r-1}} \sum_{x \in \mathbb{F}_{p^r}^*} \zeta_{p^r-1}^{\alpha \log_g(x)} |x\rangle|\hat{1}\rangle = |\chi\rangle|\hat{1}\rangle. \quad (7)$$

Given this construction, we can also create its Fourier transform  $|\hat{\chi}\rangle$  by using the quantum Fourier transform on  $|\chi\rangle$ .  $\blacksquare$

## 4 Estimating Gauss Sums over Finite Fields

With the ingredients of the last two sections, we are now ready to describe the quantum algorithm that estimates the angle  $\gamma$  of the Gauss sum  $G = |G| \cdot e^{i\gamma}$  over finite fields. The crucial part of our algorithm relies on the interaction between the Fourier transform  $\mathcal{F}_\beta$  and the multiplicative character  $\chi$ . Using the fact that for nontrivial characters  $\hat{\chi} = G(\mathbb{F}_{p^r}, \chi, \beta) / \sqrt{p^r} \cdot \bar{\chi}$ , we are able to perform a  $\gamma$ -phase change. By sampling this unknown phase factor we can obtain an arbitrary precise estimation of  $\gamma$  and thus of  $G(\mathbb{F}_{p^r}, \chi, \beta)$ .

**Algorithm 1** Consider a finite field  $\mathbb{F}_{p^r}$ , a nontrivial character  $\chi$  and a  $\beta \in \mathbb{F}_{p^r}^*$ . If we apply the quantum Fourier transform ( $\mathcal{F}_\beta$ ) over this field to the state  $|\chi\rangle$ , followed by a phase change  $|y\rangle \mapsto \chi^2(y)|y\rangle$ , then we generate an overall phase change according to

$$|\chi\rangle := \frac{1}{\sqrt{p^r-1}} \sum_{x \in \mathbb{F}_{p^r}} \chi(x)|x\rangle \mapsto \frac{G(\mathbb{F}_{p^r}, \chi, \beta)}{\sqrt{p^r}} |\chi\rangle. \quad (8)$$

**Proof:** First, we note that the output after the Fourier transform  $\mathcal{F}_\beta$  looks like

$$|\hat{\chi}\rangle := \frac{1}{\sqrt{p^r(p^r-1)}} \sum_{y \in \mathbb{F}_{p^r}} \left( \sum_{x \in \mathbb{F}_{p^r}} \chi(x) \zeta_p^{\text{Tr}(\beta xy)} \right) |y\rangle. \quad (9)$$

The expression between the big parentheses equals  $G(\mathbb{F}_{p^r}, \chi, \beta y)$ , which equals  $\chi(y^{-1})G(\mathbb{F}_{p^r}, \chi, \beta)$  for  $y \neq 0$  and is zero if  $y = 0$ . In sum, we thus see that

$$|\hat{\chi}\rangle = \frac{G(\mathbb{F}_{p^r}, \chi, \beta)}{\sqrt{p^r(p^r-1)}} \sum_{y \in \mathbb{F}_{p^r}^*} \chi(y^{-1})|y\rangle, \quad (10)$$

such that indeed after  $|y\rangle \mapsto \chi^2(y)|y\rangle$  we have created the eigenstate  $|\chi^2 \circ \hat{\chi}\rangle = \frac{G(\mathbb{F}_{p^r}, \chi, \beta)}{\sqrt{p^r}} |\chi\rangle$ . ■

With the above algorithm we are now able to efficiently estimate the angle  $\gamma$  in the equation  $G(\mathbb{F}_{p^r}, \chi, \beta) = \sqrt{p^r} \cdot e^{i\gamma}$ .

**Theorem 1 (Quantum Algorithm for Gauss Sum Estimation over  $\mathbb{F}_{p^r}$ )** *For any  $\varepsilon > 0$ , there exists a quantum algorithm that estimates the phase  $\gamma$  in  $G(\mathbb{F}_{p^r}, \chi, \beta) = \sqrt{p^r} \cdot e^{i\gamma}$ , with expected error  $\mathbb{E}[|\gamma - \tilde{\gamma}|] < \varepsilon$ . The time complexity of this algorithm is bounded by  $O(\frac{1}{\varepsilon} \cdot \text{polylog}(p^r))$ .*

**Proof:** By the earlier algorithm, we know that we can induce the phase change  $|\chi\rangle \mapsto e^{i\gamma}|\chi\rangle$  in  $\text{polylog}(p^r)$  time. If we do this in superposition with a ‘stale’ component  $\emptyset$ , then we have produced the relative phase shift  $\frac{1}{\sqrt{2}}(|\emptyset\rangle + |\chi\rangle) \mapsto \frac{1}{\sqrt{2}}(|\emptyset\rangle + e^{i\gamma}|\chi\rangle)$ . As described in Fact 2, we can estimate this phase by measuring the states along the axis  $|m_\phi\rangle := \frac{1}{\sqrt{2}}(|\emptyset\rangle + e^{i\phi}|\chi\rangle)$  for different  $\phi$ . After  $O(\frac{1}{\varepsilon})$  of such observations, the estimate  $\tilde{\gamma}$  of the true  $\gamma$  will have expected error  $\mathbb{E}[|\gamma - \tilde{\gamma}|] < \varepsilon$ . ■

### 4.1 Estimation of Jacobi Sums over Finite Fields

Closely related to Gauss sums are the *Jacobi sums*, which play an especially important role in primality testing [4].

**Definition 3 (Jacobi Sums over Finite Fields)** *For a finite field  $\mathbb{F}_{p^r}$  and two multiplicative characters  $\chi$  and  $\psi$ , the Jacobi sum  $J(\chi, \psi)$  is defined by*

$$J(\chi, \psi) := \sum_{x \in \mathbb{F}_{p^r}} \chi(x)\psi(1-x). \quad (11)$$

Clearly,  $J(\chi, \psi) = J(\psi, \chi)$ . With  $\chi^0$  the trivial character and  $\psi$  a nontrivial character we have  $J(\chi^0, \chi^0) = p^r - 2$ ,  $J(\psi, \psi^{-1}) = -\psi(-1)$ , and  $J(\chi^0, \psi) = -1$ . (Note that we use the convention  $\chi^0(0) = 0$  for the primitive character, not  $\chi^0(0) = 1$ .) The other, less trivial, cases have the following connection with Gauss sums, which is proven in Section 2 of [1].

**Fact 7** For  $\chi$  and  $\psi$  be nontrivial multiplicative characters over  $\mathbb{F}_{p^r}$ , with  $\chi\psi$  nontrivial as well, it holds that  $J(\chi, \psi) = G(\mathbb{F}_{p^r}, \chi, 1)G(\mathbb{F}_{p^r}, \psi, 1)/G(\mathbb{F}_{p^r}, \chi\psi, 1)$ . As a result,  $J(\chi, \psi) = e^{i\lambda} \cdot \sqrt{p^r}$ .

**Corollary 1 (Quantum Algorithm for Jacobi Sum Estimation)** Using the Gauss sum estimation algorithm of Theorem 1, there exists a quantum algorithm that estimates the angle  $\lambda \bmod 2\pi$  in  $J(\chi, \psi) = e^{i\lambda} \cdot \sqrt{p^r}$  with expected error  $\varepsilon$  with time complexity  $O(\frac{1}{\varepsilon} \cdot \text{polylog}(p^r))$ .

## 5 The Classical Complexity of Approximating Gauss Sums

The obvious next question now is: How difficult it is to estimate Gauss sums with classical computers? Although we are not able to prove that this is hard, we can give the following reduction, which indicates that a classical polynomial time algorithm is unlikely.

### 5.1 Reducing the Discrete Log Problem to Gauss Sum Estimation

**Lemma 2 (Reduction from Discrete Log to Gauss Sum Estimation)** Let  $\mathbb{F}_{p^r}$  be a finite field with primitive element  $g$ ,  $\chi(g^j) := \zeta_{p^r-1}^j$  a multiplicative character and  $x$  an element of  $\mathbb{F}_{p^r}^*$ . With an oracle that  $\varepsilon$ -approximates the angle  $\gamma$  of the Gauss sum  $G(\mathbb{F}_{p^r}, \chi, \beta)$  for arbitrary  $\beta$ , we can efficiently determine, classically, the discrete  $\log_g(x)$ .

**Proof:** With  $x = g^\ell$ , we try to determine this  $0 \leq \ell \leq p^r - 2$ . For  $k = 1, 2, 3, \dots$  we observe, using Lemma 1, that:  $G(\mathbb{F}_{p^r}, \chi, x^k)/G(\mathbb{F}_{p^r}, \chi, 1) = \chi(g^{-k\ell}) = e^{-2\pi i k\ell/(p^r-1)}$ ; call this angle  $\gamma_k := -2\pi k\ell/(p^r-1)$ . Using the ‘powering algorithm’ ( $x \mapsto x^2 \mapsto x^4 \dots$  et cetera) we can calculate  $x^k$  for any  $0 \leq k \leq p^r - 2$  in  $\text{polylog}(p^r)$  time, hence we can use our oracle to  $\varepsilon$ -approximate  $\gamma_k$  for any such  $k$ . Via the equality  $-\frac{2k}{2\pi}(p^r-1) = k\ell \bmod (p^r-1)$  this will give us information on the value of  $\ell \bmod (p^r-1)$  depending on  $k$ . By estimating  $\gamma_k$  for  $k = 1, 2, 4, 8, \dots, \approx p^r$ , we can thus get a reliable estimation of all  $\log(p^r)$  bits of  $\ell$ , thereby calculating the desired value  $\log_g(x) = \ell$ .  $\blacksquare$

### 5.2 Galois Automorphisms and Other Homomorphisms of $\mathbb{Q}(\zeta_{p^r-1}, \zeta_{p^r})$

The previous lemma shows that it is not trivial to estimate the Gauss sum  $G(\mathbb{F}_{p^r}, \chi, \beta)$  even if we already know the value  $G(\mathbb{F}_{p^r}, \chi, 1)$ . A similar result seems to hold for the estimation of  $G(\mathbb{F}_{p^r}, \chi^\alpha, \beta)$  while having information on  $G(\mathbb{F}_{p^r}, \chi, \beta)$ .

As noted earlier,  $G(\mathbb{F}_{p^r}, \chi, \beta)$  is an element of  $\mathbb{Q}(\zeta_{p^r-1}, \zeta_p)$ . Compare now the two expressions for  $G(\mathbb{F}_{p^r}, \chi, \beta)$  and  $G(\mathbb{F}_{p^r}, \chi^\alpha, \beta)$ , respectively,  $\sum_{j=0}^{p^r-2} \zeta_{p^r-1}^j \zeta_p^{\text{Tr}(\beta g^j)}$  and  $\sum_{j=0}^{p^r-2} \zeta_{p^r-1}^{\alpha j} \zeta_p^{\text{Tr}(\beta g^j)}$ . This shows that under the homomorphism  $\sigma_\alpha : \mathbb{Q}(\zeta_{p^r-1}, \zeta_p) \rightarrow \mathbb{Q}(\zeta_{p^r-1}, \zeta_p)$ , with  $\sigma_\alpha : \zeta_{p^r-1} \mapsto \zeta_{p^r-1}^\alpha$ , we have  $\sigma_\alpha : G(\mathbb{F}_{p^r}, \chi, \beta) \mapsto G(\mathbb{F}_{p^r}, \chi^\alpha, \beta)$ . (If  $\text{gcd}(\alpha, p^r-1) = 1$  then this mapping is a Galois automorphism  $\sigma_\alpha \in \text{Gal}(\mathbb{Q}(\zeta_{p^r-1}, \zeta_p)/\mathbb{Q}(\zeta_p))$ . If  $\text{gcd}(\alpha, p^r-1) \neq 1$  then the mapping  $\sigma_\alpha$  is not necessarily bijective, and hence not an automorphism.)

This result suggests that knowledge about the Gauss sum  $G(\mathbb{F}_{p^r}, \chi, \beta)$  is sufficient to efficiently determine  $G(\mathbb{F}_{p^r}, \chi^\alpha, \beta)$  for all other  $\alpha$ . However, it should be noted that the degree  $[\mathbb{Q}(\zeta_{p^r-1}, \zeta_p) : \mathbb{Q}(\zeta_p)]$  equals  $\phi(p^r-1)$ , which is exponential in the input size  $\log(p^r)$ . As a result, the  $\sigma_\alpha$  mapping concerns an exponential number of coefficients, and is hence not efficient.

### 5.3 Gauss Sums as Pseudorandom Walks in $\mathbb{C}$

Let the finite field be a base field  $\mathbb{F}_p$ . For every  $x \neq 0$ , the terms  $\chi(x)e_\beta(x)$  in the summation  $\sum_x \chi(x)e_\beta(x)$  are unit norm vectors in  $\mathbb{C}$  that together describe a walk in  $\mathbb{C}$  (of  $p-1$  steps) from 0 to the final outcome  $G(\mathbb{F}_p, \chi, \beta)$ . Viewed like this, an obvious classical attempt to approximate  $G$  consists of trying to estimate the ‘average direction’ of the terms  $\chi(x)e_\beta(x)$  by sampling a small number of  $x$  values. It should also be obvious that this method will not work for random samples that are not polynomial in  $p$ . As the final destination  $G$  is only  $\sqrt{p}$  away from the origin, a significant average direction can only be obtained with a sample size that is polynomial in  $p$ .

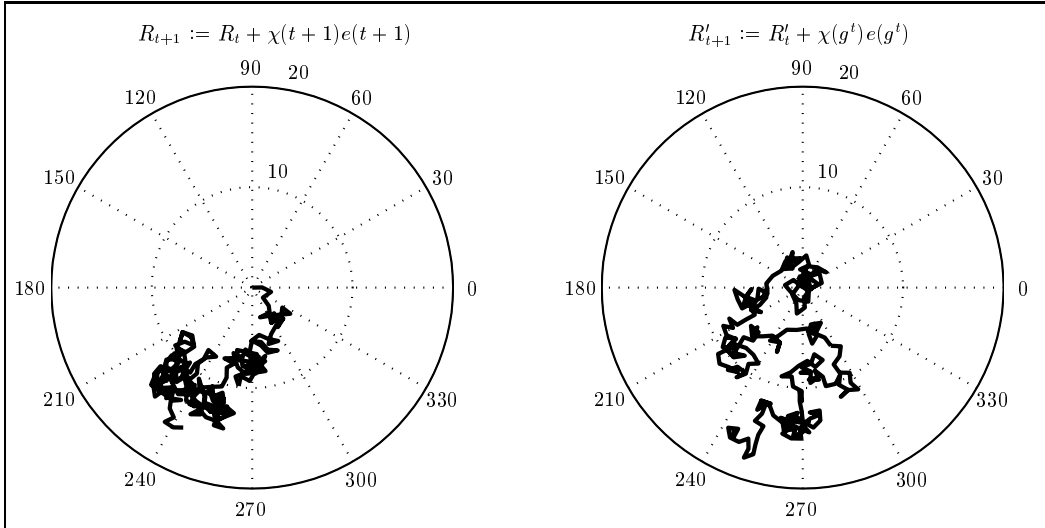


Figure 1: Illustration of the pseudorandom walks described in Example 2. The walk  $R$  on the left is defined by the equation  $R(t) := \sum_{x=0}^t \chi(x)e(x)$ , while the walk  $R'$  on the right obeys  $R'(t) := \sum_{j=0}^{t-1} \chi(7^j)e(7^j)$ .

In fact, the just described walk shares many of the properties that a truly random walk in  $\mathbb{C}$  would have. Not only does the final distance coincide with the expected distance norm of a random walk, but also the sequence of steps exhibits the nonregularity of a random process. It is easy to verify that the autocorrelation of the sequence  $\chi(1)e(1), \chi(2)e(2), \dots$  is near-zero:  $\mathbb{E}[\chi(j)e(j)\bar{\chi}(j+s)\bar{e}(j+s)] = \frac{-e(-s)}{p-1}$  for  $s \neq 0$ . These pseudorandom characteristics do not change when we indexing of the summation (and hence of the sequence) to  $\chi(1)e(1), \chi(g)e(g), \chi(g^2)e(g^2), \dots$  with  $g$  a generator of  $\mathbb{F}_p^*$ , as this sequence obeys  $\mathbb{E}[\chi(g^j)e(g^j)\bar{\chi}(g^{j+s})\bar{e}(g^{j+s})] = \frac{-\chi(-s)}{p-1}$ . See the following example for an illustration of this pseudorandom behavior.

**Example 2** Consider the Gauss sum for the finite field  $\mathbb{F}_{241}$ , with multiplicative generator 7, and the character defined by  $\chi(7^j) := \zeta_{240}^{10j}$ . Calculations show that  $G(\mathbb{F}_{241}, \chi, 1) = \sqrt{241} \cdot e^{2\pi i \cdot 0.6772\dots} \approx -6.85 + 13.9i$ . Figure 1 shows the two pseudorandom walks that are defined by the sequences  $\chi(1)e(1), \chi(2)e(2), \dots$  (left) and  $\chi(7^0)e(7^0), \chi(7^1)e(7^1), \dots$  (right).

## 6 Gauss Sums over Finite Rings

Although the final quantum algorithm for estimating Gauss sums over rings  $\mathbb{Z}/n\mathbb{Z}$  is not much more complicated than the finite field algorithm, the theory surrounding it is somewhat more elaborate. A large part of this section concerns the proper description of a multiplicative character over  $\mathbb{Z}/n\mathbb{Z}$  and its various properties. These details are necessary to get a valid definition for the input size of the Gauss sum problem over  $\mathbb{Z}/n\mathbb{Z}$  (see Definition 5).

### 6.1 Definitions and Notation: Dirichlet Characters

Again, the  $n$ th root of unity is denoted by  $\zeta_n^x := e^{2\pi i x/n}$ . From [4], Section 1.4 we copy the following facts. Consider the multiplicative subgroup  $(\mathbb{Z}/n\mathbb{Z})^*$  with the prime decomposition  $n = p_1^{r_1} \cdots p_k^{r_k}$ . Following the Chinese remainder theorem we have  $(\mathbb{Z}/n\mathbb{Z})^* \simeq (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*$ , such that  $|(\mathbb{Z}/n\mathbb{Z})^*| = \phi(n)$ ,



with  $\phi$  Euler's totient function. Furthermore we have

$$\begin{cases} (\mathbb{Z}/p^r\mathbb{Z})^* \simeq \mathbb{Z}/(p-1)p^{r-1}\mathbb{Z} \text{ if } p \geq 3, \\ (\mathbb{Z}/2\mathbb{Z})^* \simeq \mathbb{Z}/\mathbb{Z} \simeq \{0\}, \\ (\mathbb{Z}/4\mathbb{Z})^* \simeq \mathbb{Z}/2\mathbb{Z}, \\ (\mathbb{Z}/2^r\mathbb{Z})^* \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r-2}\mathbb{Z} \text{ if } r \geq 3, \end{cases} \quad (12)$$

hence  $(\mathbb{Z}/n\mathbb{Z})^*$  is cyclic if and only if  $n = 2, 4, p^r$  or  $2p^r$ , with  $p$  an odd prime.

**Definition 4 (Dirichlet Characters)** *A function  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  is a Dirichlet character if for all  $x, y \in \mathbb{Z}/n\mathbb{Z}$  we have  $\chi(x)\chi(y) = \chi(xy)$  and  $\chi(x) = 0$  if and only if  $\gcd(n, x) \neq 1$ .*

Using the multiplicative decomposition  $(\mathbb{Z}/n\mathbb{Z})^* = (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*$ , we see that all Dirichlet characters  $\chi : (\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{C}$  can be decomposed as  $\chi = \chi_1 \cdots \chi_k$  with  $\chi_i : (\mathbb{Z}/p_i^{r_i}\mathbb{Z}) \rightarrow \mathbb{C}$  for every  $1 \leq i \leq k$ , and thus  $\chi(x) := \chi_1(x \bmod p_1^{r_1}) \cdots \chi_k(x \bmod p_k^{r_k})$ .

For  $p$  an odd prime the character  $\chi : \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathbb{C}$  can be described by the expression  $\chi(x) := \zeta_{\phi(p^r)}^{\alpha \log_g(x)}$ , where  $g$  is a generator of the cyclic  $(\mathbb{Z}/p^r\mathbb{Z})^*$ ,  $\phi(p^r) = p^{r-1}(p-1)$  and  $\alpha \in \mathbb{Z}/\phi(p^r)\mathbb{Z}$ . For  $(\mathbb{Z}/2\mathbb{Z})^*$  we only have the trivial character  $\chi^0$ , while for  $(\mathbb{Z}/4\mathbb{Z})^*$  we have two possibilities:  $\chi^0$  and  $\chi^1$  with  $\chi^\alpha(3) = (-1)^\alpha$  and  $\chi^\alpha(1) = 1$ . If  $\chi$  is a character over  $(\mathbb{Z}/2^r\mathbb{Z})^*$  with  $r \geq 3$ , then we have to decompose the character in two terms. The group  $(\mathbb{Z}/2^r\mathbb{Z})^*$  is generated by 3 and 5 (see [4]), hence the character can be described by the pair  $(\alpha, \alpha') \in (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{r-2}\mathbb{Z})$  such that for all  $i$  and  $i'$  we have  $\chi(3^i 5^{i'} \bmod 2^r) := (-1)^{\alpha i} \zeta_{2^{r-2}}^{\alpha' i'}$  (while  $\chi(x) = 0$  if  $x$  is even).

**Definition 5 (Specification of Dirichlet Characters)** *Let  $(\mathbb{Z}/n\mathbb{Z})^* \simeq (\mathbb{Z}/2^{r_0}\mathbb{Z})^* \times (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^* \simeq (\mathbb{Z}/2^{r_0}\mathbb{Z})^* \times (\mathbb{Z}/\phi_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/\phi_k\mathbb{Z})$ , with  $\phi_j := (p_j - 1)p_j^{r_j-1}$  (see Equation 12). The specification of a Dirichlet character  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  is done by three sequences  $(p, g, \alpha)$ , with the prime decomposition  $p = (p_1, \dots, p_k)$  of  $n$ , the generators  $g = (g_1, \dots, g_k) \in (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*$  of the multiplicative groups, and  $\alpha = ((\alpha_0, \alpha'_0), \alpha_1, \dots, \alpha_k) \in (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r_0-2}) \times (\mathbb{Z}/\phi_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/\phi_k\mathbb{Z})$  the specification of the characters  $\chi_j$  in the definition  $\chi(x) := \chi_0(x \bmod 2^{r_0}) \chi_1(x \bmod p_1^{r_1}) \cdots \chi_k(x \bmod p_k^{r_k})$  with*

$$\chi_0(3^i 5^{i'} \bmod 2^{r_0}) := (-1)^{\alpha_0 i} \zeta_{2^{r_0-2}}^{\alpha'_0 i'} \quad \text{and} \quad \chi_j(x_j) := \zeta_{\phi_j}^{\alpha_j \log_{g_j}(x_j)} \text{ if } x_j \in (\mathbb{Z}/p_j^{r_j}\mathbb{Z})^*, \quad (13)$$

while  $\chi_j(x_j) := 0$  if  $\gcd(x, p_j) \neq 1$ .

With this definition, we see that the specification of a Dirichlet character  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  requires only  $O(\log n)$  bits of information. Hence, in the context of such characters, an algorithm that requires  $\text{polylog}(n)$  steps is efficient, while a running time polynomial in  $n$  is inefficient.

**Lemma 3 (Calculation of Dirichlet Character Values)** *Let  $(p, g, \alpha)$  be the specification of a character  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$ . Given  $n$  and  $(p, g, \alpha)$ , we can induce the phase change  $|x\rangle \mapsto \chi(x)|x\rangle$  for any  $x \in (\mathbb{Z}/n\mathbb{Z})^*$  efficiently with a quantum algorithm.*

**Proof:** To compute the phase change  $|x\rangle \mapsto \chi_j(x \bmod p_j^{r_j})|x\rangle$  we perform the following two steps (we use a similar protocol for the  $\chi_0$  part of  $\chi$ ):

1. Use Shor's discrete log algorithm (Fact 3) to determine the value  $s_j := \log_{g_j}(x \bmod p_j^{r_j})$ .
2. Use the phase kickback trick (Fact 5) to induce the phase change  $|x\rangle \mapsto \zeta_{\phi(p_j^{r_j})}^{\alpha_j s_j}$ .

Perform the phase changes for all  $\chi_j$  to the same  $x$  state, such that the overall transformation will be:  $|x\rangle \mapsto \chi_0(x)|x\rangle \mapsto \chi_0(x)\chi_1(x) \mapsto \cdots \mapsto \chi_0(x)\chi_1(x) \cdots \chi_k(x)|x\rangle = \chi(x)|x\rangle$ .  $\blacksquare$

**Definition 6 (Conductance and Triviality of Dirichlet Characters)** *A character  $\chi$  is trivial if  $\chi(x) = 1$  for all  $x \in (\mathbb{Z}/n\mathbb{Z})^*$ , that is, if  $\alpha$  is the zero vector. The conductor  $c$  of a character  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  is the minimum value  $c > 1$  for which there is a character  $\chi_c : \mathbb{Z}/c\mathbb{Z} \rightarrow \mathbb{C}$  such that  $\chi = \chi_c \chi^0$  where  $\chi^0$*

is the trivial character over  $\mathbb{Z}/n\mathbb{Z}$ . We call  $\chi$  a primitive character if  $\chi$  has the maximum conductance  $c = n$ . The trivial character has conductance 1 and is not primitive. The field  $\mathbb{Z}/p\mathbb{Z}$  has  $p - 2$  primitive characters, whereas the ring  $\mathbb{Z}/p^r\mathbb{Z}$  with  $r \geq 2$  has  $p^{r-2}(p - 1)^2$  such characters. For  $\mathbb{Z}/2^r\mathbb{Z}$  with  $r \geq 2$  this means that all characters  $\chi^{(1, \alpha')}$  are primitive ( $\mathbb{Z}/2\mathbb{Z}$  has no primitive characters, and  $\mathbb{Z}/4\mathbb{Z}$  has one). For  $\mathbb{Z}/p^r\mathbb{Z}$  with  $p$  and odd prime, a character  $\chi(x) = \zeta_{\phi(p^r)}^{\alpha \log_g(x)}$  has conductance  $p^{r-s}$  where  $p^s \mid \alpha$ , while  $p^{s+1} \nmid \alpha$ . As a result  $\chi^\alpha$  is primitive, if and only if  $p \nmid \alpha$ . In general, a character over the ring  $\mathbb{Z}/n\mathbb{Z}$  with  $(\mathbb{Z}/n\mathbb{Z})^* \simeq (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*$ , and accordingly  $\chi = \chi_1 \cdots \chi_k$ , is primitive if and only if all  $\chi_i$  factors are primitive.

## 6.2 Definition and Properties of Gauss Sums over Rings $\mathbb{Z}/n\mathbb{Z}$

The definition of Gauss sums over  $\mathbb{Z}/n\mathbb{Z}$  is a natural generalization of the definition for finite fields.

**Definition 7 (Gauss sums over Finite Rings)** For the ring  $\mathbb{Z}/n\mathbb{Z}$ , the Dirichlet character  $\chi$ , and the additive character  $e_\beta(x) := \zeta_n^{x\beta}$ , we define the Gauss sum  $G$  by

$$G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) := \sum_{x \in \mathbb{Z}/n\mathbb{Z}} \chi(x) \zeta_n^{\beta x}. \quad (14)$$

Note that the Gauss sum terms are the coefficients of the Fourier transform of the character:  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = \hat{\chi}(\beta)$ , with the Fourier transform over the additive group  $\mathbb{Z}/n\mathbb{Z}$ . See Section 1.6 in [1] for the proofs of the facts below.

**Fact 8** Let  $\chi$  be a nontrivial Dirichlet character over  $\mathbb{Z}/n\mathbb{Z}$ . The summation of all  $\chi$  values obeys  $\sum_{x=0}^{n-1} \chi(x) = 0$ . If  $\chi$  is trivial, then the sum equals  $\phi(n)$ .

**Fact 9** Let  $\chi$  be a character over the ring  $(\mathbb{Z}/n\mathbb{Z})^* \simeq (\mathbb{Z}/p_0^{r_0}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*$ , such that  $\chi = \chi_0 \cdots \chi_k$  with  $\chi_i$  a multiplicative character over  $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^*$  for every  $0 \leq i \leq k$ . With  $J_i \in (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^*$  the integers such that  $J_i n/p_i^{r_i} = 1 \pmod{p_i^{r_i}}$ , it holds that  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = \prod_{i=0}^k G(\mathbb{Z}/p_i^{r_i}\mathbb{Z}, \chi_i, \beta J_i)$ . (Such  $J_i$  always exist because  $\gcd(p_i, n/p_i^{r_i}) = 1$ , and hence  $n/p_i^{r_i} \in (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^*$ , for all  $i$ .)

This last lemma shows that we should only be concerned about Gauss sums over rings  $\mathbb{Z}/p^r\mathbb{Z}$ , the size of a prime power. For trivial characters, such Gauss sums are easily calculated.

**Fact 10** Let  $\chi^0 : \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathbb{C}$  be the trivial character, and  $p^j \mid \beta$  with  $p^{j+1} \nmid \beta$ , then

$$G(\mathbb{Z}/p^r\mathbb{Z}, \chi^0, \beta) = \begin{cases} p^{r-1}(p-1) & \text{if } j = r, \\ -p^{r-1} & \text{if } j = r-1, \\ 0 & \text{if } j < r-1. \end{cases} \quad (15)$$

Nontrivial characters that are not primitive can be reduced to primitive characters over smaller groups in the following way.

**Fact 11** Let  $\chi : \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathbb{C}$  be a non-primitive character with conductance  $p^{r-s}$ , then the corresponding Gauss sum obeys (note that  $\chi$  modulo  $p^{r-s}$  will be primitive):

$$G(\mathbb{Z}/p^r\mathbb{Z}, \chi, \beta) = \begin{cases} p^{s-1}(p-1) \cdot G(\mathbb{Z}/p^{r-s}\mathbb{Z}, \chi, \beta/p^s) & \text{if } \beta \mid p^s, \\ 0 & \text{if } \beta \nmid p^s. \end{cases} \quad (16)$$

Similar to the finite field case, the  $\beta$  index of the additive character can be ‘factored out’ as  $\chi^{-1}(\beta)$ :

**Fact 12** Let  $\chi$  be a primitive character over  $\mathbb{Z}/n\mathbb{Z}$ , then  $\hat{\chi}(\beta) = \chi^{-1}(\beta)\hat{\chi}(1)$ , and hence, equivalently,  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = \chi^{-1}(\beta)G(\mathbb{Z}/n\mathbb{Z}, \chi, 1)$ . Also,  $|G(\mathbb{Z}/n\mathbb{Z}, \chi, 1)| = \sqrt{n}$  holds.

## 7 Estimating Gauss Sums over Finite Rings

**Theorem 2 (Quantum Algorithm for Gauss Sum Estimation over  $\mathbb{Z}/n\mathbb{Z}$ )** For any  $\varepsilon > 0$ , there exists a quantum algorithm that estimates the phase  $\gamma$  in  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = |G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta)| \cdot e^{i\gamma}$ , with expected error  $\mathbb{E}[|\gamma - \hat{\gamma}|] < \varepsilon$ . The time complexity of this algorithm is bounded by  $O(\frac{1}{\varepsilon} \cdot \text{polylog}(n))$ . Also the norm  $|G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta)|$  can be determined in  $\text{polylog}(n)$  time.

**Proof:**

1. Determine the integers  $J_0, \dots, J_k$  as mentioned in Fact 9.
2. Calculate the Gauss sums  $G(\mathbb{Z}/p_i^{r_i}\mathbb{Z}, \chi_i, \beta J_i)$  for the trivial characters  $\chi_i$ , using Fact 10. Continue with the reduced product of nontrivial characters.
3. Re-express the Gauss sums terms for the periodic characters, using Fact 11. Continue with the reduced product of primitive characters.
4. Use Algorithm 2 to calculate the norms and estimate the phases of the Gauss sums of the remaining primitive characters (using the standard phase estimation technique of Fact 2 this requires  $O(\frac{1}{\varepsilon} \cdot \text{polylog}(n))$  steps).

**Algorithm 2** Let  $\chi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  be a primitive character and  $\beta \in \mathbb{Z}/n\mathbb{Z}$ . The following algorithm calculates the norm and estimates the phase  $\gamma$  in the Gauss sum  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = |G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta)| \cdot e^{i\gamma}$ .

1. If  $\beta \notin (\mathbb{Z}/n\mathbb{Z})^*$  then conclude that  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = 0$  (see Fact 12).
2. Otherwise, use Shor's discrete log algorithm to determine the  $\chi(\beta^{-1})$  factor in the equality  $G(\mathbb{Z}/n\mathbb{Z}, \chi, \beta) = \chi(\beta^{-1})G(\mathbb{Z}/n\mathbb{Z}, \chi, 1)$  (Fact 12). Continue with the estimation of  $\gamma$  in  $G(\mathbb{Z}/n\mathbb{Z}, \chi, 1) = e^{i\gamma}\sqrt{n}$ .
3. Apply the quantum Fourier transform  $\mathcal{F}$  over the ring  $\mathbb{Z}/n\mathbb{Z}$  to the state  $|\chi\rangle$ , followed by a phase change  $|y\rangle \mapsto \chi^2(y)|y\rangle$  for all  $y \in (\mathbb{Z}/n\mathbb{Z})^*$  in the superposition  $|\hat{\chi}\rangle = \sum_y \hat{\chi}(y)|y\rangle$ . Because for primitive characters  $\hat{\chi}(y) = \chi^{-1}(y)\hat{\chi}(1)$  (Fact 12), this transformation generate an overall phase change according to  $|\chi\rangle \mapsto e^{i\gamma}|\chi\rangle$ . Like in Theorem 1, we use this phase change to estimate  $\gamma$  to the required precision  $\varepsilon$ . ■

It should be noted that for primitive characters over rings  $\mathbb{Z}/p^r\mathbb{Z}$  with  $r \geq 2$  the Gauss sums  $G(\mathbb{Z}/p^r\mathbb{Z}, \chi, 1)$  are known (Section 1.6 in [1]). Hence step 3 of the above algorithm is not always necessary.

## 8 Conclusion and Discussion

The algorithms that we presented in this article rely on the specific interaction between multiplicative characters and Fourier transformations, some of which have been described earlier in [5] and [9]. Typical for these results is the fact they are defined for finite fields or finite rings but not for groups, which indicates a departure from the Hidden Subgroup framework for quantum algorithms [10]. What is new about the results presented here, is that they describe quantum algorithms for a natural problem that does not assume the presence of a black box function. The only other natural problems for which an efficient quantum algorithm has been constructed are those described by Shor [12] and Hallgren [8], both of which deal with number theory as well.

For the results of this article it remains therefore an important open question if Gauss sum estimation is hard classically, even under the assumption that factoring and discrete logarithms are easy. If this is indeed the case, another related question remains: Which problems reduce to Gauss sum estimation that do not reduce to factoring or the discrete logarithm problems?

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