Maximally entangled mixed states for two qubits

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The recent developments in quantum information have lead to a renewed interest in multi particle quantum mechanics. A two-qubit system displays many of the paradoxical features of quantum mechanics such as superposition and entanglement. Such states can be partially characterized by their degree of impurity and degree of entanglement. We specifically examine the class of states that have the maximum amount of entanglement (EOF) for a given degree of impurity. We show how these states are more entangled than the Werner like state for a given degree of mixture for several measures of entanglement.

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Maximally Entangled Mixed States for Two Qubits

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Abstract: The recent developments in quantum information have lead to a renewed interest in multi particle quantum mechanics. A two-qubit system displays many of the paradoxical features of quantum mechanics such as superposition and entanglement. Such states can be partially characterized by their degree of *impurity* and *degree of entanglement*. We specifically examine the class of states that have the maximum amount of entanglement (EOF) for a given degree of impurity. We show how these states are more entangled than the Werner like state for a given degree of mixture for several measures of entanglement.

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With the recent rapid developments in quantum information there has been a renewed interest in multi particle quantum mechanics and entanglement. The properties of states between the pure, maximally-entangled, and completely mixed (separable) limits are not completely known and have not been fully characterized. The physically allowed degree of entanglement and mixture is a timely issue, given that entangled qubits are a critical resource in many quantum information applications (such as quantum computation[1], quantum communication[2], quantum cryptography[3] and teleportation[4]), and that entangled mixed states could be advantageous for certain quantum information situations[5].

In this article we explore theoretically the domain between pure, highly entangled states, and highly mixed, weakly entangled states. We partially characterize such two-qubit states by their *purity* and *degree of entanglement* [6]. We examine the class of maximally-entangled mixed states,that is, states with the maximum amount of entanglement for a given degree of impurity? We show these states have significantly more entanglement that the Werner state[7]. The entanglement of formation can be viewed as the amount of entanglement necessary to create the state of interest from the pure Bell state, but to what extend can this entanglement be used?

Currently a variety of measures are known for quantifying the degree of entanglement in a bipartite system. These include the entanglement of distillation[6], the relative entropy of entanglement[8] and the *entanglement of formation* (EOF) [6]. For an arbitrary two-qubit system this is given by the analytical formula[9]

$$E_F(\hat{\rho}) = h\left(\frac{1+\sqrt{1-\mathcal{C}^2}}{2}\right), \tag{1}$$

where $h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is Shannon's entropy function and $\mathcal{C}$, the "concurrence" [9] is given by $\mathcal{C} = [\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}]$. Here the λ's are the square roots of the eigenvalues, in decreasing order, of the matrix, $\hat{\rho}\tilde{\hat{\rho}} = \hat{\rho}\ \sigma_y^A \otimes \sigma_y^B \hat{\rho}^* \sigma_y^A \otimes \sigma_y^B$, where $\hat{\rho}^*$ denotes the complex conjugation of $\hat{\rho}$ in the computational basis $\{|00, |01, |10, |11\}$. The entanglement of distillation is the logical means for characterizing the usable entanglement in the state ρ. This is extremely hard to calculate but an upper bound has been given by the relative entropy of entanglement[8],

$$E_N(\hat{\rho}) = \mathrm{Tr}\left[\rho\ln\rho - \rho\ln\sigma\right] \tag{2}$$

where σ is the closet separable density matrix to the state ρ which minimizes E_N. No simply analytical expression exists for an arbitrary two qubit state but its has been calculated for several classes of states[8]. For pure states $E_N(\rho) = E_F(\rho)$.

Here we will investigate a recent class of maximally entangled mixed states[10]. These states have the form

$$\hat{\rho_M} = \begin{pmatrix} g(\gamma) & 0 & 0 & \frac{\gamma}{2} \\ 0 & 1-2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\gamma}{2} & 0 & 0 & g(\gamma) \end{pmatrix} \quad \text{where} \quad g(\gamma) = \begin{cases} \gamma/2 & \gamma \geq 2/3 \\ 1/3 & \gamma < 2/3 \end{cases}, \tag{3}$$

and has been shown to have the maximal amount of entanglement (as characterized by the entanglement of formation) for a certain degree of impurity (as measured by the linear entropy $S_L = \frac{4}{3}\left[1 - Tr(\rho^2)\right]$, or vice versa. Such a state is entangled for all nonzero γ, and in fact has a concurrence given by $\mathcal{C} = \gamma$. Their is no state possible with a higher entanglement of formation for a given linear entropy. If our impurity measure is the conventional entropy then the state (3) is not optimal but a solution of a similar form has been shown to exist. It is useful to have a entangled mixed state to compare this maximally entangled mixed state to. As a comparison consider the well known Werner state[7] given by

$$\hat{\rho_W} = \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & \frac{\gamma}{2} \\ 0 & \frac{1-\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\gamma}{4} & 0 \\ \frac{\gamma}{2} & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix} \tag{4}$$

which is entangled for $\gamma > 1/3$ and has a concurrence $\mathcal{C} = (3\gamma - 1)/2$. For a given degree of mixture, the maximally entangled mixed state is always more entangled that the Werner state[7] at the same degree of mixture apart from at the extreme endpoints. This is shown in Figure (1a) where we plot the entanglement of formation versus the linear entropy for the MEMS state (curve i) and the Werner state (curve iii). These curves indicate the amount of entanglement required to create the mixed state but for both of these states what is the usable entanglement? Is this maximally entangled mixed state more entangled that the Werner state in other entanglement measures? To this end, we examine the relative entropy of entanglement for these states. It provides an upper bound on the entanglement of distillation. In Figure (1a) we show the relative entropy of entanglement for the MEMS (curve ii) and Werner (curve iv) states. Several

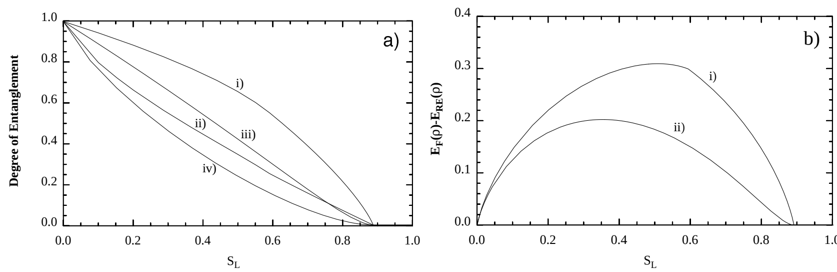

Fig. 1. Plot of entanglement various impurity for the maximally entangled mixed state and the Werner state. Curve i) in a) displays the entanglement of formation for ρ_M while curve ii) displays the relative entropy of entanglement. Curve iii) displays the entanglement of formation for the Werner state ρ_W while curve iv)displays the relative entropy of entanglement. In plot b) the entanglement difference between the entanglement of formation and relative entropy of entanglement is displayed for the maximally entangled mixed state (curve i) and the Werner state (curve ii).

observations can be drawn from this figure. First it is clear that for a given impurity, the relative entropy of entanglement for the maximally entangled mixed state is significantly less that the corresponding value of the entanglement of formation. In fact for a small degree of impurity the entanglement of formation for the MEMS state is quite flat, however the relative entropy of entanglement falls quite rapidly. For a change in linear entropy of $\Delta S_L = 0.1$, $\Delta E_F(\rho_M) \approx 0.05$ and $\Delta E_N(\rho_M) \approx 0.2$. While it is clear that $E_N(\rho) \leq E_F(\rho)$ the difference for certain impurity values is quite large (Fig 1b curve i). This difference is larger than the difference for the Werner state and raises several interesting questions about whether the difference between the entanglement of formation and the relative entropy of entanglement increases as states with a fixed E_F become more mixed. What is the optimal form of the maximally entangled mixed states when the entanglement is measured by the relative entropy of entanglement. Both of these questions are left for later discussion.

There is also a region in Fig (1a) where the relative entropy of the maximally entangled mixed state is greater than the entanglement of formation for the Werner state. This occurs for very small degrees of entanglement where the linear entropy is between $S_L = 0,8 - 0.9$ Another means that has been proposed for establishing the usefulness of the entanglement resource is in teleportation. Here one finds that the maximally entangled mixed state for single qubit teleportation at a fixed linear entropy produces with a higher fedility than they achieved by the Werner state.

To summarize, we have investigated several properties of the maximally entangled mixed state. This work was funded in part by the European project EQUIP(IST-1999-11053) and QUIPROCONE (IST-1999-29064)

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