Bell inequalities for arbitrarily high dimensional systems

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We develop a novel approach to Bell inequalities based on a constraint that the correlations exhibited by local variable theories must satisfy. This is used to construct a family of Bell inequalities for bipartite quantum systems of arbitrarily high dimensionality which are strongly resistant to noise. In particular our work gives an analytic description of previous numerical results, and generalizes them to arbitrarily high dimensionality.

One of the most remarkable aspect of quantum mechanics is its predicted correlations. Indeed, the correlations between outcomes of measurements performed on systems composed of several parts in an entangled state have no classical analog. The most striking aspect of this characteristic feature of quantum physics is revealed when the parts are spatially separated: no classical theory based on local variables can reproduce the quantum correlations. Historically, this became known as the EPR paradox and was formulated in terms of measurable quantities by Bell [1] and by Clauser, Horn, Shimony and Holt [2] as the nowadays famous inequalities. Other aspects of quantum correlation were analyzed in the form of paradoxes, like, e.g. Schrödinger's cat and the measurement problem. In recent years, these paradoxical aspects have been overthrown by a more effective approach: *let's exploit "quantum strangeness" to perform tasks that are classically impossible* has become the new leitmotiv! From this "conceptual revolution", the field of quantum information emerged. Old words became fashionable, like "entanglement". Old questions were revisited, like the classifications of quantum correlation.

The variety of known partial results, in particular about entanglement measures, makes it today obvious that there is no one-parameter classification of entanglement. This letter concerns classifications related to what is called quantum non-locality, i.e. the impossibility to reproduce quantum correlations with theories based on local variables (often called local realistic theories). Specifically we develop a powerful new approach to Bell inequalities which we then use to write several families of Bell inequalities for higher dimensional systems.

Local variable theories cannot exhibit arbitrary correlations. Rather the conditions these correlations must obey can always be written as inequalities (the Bell inequalities) which the joint probabilities of outcomes must satisfy. Our approach to Bell inequalities is based on a logical constraint the correlations must satisfy in the case of local variable theories. In order to introduce this constraint, let us suppose that one of the parties, Alice, can carry out two possible measurements, A_1 or A_2, and that the other party, Bob, can carry out two possible measurements, B_1 or B_2. Each measurement may have d possible outcomes: $A_1, A_2, B_1, B_2 = 0, \dots, d-1$. Without loss of generality a local variable theory can be described by d^4 probabilities c_{jklm} $(j,k,l,m = 0,\dots,d-1)$ that Alice's local variable (jk) specifies that measurement A_1 gives outcome j and that measurement A_2 gives outcome k; and that Bob's local variable (lm) specifies that measurement B_1 gives outcome l and that measurement B_2 gives outcome m. (In this formulation Alice and Bob's strategy is deterministic since it is completely determined by the value of their variables jk and lm. Any non deterministic local theory can be rephrased in the above way by incorporating the local randomness in the probabilities c_{jklm}, see for instance [3]). Since they are probabilities the c_{jklm} are positive ($c_{jklm} \geq 0$) and sum to one ($\sum_{jklm} c_{jklm} = 1$). The joint probabilities take the form $P(A_1 = j, B_1 = l) = \sum_{km} c_{jklm}$, and similarly for $P(A_1 = j, B_2 = m)$, $P(A_2 = k, B_1 = l)$ and $P(A_2 = k, B_2 = m)$.

Let us consider a particular choice of local variables $jklm$ (this choice occurs with probability c_{jklm}). Since $A_1 = j$, $A_2 = k$, $B_1 = l$, $B_2 = m$ we have

$$\begin{aligned} r' &\equiv B_1 - A_1 = l - j \ , \\ s' &\equiv A_2 - B_1 = k - l \ , \\ t' &\equiv B_2 - A_2 = m - k \ , \\ u' &\equiv A_1 - B_2 = j - m \ . \end{aligned} \tag{1}$$

We see that the difference, r', between A_1 and B_1 can be freely chosen by choosing j and l. Similarly the difference, s', between B_1 and A_2 and the difference, t', between A_2 and B_2 can be freely chosen. But then the difference u' between B_2 and A_1 is constrained since we necessarily have

$$r' + s' + t' + u' = 0 \ . \tag{2}$$

Thus in a local variable theory the relation between three pairs of operators can be freely chosen, but then the last relation is constrained.

This constraint plays a central role in our Bell inequalities. Indeed they are written in such a way that their maximum value can be attained only if this constraint is frustrated. The simplest such Bell expression is

$$I \equiv P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) \quad (3)$$

where we have introduced the probability $P(A_a = B_b + k)$ that the measurements A_a and B_b have outcomes that differ, modulo d, by k:

$$P(A_a = B_b + k) \equiv \sum_{j=0}^{d-1} P(A_a = j, B_b = j + k \text{ mod d}) \ . \quad (4)$$

Because the difference between A_a and B_b is evaluated modulo d, all the outcomes of A_a and B_b are treated on an equal footing. As we see in eq. (3) this symmetrization is the key to reducing Bell inequalities to the logical constraint that is imposed by local variable theories. Indeed because of the constraint eq. (2) any choice of local variables $jklm$ can satisfy only three of the relations appearing in eq. (3), eg. $A_1 = B_1$, $B_1 = A_2 + 1$, etc.... Hence I(local realism) ≤ 3. On the other hand non-local correlations can attain $I = 4$ since they can satisfy all 4 relations.

In the case of two dimensional systems the inequality I(local variable) ≤ 3 is equivalent to the CHSH inequality [2]. But the power of our reformulation is already apparent since this inequality generalizes the CHSH inequality to arbitrarily large dimensions. In fact the above formulation of the constraint imposed by local realistic theories allows one to write in a unified way all previously known Bell inequalities [4]. It can also serve to write completely new Bell inequalities and this is the subject of the present article. Specifically we have generalised in a non trivial way (see eqs. (5) and (6) below) the Bell expression (3) to d dimensional systems (for any $d \geq 2$).

One of the interests of these new Bell expressions is that they are highly resistant to noise. Indeed Bell inequalities are sensitive to the presence of noise and above a certain amount of noise the Bell inequalities will cease to be violated by a quantum system. However it has been shown by numerical optimization [5] that using higher dimensional systems can increase the resistance to noise. The measurements that are carried out on the quantum system in order to obtain an increased violation have been described analytically in [6]. And an analytical proof of the greater robustness of quantum systems of dimension 3 was given in [7]. One of the interests of our new Bell inequalities is that when we apply them to the quantum state and measurement described in [6] for those dimensions ($d \leq 16$) for which a numerical optimisation was carried out in [6], we obtain the same resistance to noise as in [6].

The first generalisation of the Bell expression eq. (3) is

$$\begin{aligned} I_3 = &+ [P(A_1 = B_1) + P(B_1 = A_2 + 1) \\ &+P(A_2 = B_2) + P(B_2 = A_1)] \\ &- [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ &+P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \ . \end{aligned} \quad (5)$$

The maximum value of I_3 for non-local theories is 4 since a non local theory could satisfy all 4 relations that have a + sign in (5). On the other hand for a local variable theory $I_3 \leq 2$. This should be compared to the constraint I(local variable) ≤ 3 for the expression (3). The origin of this difference is the $-$ signs in (5). Indeed we have seen when analyzing (3) that only three of the relations with a + sign can be satisfied by local realistic theories. But if 3 relations with + are satisfied in (5), then necessarily one relation with $-$ is also satisfied giving a total of $I_3 = 2$. Alternatively one can satisfy 2 relations with + and two relations with weight zero (if the dimension is larger than 2), once more giving a total of $I_3 = 2$.

For $d = 2$ the inequality I_3(local variable) ≤ 2 is equivalent to the inequality I(local variable) ≤ 3 and therefore to the CHSH inequality. But for $d \geq 3$ the inequality based on I_3 is not equivalent to that based on I. For the quantum measurement described below (when $d \geq 3$) the inequality based on I_3 (and its generalisations I_d given below) is more robust than that based on I.

The Bell expression I_3 can be further generalised when the dimensionality is greater than 3 by adding extra terms. The extra terms in I_d do not change the maximum value attainable by local variable theories (I_d^{max}(local variable) $= 2$), nor do they change the maximum value attainable by completely non local theories ($I_d^{max} = 4$). However these extra terms allow a better exploitation of the correlations exhibited by quantum systems.

These new Bell expressions have the form:

$$\begin{aligned} I_d = &\sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \\ &\Big(+ [P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ &\quad +P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] \\ &\quad - [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\ &\quad +P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]\Big) \ . \end{aligned} \quad (6)$$

As mentioned above the maximum value of I_d is 4. This follows immediatly from the fact that the maximum weight of the terms in (6) is +1. And the maximum value of I_d for local variable theories is 2. We now prove this last result.

The proof consists of enumerating all the possible relations between A_1, B_1, A_2, B_2 allowed by the constraints (2). This is most easily done by first changing notation. We do not use the coefficients r', s', t', u' defined in (1), but use new coefficients r, s, t, u defined by the relation

$$A_1 = B_1 + r \ , \ B_1 = A_2 + s + 1 \ ,$$
$$A_2 = B_2 + t \ , \ B_2 = A_1 + u \ , \tag{7}$$

which obey the constraint

$$r + s + t + u + 1 = 0 \text{ mod } d \ . \tag{8}$$

Furthermore we restrict (without loss of generality) r, s, t, u to lie in the interval

$$-[d/2] \leq r, s, t, u \leq [(d-1)/2] \tag{9}$$

With this notation the value of the Bell inequality for a given choice of r, s, t, u is

$$I_d(r, s, t, u) = f(r) + f(s) + f(t) + f(u) \tag{10}$$

where f is given by

$$f(x) = \begin{cases} -\frac{2x}{d-1} + 1 & , \ x \geq 0 \\ -\frac{2x}{d-1} - \frac{d+1}{d-1} & , \ x < 0 \end{cases} \tag{11}$$

We now consider different cases according to the signs of r, s, t, u.

1. r, s, t, u are all positive. Then (8) and (9) imply that $r + s + t + u = d - 1$. Inserting into (10) and using (11) one finds $I_d = 2$.

2. Three of the numbers r, s, t, u are positive, one is strictly negative. Then (8) and (9) imply that either $r + s + t + u = d - 1$ or $r + s + t + u = -1$. Inserting into (10) and using (11) one finds either $I_d = -2/(d-1)$ or $I_d = 2$.

3. Two of the numbers r, s, t, u are positive, two are strictly negative. Then (8) and (9) imply that $r + s + t + u = -1$. Inserting into (10) and using (11) one finds $I_d = -2/(d-1)$.

4. One of the numbers r, s, t, u is positive, three are strictly negative. Then (8) and (9) imply that either $r + s + t + u = -1$ or $r + s + t + u = -d - 1$. Inserting into (10) and using (11) one finds either $I_d = -2(d+1)/(d-1)$ or $I_d = -2/(d-1)$.

5. The numbers r, s, t, u are all strictly negative. Then (8) and (9) imply that $r + s + t + u = -d - 1$. Inserting into (10) and using (11) one finds $I_d = -2(d+1)/(d-1)$.

(Note that for small dimensions d not all the possibilities enumerated above can occur. For instance for $d = 2$, the only possible values are $I_d = \pm 2$.) Thus for all possible choices of r, s, t, u, I_d(local realism) ≤ 2. This concludes the proof.

Let us now consider the maximum value that can be attained for the Bell expressions I_d for quantum measurements on an entangled quantum state. We have carried out a numerical search for the optimal measurements. It turns out that the best measurements that we have found numerically give the same value as the measurements described in [6]. We do not have a proof that these measurements are optimal, but our numerical work and the numerical work that inspired [6] suggests that this is the case.

We therefore first recall the state state and the measurement described in [6]. The quantum state is the maximally entangled state of two d-dimensional systems

$$\psi = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes |j\rangle_B \ . \tag{12}$$

The measurements is carried out in 3 steps. First Alice and Bob give each of the states $|j\rangle$ a variable phase, $e^{i\phi_a(j)}$ for Alice and $e^{i\varphi_b(j)}$ for Bob, which depends on the measurement they want to carry out. The state thus becomes

$$\psi = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_a(j)} e^{i\varphi_b(j)} |j\rangle_A \otimes |j\rangle_B \ . \tag{13}$$

where $\phi_1(j) = \frac{2\pi}{d}\alpha_1 j$, $\phi_2(j) = \frac{2\pi}{d}\alpha_2 j$, $\varphi_1(j) = \frac{2\pi}{d}\beta_1 j$ and $\varphi_2(j) = \frac{2\pi}{d}\beta_2 j$ with $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\beta_1 = 1/4$ and $\beta_2 = -1/4$. The second step consists of each party carrying out a discrete Fourier transform to bring the state to the form

$$\psi = \frac{1}{d^{3/2}} \sum_{j,k,l=0}^{d-1} \exp\left[i\left(\phi_a(j) + \varphi_b(j) + \frac{2\pi}{d} j(k-l)\right)\right]$$
$$|k\rangle_A \otimes |l\rangle_B \ . \tag{14}$$

The final step is for Alice to measure the k basis and Bob to measure the l basis. Thus the joint probabilities are

$$\begin{aligned} & P_{QM}(A_a = k, B_b = l) \\ &= \frac{1}{d^3} \left| \sum_{j=0}^{d-1} \exp\left[i\frac{2\pi j}{d}(k - l + \alpha_a + \beta_b)\right] \right|^2 \\ &= \frac{1}{d^3} \frac{\sin^2[\pi(k - l + \alpha_a + \beta_b)]}{\sin^2[\pi(k - l + \alpha_a + \beta_b)/d]} \\ &= \frac{1}{2d^3 \sin^2[\pi(k - l + \alpha_a + \beta_b)/d]} \end{aligned} \tag{15}$$

where in the last line we have used the values of α_a and β_b given above.

Equation (15) shows that these joint probabilities have several symmetries. First of all we have the relation

$$P_{QM}(A_a = k, B_b = l) = P_{QM}(A_a = k + c, B_b = l + c)$$

for all integers c. This symmetry property justifies us considering, as in (4), only the probabilities that A_a and B_b differ by a given constant integer c:

$$\begin{aligned} P_{QM}(A_a = B_b + c) &= \sum_{j=0}^{d-1} P_{QM}(A_a = j + c, B_b = j) \\ &= d P_{QM}(A_a = c, B_b = 0) \ . \end{aligned} \tag{16}$$

Furthermore we have the relation

$$\begin{aligned} &P_{QM}(A_1 = B_1 + c) = P_{QM}(B_1 = A_2 + c + 1) \\ &= P_{QM}(A_2 = B_2 + c) = P_{QM}(B_2 = A_1 + c) \ . \end{aligned} \quad (17)$$

Using eqs. (15 to 17) we can rank these probabilities by decreasing order. Let us denote

$$q_c = P_{QM}(A_1 = B_1 + c) = 1/\left(2d^3 \sin^2[\pi(c + 1/4)/d]\right) \ . \quad (18)$$

Then we have

$$q_0 > q_{-1} > q_1 > q_{-2} > q_2 > \ldots > q_{-[d/2]} \ (> q_{[d/2]}) \quad (19)$$

where $[x]$ denotes the integer part of x and the last term between parenthesis occurs only for odd dimension d. This suggests that the quantum probabilities violate the constraints imposed by local variable theories. Indeed the probabilities in (17) are maximized by taking $c = 0$, but then the 4 relations that appear in (17) are incompatible with local realism. In fact replacing the above probabilities in the expression (3) yields a value $I_{QM} = 4dq_0 > 3$ for all dimensions d.

However a stronger violation is obtained if instead of using the Bell expression I, one uses the Bell expressions I_d. In fact for a d dimensional quantum systems, one can use all the Bell expressions I_k for $k \leq d$, but the strongest violation is obtained by using the Bell expression I_d. This value, denoted $I_d(QM)$, is given by

$$I_d(QM) = 4d \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \left(q_k - q_{-(k+1)}\right) \ . \quad (20)$$

For instance we find

$$\begin{aligned} I_3(QM) &= 4/\left(-9 + 6\sqrt{3}\right) \simeq 2.87293 \ , \\ I_4(QM) &= \frac{2}{3}\left(\sqrt{2} + \sqrt{10 - \sqrt{2}}\right) \simeq 2.89624 \ , \\ \lim_{d\to\infty} I_d(QM) &= \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(k + 1/4)^2} - \frac{1}{(k + 3/4)^2} \\ &= 32 \text{ Catalan}/\pi^2 \simeq 2.6981 \end{aligned}$$

where Catalan $\simeq 0.9159$ is Catalan's constant.

In the presence of uncolored noise the quantum state becomes

$$\rho = p|\psi\rangle\langle\psi| + (1 - p)\frac{\mathbb{1}}{d^2} \quad (21)$$

where p is the probability that the state is unaffected by noise. The value of the Bell inequality for the state ρ is

$$I_d(\rho) = pI_d(QM) \quad (22)$$

Hence the Bell inequality I_d is certainly violated if

$$p > \frac{2}{I_d(QM)} = p_d^{min} \ . \quad (23)$$

(If there is a quantum measurement giving a value of I_d greater than that given by eq. (20), then of course the Bell inequality would be violated with even more noise. This remark applies to the various p^{min} below).

As a function of d one finds that p_d^{min} is a decreasing function of d. For instance:

$$\begin{aligned} p_3^{min} &= (6\sqrt{3} - 9)/2 \simeq 0.69615 \\ p_4^{min} &= 3/(\sqrt{2} + \sqrt{10 - \sqrt{2}}) \simeq 0.69055 \\ \lim_{d\to\infty} p_d^{min}(d) &= \pi^2/\left(16 \text{ Catalan}\right) \simeq 0.67344 \end{aligned}$$

For $d = 3$ this reproduces the analytical result of [7]. And combining eqs. (20) and (23) reproduces the numerical results of [6] for all dimensions ($2 \leq d \leq 16$) for which a numerical optimization was carried out.

In summary our reformulation of Bell inequalities in terms of a logical constraint local variable theories must satisfy has provided the basis for constructing a large family of Bell inequalities for systems of large dimension. The numerical work of [5,6] and a numerical search of our own suggest that these Bell inequalities are optimal in the same sense that the CHSH inequality is optimal for 2 dimensional systems, namely both the resistance to noise and the amount by which the inequality are violated are maximal. For this reason we hope that the Bell inequalities presented here will have as much interest for physicists studying entanglement of systems of large dimensionality as the CHSH inequalities have had for bidimensional systems.

Note: while completing this paper we learned of a Bell inequality for qutrits [8] that exhibits the same resistance to noise as that obtained in [5–7].

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