A Group-Theoretic Approach to Constructions Of Nonrelativistic Spin-Statistics

J.M. Harrison[1], J.M. Robbins
Basic Research Instititute in the Mathematical Sciences
HP Laboratories Bristol
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[1] Department of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK

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J. M. Harrison[a] & J. M. Robbins[a b]

[a]*Department of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, U.K.*
[b]*Basic Research Institute in the Mathematical Sciences, Hewlett-Packard Laboratories Bristol, Filton Road, Stoke Gifford, Bristol BS34 8QZ, U.K.*

Abstract. We give a group-theoretical generalization of Berry and Robbins' treatment of identical particles with spin. The original construction, which leads to the correct spin-statistics relation, is seen to arise from particular irreducible representations – the totally symmetric representations - of the group *SU*(4). Here we calculate the exchange signs and corresponding statistics for all irreducible representations of *SU*(4).

INTRODUCTION

Berry and Robbins [1 - 3] formulate quantum mechanics for identical spin-s particles – here we suppose there are two - on a configuration space in which permuted configurations are identified. The two-particle wavefunctions are expanded in a basis of transported spin states $\left|m_1\, m_2(\boldsymbol{r})\right\rangle\!\rangle$ which are made to vary smoothly with the relative coordinate $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ so that spins are exchanged with positions up to a sign $(-1)^k$, i.e.

$$\left|m_1\, m_2(-\boldsymbol{r})\right\rangle\!\rangle=(-1)^k\left|m_2\, m_1(\boldsymbol{r})\right\rangle\!\rangle . \qquad (1)$$

The position, momentum and spin operators are defined so as to satisfy the standard commutation relations, and the statistics are determined by the exchange sign: $+1$ for bosons and -1 for fermions.

The construction in [1] is based on the Schwinger model for spin, in which each spin is described in terms of a pair of harmonic oscillators. The spin angular momentum $\boldsymbol{S}$ is expressed in terms of the creation and annihilation operators, and the spin-s basis states $\left|m\right\rangle$ - eigenstates of S_z - correspond to number states with $s+m$ (resp. $s-m$) quanta in the oscillators. Two-spin eigenstates $\left|m_1\, m_2\right\rangle$ are analogously represented in terms of two pairs of oscillators.

The transported basis $\left|m_1\, m_2(\boldsymbol{r})\right\rangle\!\rangle$ is obtained from the Schwinger states $\left|m_1\, m_2\right\rangle$ by a unitary *exchange rotation*,

$$\left|m_1\, m_2(\boldsymbol{r})\right\rangle\!\rangle=U(\boldsymbol{r})\left|m_1\, m_2\right\rangle, \quad \text{where} \quad U(\boldsymbol{r})=\exp\left(-iq\,\hat{\boldsymbol{n}}(\boldsymbol{r}).\boldsymbol{E}\right). \qquad (2)$$

The *exchange angular momentum* $\boldsymbol{E}$, which generates the exchange rotation, couples quanta between pairs of oscillators, just as spin angular momentum couples quanta within a single oscillator pair. An exchange rotation by p about an axis perpendicular to z takes $\left|m_1\, m_2\right\rangle$, up to a phase factor, to $\left|m_2\, m_1\right\rangle$ (in analogy with the fact that a spin rotation by p about an axis perpendicular to z takes $\left|m\right\rangle$, up to a phase factor, to $\left|-m\right\rangle$). Because the $\left|m_2\, m_1\right\rangle$ states are null eigenstates of E_z, i.e.

$$E_z\left|m_1\, m_2\right\rangle = 0\,, \tag{3}$$

the phase factor accompanying exchange is an axis-independent sign. Calculation gives $\exp\left(-ip\ E_y\right)\left|m_1\, m_2\right\rangle = \left(-1\right)^{2s}\left|m_2\, m_1\right\rangle$. It follows that the exchange condition (1) is satisfied, with $k = 2s$, if and only if $U\left(-\boldsymbol{r}\right)$ and $U\left(\boldsymbol{r}\right)$ differ on the right by the exchange rotation $\exp\left(-ip\ E_y\right)\times\exp\left(-i\,a\left(\boldsymbol{r}\right)E_z\right)$. A simple choice which satisfies this condition is to let $U\left(\boldsymbol{r}\right)$ be an exchange rotation corresponding to any spatial rotation which maps $\hat{\boldsymbol{z}}$ to $\hat{\boldsymbol{r}}$.

In [1] it was suggested that certain features of the transported basis (smooth exchange, parallel transport) might characterize the spin-statistics relation. In [2] it was pointed out that this is not so; there are other constructions of the transported basis which have these features and yet yield different statistics. While the Schwinger construction appears to be the simplest in certain respects, specific principles are needed to establish a relation between spin and statistics.

Here we examine in a systematic way a family of alternative constructions motivated by group-theoretical considerations. We observe that the spin rotations constitute a group, $SU(2)\times SU(2)$ (there is one $SU(2)$ factor for each spin), and the exchange rotations form another $SU(2)$ group. Together, the spin and exchange rotations generate, and are subgroups of, a larger group, $SU(4)$. The Schwinger representation corresponds to particular irreducible representations, one for each s, of $SU(4)$, but there are many others, and the construction (2) of the transported basis generalizes to them. What is required is that the states $\left|m_1\, m_2\right\rangle$ transform under spin rotations as an (s,s)-multiplet, and that they are null eigenstates of E_z (cf. (3)). The exchange sign in

$$\exp\left(-ip\ E_y\right)\left|m_1\, m_2\right\rangle = \left(-1\right)^{k}\left|m_2\ m_1\right\rangle, \tag{4}$$

which determines the particle statistics as above, depends on the chosen representation and multiplet. It turns out that not all irreducible representations support the construction, but for those that do the exchange sign need not be unique. A detailed account of this work will be given elsewhere [4].

REPRESENTATIONS OF $SU(4)$

The defining representation of the generators of $SU(4)$ is

$$S_{1i} = \tfrac{1}{2}\begin{pmatrix} s_i & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{2i} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & s_i \end{pmatrix}$$
$$E_x = \tfrac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad E_y = \tfrac{1}{2}\begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \qquad E_z = \tfrac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

where s_i are the Pauli matrices and I is the 2×2 identity matrix. The operators $S_{1i}\, S_{2i}$ are the usual spin operators for the two particles and generate the $SU(2)\times SU(2)$ spin subgroup. The irreducible representations of $SU(4)$ are constructed by applying maximal symmetry conditions to a tensor product of basis states of the defining representation [5]. The symmetry conditions applied to the tensor product are recorded in a Young tableau. A Young tableau, denoted by $(n_1, n_2, \ldots, n_r)$, is a series of rows of boxes where row i has length n_i and $n_1 \ge n_2 \ge \ldots \ge n_r$. The irreducible representations of $SU(4)$ are in one-to-one correspondence with Young tableaux containing up to three rows, and the representations of $SU(2)$, with tableaux of one row. We refer to the representations of $SU(4)$ by using the notation (n_1, n_2, n_3), where $n_1 + n_2 + n_3 = n$. $SU(2)$ tableaux can be augmented on the left by any number of columns of two; an $SU(2)$ tableau with two rows (l_1, l_2) is equivalent to one with a single row $(l_1 - l_2)$. This equivalence is used in $SU(2)\times SU(2)$ content of the irreducible representation of $SU(4)$.

CALCULATING THE EXCHANGE SIGN

An irreducible representation (n_1, n_2, n_3) of $SU(4)$, when restricted to the subgroup of spin rotations $SU(2)\times SU(2)$, is reducible, and can be decomposed into its irreducible spin components. These are labeled by two $SU(2)$ tableaux (k_1, k_2) and (l_1, l_2), corresponding to spins $(l_1 - l_2)/2$ and $(k_1 - k_2)/2$. Requiring the two spins to have the same value s and that the component be a nullspace of E_z, we find that (k_1, k_2) and (l_1, l_2) must coincide, with

$$s = (l_1 - l_2)/2\,. \tag{6}$$

Using the rules for decomposing the $SU(2)\times SU(2)$ subspace [6], we find that the number p of these (s, s) multiplets in a representation of $SU(4)$ is the minimum of the eleven integers

$$
\begin{array}{llll}
1) & l_2+1 & 7) & n_1+n_2-2l_1+1 \\
2) & n_1-n_2+1 & 8) & l_1+l_2-n_2+1 \\
3) & n_1-l_1+1 & 9) & n_1+n_2-2l_2-l_1+1 \\
4) & n_1-2l_1+1 & 10) & n_1+2n_2-2l_1-2l_2+1 \\
5) & n_2-l_2+1 & 11) & 2l_1+2l_2-n_1-n_2+1 \\
6) & l_1-l_2+1 & &
\end{array}
. \qquad (7)
$$

(Interestingly, none of the quantities 1) – 11) are redundant). If the minimum of (7) is negative or zero, there are no allowed spin-s multiplets in the representation. If p is even, there are equal numbers of bosonic (exchange sign = +1) and fermionic (exchange sign = -1) multiplets. If p is odd, there is either an extra bosonic or fermionic multiplet; the exchange sign of the extra multiplet can be calculated explicitly.

Table 1 shows an example of the allowed (s,s) multipets in the $(8,5,3)$ representation of $SU(4)$, along with their exchange signs. For the case $s=1$, the representation of $SU(2)$ is described by a tableau with two rows, $l_1=5$ and $l_2=3$. There are three $s=1$ multiplets, one with exchange sign $+1$ (the physically correct sign) and two with exchange sign -1.

TABLE 1. Decomposing the $(8,5,3)$ representation of $SU(4)$ into its (s,s) multiplets.

Spin	Number of multiplets	Exchange signs
2	3	-1, +1, -1
1	3	-1, +1, -1
0	1	-1

The derivation of these results will be given elsewhere [4].

CONCLUSION

From the perspective of representation theory, the expression (2) for the transported basis for two spins generalizes to other irreducible representations of $SU(4)$, and the exchange signs for all admissible representations can be calculated. These generalized results do not yield any spin-statistics relation, much less the physically correct one. A typical irreducible representation gives rise to transported bases for several values of spin, and for each value may lead to transported bases with either exchange sign. The only systematic scheme for associating representations to spin to have suggested itself is the one provided implicitly by the Schwinger model of [1]. The completely symmetric representations, the simplest from the point of view of representation theory, give rise to a single transported basis for just one value of spin, and thus lead to a definite spin-statistics relation.

Within this representation-theoretic framework, the completely symmetric representations provide a natural mechanism for incorporating the indistinguishability of spins along with positions of identical particles in nonrelativistic quantum mechanics. But the framework itself requires justification. A compelling derivation of the spin-statistics relation should proceed from general principles motivated by considerations of physics and/or mathematical simplicity. The models introduced here should prove useful in formulating and testing such principles.

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