Eigenvalues of the Laguerre Process as Non-colliding Squared Bessel Processes

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Let $A(t)$ be a nxp matrix with independent standard complex Brownian entries and set $M(t) = A(t)^* A(t)$. This is a process version of the Laguerre ensemble and as such we shall refer to it as the *Laguerre process*. The purpose of this note is to remark that, assuming $n > p-1$, the eigenvalues of $M(t)$ evolve like p independent squared Bessel processes of dimension $2(n-p+1)$, conditioned (in the sense of Doob) never to collide. More precisely, the function $h(\chi) = \prod_{i<j}(\chi_i - \chi_j)$ is harmonic with respect to p independent squared Bessel processes of dimension $2(n-p+1)$, and the eigenvalue process has the same law as the corresponding Doob h-transform. In the case where the entries of $A(t)$ are *real* Brownian motions, $(M(t))_{t \geq 0}$ is the Wishart process considered by Bru [Br91]. There it is shown that the eigenvalues of $M(t)$ evolve according to a certain diffusion process, the generator of which is given explicitly. An interpretation in terms of non-colliding processes does not seem to be possible in this case. We also identify a class of processes (including Brownian motion, squared Bessel processes and generalised Ornstein-Uhlenbeck processes) which are all amenable to the same h-transform, and compute the corresponding transition densities and upper tail asymptotics for the first collision time.

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EIGENVALUES OF THE LAGUERRE PROCESS AS NON-COLLIDING SQUARED BESSEL PROCESSES

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Abstract: Let $A(t)$ be a $n \times p$ matrix with independent standard complex Brownian entries and set $M(t) = A(t)^* A(t)$. This is a process version of the Laguerre ensemble and as such we shall refer to it as the *Laguerre process*. The purpose of this note is to remark that, assuming $n > p-1$, the eigenvalues of $M(t)$ evolve like p independent squared Bessel processes of dimension $2(n-p+1)$, conditioned (in the sense of Doob) never to collide. More precisely, the function $h(x) = \prod_{i<j}(x_i - x_j)$ is harmonic with respect to p independent squared Bessel processes of dimension $2(n-p+1)$, and the eigenvalue process has the same law as the corresponding Doob h-transform.

In the case where the entries of $A(t)$ are *real* Brownian motions, $(M(t))_{t \geq 0}$ is the Wishart process considered by Bru [Br91]. There it is shown that the eigenvalues of $M(t)$ evolve according to a certain diffusion process, the generator of which is given explicitly. An interpretation in terms of non-colliding processes does not seem to be possible in this case.

We also identify a class of processes (including Brownian motion, squared Bessel processes and generalised Ornstein-Uhlenbeck processes) which are all amenable to the same h-transform, and compute the corresponding transition densities and upper tail asymptotics for the first collision time.

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1. Introduction

Let $A(t)$ be a $n \times p$ matrix with independent standard complex Brownian entries (so that each entry of $A(t)$ has variance $2t$) and set $M(t) = |A(t)|^2 = A(t)^* A(t)$. We shall refer to $\mathbb{M} = (M(t))_{t\in[0,\infty)}$ as the *Laguerre process*. In the case $p = 1$, $\mathbb{M}$ is a squared Bessel process of dimension $2n$, usually denoted by BESQ^{2n}.

Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_p(t))$ be the vector of eigenvalues of $M(t)$, ordered decreasingly such that $\lambda_p(t) \geq \dots \geq \lambda_1(t) \geq 0$. (Note that $M(t)$ is almost surely nonnegative definite for any $t \geq 0$.) The process $(\lambda(t))_{t\geq 0}$ is a diffusion on $[0,\infty)^p$ with generator given by

$$H_{n,p} = 2\sum_{i=1}^p x_i \partial_i^2 + 2\sum_{i=1}^p \Big[n + \sum_{\substack{j=1\\ j\neq i}}^p \frac{x_i + x_j}{x_i - x_j}\Big]\partial_i. \tag{1.1}$$

This follows from the arguments given by Bru [Br91] for the Wishart case, with minor modifications. We remark that the Focker-Planck equation associated with $(\lambda(t))_{t\geq 0}$ was formally derived in [AW97].

We will assume that $n > p - 1$. Our main observation is that the process $(\lambda(t))_{t\geq 0}$ can be identified as the h-transform of p independent squared Bessel processes of dimension $2(n-p+1)$, where the function $h\colon [0,\infty)^p \to \mathbb{R}$ is given by

$$h(x) = \prod_{\substack{i,j=1\\ i<j}}^p (x_j - x_i), \qquad x = (x_1, \dots, x_p) \in [0,\infty)^p. \tag{1.2}$$

In other words, the process λ behaves like p independent $\mathrm{BESQ}^{2(n-p+1)}$ processes conditioned never to collide.

To justify this claim, we will show that the function h given by (1.2) is harmonic with respect to the generator

$$G_{p,d} = 2\sum_{i=1}^p x_i \partial_i^2 + d\sum_{i=1}^p \partial_i \tag{1.3}$$

of a vector of p independent BESQ^d, and use standard methods to compute the generator $\widehat{G}_{p,d}$ of the h-transform. We obtain

$$\widehat{G}_{p,d} = 2\sum_{i=1}^p x_i \partial_i^2 + d\sum_{i=1}^p \partial_i + 2\sum_{i=1}^p \Big[\sum_{\substack{j=1\\ j\neq i}}^p \Big(\frac{x_i + x_j}{x_i - x_j} + 1\Big)\Big]\partial_i. \tag{1.4}$$

It is now easy to see that $H_{n,p} = \widehat{G}_{p,2(n-p+1)}$. This will be presented carefully in the next section.

As is well-known, the function h is also harmonic with respect to the generator of p-dimensional Brownian motion. This also arises in the context of random matrices. It is a classical result, due to Dyson [Dy62], that the eigenvalues of Hermitian Brownian motion (the process-version of the Gaussian unitary ensemble) evolve like independent Brownian motions conditioned never to collide (see also [Gr00]). In Lemma 3.1 below we identify a class of generators for which the function h is harmonic which includes both of the above. We remark that Dyson also considered unitary Brownian motion, and showed that the eigenvalues in this case behave like independent Brownian motions on the circle conditioned never to collide via the complex analogue of the function h. (For more detailed information about this process see [HW96].)

In the Wishart case, where the entries of $\mathbb{A} = (A(t))_{t\geq 0}$ are independent standard *real* Brownian motions, we do not see how to give a similar interpretation for the eigenvalue process. In this case, Bru [Br91] identified the generator of the process of eigenvalues of $M(t)$ as

$$2\sum_{i=1}^{p} x_i \partial_i^2 + \sum_{i=1}^{p}\Big[n + \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{x_i + x_j}{x_i - x_j}\Big]\partial_i. \tag{1.5}$$

Note the missing factor of 2 in front of the drift term.

Similar remarks apply to the Gaussian ensembles: in Dyson's work [Dy62] it turned out that, in contrast to the complex case, the process version of the Gaussian orthogonal ensemble (the real case) does not admit a representation of the eigenvalue process in terms of a system of independent particles conditioned never to collide.

The interpretation of the Laguerre eigenvalue processes as h-transforms can be applied to obtain alternative derivations for the eigenvalue densities of the corresponding ensemble. As is known from the theory of random matrices (see, e.g., [Ja64]), these densities are given in the following closed form. We have

$$\mathbb{P}(\lambda(1) \in \mathrm{d}x) = \frac{1}{Z_{p,\nu}} \prod_{\substack{i,j=1\\ i<j}}^{p} (x_i - x_j)^2 \prod_{j=1}^{p} \big[x_j^{\nu} e^{-x_j}\big]\,\mathrm{d}x, \qquad x_1 > \cdots > x_p \geq 0, \tag{1.6}$$

where $\nu = n-p$ denotes the index of $\mathrm{BESQ}^{2(n-p+1)}$, and $Z_{p,\nu}$ denotes the normalisation constant. In words, $\lambda(1)$ has the distribution of p independent Gamma(ν)-distributed random variables, transformed with the density $h(x)^2$.

In Section 2 we introduce the h-transform of $(\mathrm{BESQ}^d)^{\otimes p}$ and its generator, and in Section 3 we establish the harmonicity of h for a certain class of processes having independent components, which includes Brownian motion, squared Bessel processes and generalized Ornstein-Uhlenbeck processes driven by Brownian motion. Furthermore we calculate the transition densities of the transformed process started at the origin and describe the upper tail asymptotics of the first collision time of the components.

2. Non-colliding squared Bessel processes

Fix $p \in \mathbb{N}$ and let $\mathbb{X} = (\mathbb{X}(t))_{t\in[0,\infty)} = (X_1(t),\dots,X_p(t))_{t\in[0,\infty)}$ be a diffusion on $[0,\infty)^p$ whose components are independent squared Bessel processes (BESQ^d) of dimension d. In the following, the dimension d is any nonnegative number. The process $\mathbb{X}$ has the generator $G_{p,d}$ given by (1.3). We denote the distribution of $\mathbb{X}$ when started at $x \in [0,\infty)^p$ by $\mathbb{P}_x$. Note that 0 is an entrance boundary for the BESQ^d. In dimensions $d \geq 2$, the process $\mathbb{X}$ stays in $(0,\infty)^p$ after time zero for ever, and the domain of the generator consists of the functions f such that $G_{p,d}f$ is continuous and bounded on $[0,\infty)$ and $f^+(0^+) = 0$. If the dimension d is smaller than two, then the components of $\mathbb{X}$ hit zero with probability one, the boundary point 0 is non-singular. If 0 is reflecting, then $G_{p,d}$ has the same domain as above.

As follows from the more general Lemma 3.1 below, the function h in (1.2) is harmonic with respect to $G_{p,d}$, hence the h-transform of $(\mathrm{BESQ}^d)^{\otimes p}$ is well-defined. Let us compute its generator.

Lemma 2.1. *The generator of the h-transform of $\mathbb{X}$ is given by*

$$\widehat{G}_{p,d}f(x) = G_{p,d}f(x) + 2\sum_{i=1}^p \Big[\sum_{\substack{j=1\\ j\neq i}}^p \Big(\frac{x_i+x_j}{x_i-x_j}+1\Big)\Big]\partial_i f(x). \tag{2.1}$$

Proof. We have $\widehat{G}_{p,d} = G_{p,d} + \Gamma(\log h,\cdot)$, where $\Gamma(g,f) = G_{p,d}(f\cdot g) - fG_{p,d}(g) - gG_{p,d}(f)$ is the so-called *opérateur carré du champs* (see, for example, [RY91]). Hence,

$$\begin{aligned}\widehat{G}_{p,d}f - G_{p,d}f &= G_{p,d}(f\cdot\log h) - fG_{p,d}(\log h) - \log h\, G_{p,d}(f)\\ &= 2\sum_{i=1}^p x_i\big[\partial_i^2(f\log h) - f\partial_i^2\log h - \log h\,\partial_i^2 f\big]\\ &\quad + d\sum_{i=1}^p\big[\partial_i(f\log h) - f\partial_i\log h - \log h\,\partial_i f\big]\\ &= 2\sum_{i=1}^p x_i 2(\partial_i\log h)(\partial_i f) = 4\sum_{i=1}^p x_i\frac{\partial_i h}{h}\partial_i f\\ &= 4\sum_{i=1}^p x_i\sum_{j\neq i}\frac{1}{x_i-x_j}\partial_i f = 2\sum_{i=1}^p\Big[\sum_{\substack{j=1\\ j\neq i}}^p\Big(\frac{x_i+x_j}{x_i-x_j}+1\Big)\Big]\partial_i f(x).\end{aligned} \tag{2.2}$$

□

3. Generalisations and applications

In this section, we introduce further non-colliding processes by means of h-transforms of processes with independent components. Fix $p\in\mathbb{N}$ and let $\mathbb{X} = (\mathbb{X}(t))_{t\in[0,\infty)} = (X_1(t),\dots,X_p(t))_{t\in[0,\infty)}$ be a diffusion on a (possibly infinite) interval I which contains 0. By

$$Gf(x) = \sum_{i=1}^p \frac{1}{2}\sigma^2(x_i)\,\partial_i^2 f(x) + \sum_{i=1}^p \mu(x_i)\,\partial_i f(x), \qquad x=(x_1,\dots,x_p)\in I^p. \tag{3.1}$$

we denote the generator of $\mathbb{X}$, where $\sigma^2\colon I\to(0,\infty)$ and $\mu\colon I\to\mathbb{R}$. In the following we identify a class of processes for which the function h in (1.2) is harmonic.

Lemma 3.1. *Assume that any of the following cases is satisfied: Either $\frac12\sigma^2(x) = ax+b$ and $\mu(x) = c$ with some $a,b,c\in\mathbb{R}$, or (in the case $p>2$) $\frac12\sigma^2(x) = x^2+ax+b$ and $\mu(x) = 2(p-2)x/3+c$ for some $a,b,c\in\mathbb{R}$, or (in the case $p=2$) $\frac12\sigma^2(x)$ is arbitrary and $\mu(x)$ constant. Then h is harmonic with respect to G, i.e., $Gh\equiv 0$.*

Proof. Abbreviate $G = G_\sigma + G_\mu$ with obvious notation. Using the Leibniz rule $\left(\prod_i g_i\right)' = \sum_i g_i' \prod_{j\neq i} g_j$, one easily derives that

$$\begin{aligned} G_\mu h(x) &= h(x) \sum_{i<j} \frac{\mu(x_i) - \mu(x_j)}{x_i - x_j}, \\ G_\sigma h(x) &= -h(x) \sum_{i<j<k} \frac{1}{(x_k - x_j)(x_j - x_i)} \Big[\sigma^2(x_j) - \sigma^2(x_k)\frac{x_j - x_i}{x_k - x_i} - \sigma^2(x_i)\frac{x_k - x_j}{x_k - x_i}\Big]. \end{aligned}$$

Hence, both $G_\mu h$ and $G_\sigma h$ are identically zero if μ is constant and $\frac{1}{2}\sigma^2$ a polynomial of first order. However, if $p > 2$ and $\mu(x) = cx$ and $\frac{1}{2}\sigma^2(x) = x^2$, then $G_\mu h = c\frac{p}{2}(p-1)h$, and $G_\sigma h = -\frac{1}{3}p(p-1)(p-2)h$. Hence, $Gh \equiv 0$ for the choice $c = 2(p-2)/3$. Lastly, in the case $p = 2$, we have that $G_\sigma h \equiv 0$ since h is a polynomial of first order in this case. □

Note that Lemma 3.1 covers in particular the cases of Brownian motion, squared Bessel processes and generalised Ornstein-Uhlenbeck processes driven by Brownian motion (see [CPY01]).

As an application, we compute the transition densities of the h-transform of $\mathbb{X}$, started at the origin, and the upper tails of the first collision time

$$T = \inf\{t > 0 \colon X(t) \notin W\}, \tag{3.2}$$

where

$$W = \{x = (x_1, \dots, x_p) \in I^p \colon x_p > \cdots > x_1\}. \tag{3.3}$$

Let $p_t(x, y)$ denote the transition density of the process $(X_1(t))_{t\geq 0}$, say. We will first state a general result and later discuss the special cases of Brownian motion and BESQ^d.

Lemma 3.2. *Assume that h is harmonic for the generator of $\mathbb{X}$, and assume that there is a Taylor expansion*

$$\frac{p_t(x,y)}{p_t(0,y)} = f_t(x) \sum_{m=0}^{\infty} (xy)^m a_m(t), \qquad t \geq 0, y \in I, \tag{3.4}$$

for x in a neighborhood of zero, where $a_m(t) > 0$ and $f_t(x) > 0$ satisfy $\lim_{t\to\infty} a_{m+1}(t)/a_m(t) = 0$ and $f_t(0) = 1 = \lim_{t\to\infty} f_t(x)$. Then, for any $t > 0$ and $y \in W$,

$$\lim_{\substack{x\to 0\\ x\in W}} \widehat{\mathbb{P}}_x(X(t) \in \mathrm{d}y) = C_t h(y)^2\, \mathbb{P}_0(X(t) \in \mathrm{d}y), \tag{3.5}$$

where $C_t = \prod_{m=0}^{p-1} a_m(t)$. Furthermore, for any $x \in W$,

$$\mathbb{P}_x\big(T > t\big) \sim C_t h(x)\, \mathbb{E}_0\big[h(X(t))\mathbb{1}\{X(t) \in W\}\big], \qquad t \to \infty. \tag{3.6}$$

Proof. We are going to use the formula [KM59]

$$\mathbb{P}_x(T > t; X(t) \in \mathrm{d}y) = \sum_{\sigma\in\mathfrak{S}_p} \mathrm{sign}(\sigma) \prod_{i=1}^{p} p_t(x_i, y_{\sigma(i)})\ \mathrm{d}y, \qquad x, y \in W, \tag{3.7}$$

where $\mathfrak{S}_p$ denotes the set of permutations of $1, \dots, p$, and sign denotes the signum of a permutation. Use (3.4) in (3.7) to obtain that

$$\frac{\mathbb{P}_x\big(T > t; X(t) \in \mathrm{d}y\big)}{\mathbb{P}_0\big(X(t) \in \mathrm{d}y\big)} = \prod_{i=1}^p f_t(x_i) \sum_{m_1,\dots,m_p \in \mathbb{N}_0} \prod_{i=1}^p [x_i^{m_i} a_{m_i}(t)] \sum_{\sigma \in \mathfrak{S}_p} \mathrm{sign}(\sigma) \prod_{i=1}^p y_{\sigma(i)}^{m_i}. \tag{3.8}$$

Observe that

$$\sum_{\sigma \in \mathfrak{S}_p} \mathrm{sign}(\sigma) \prod_{i=1}^p y_{\sigma(i)}^{m_i} = \det\Big[\big(y_i^{m_j}\big)_{i,j=1,\dots,p}\Big] \tag{3.9}$$

is equal to zero if $m_1, \dots, m_p$ are not pairwise distinct. Hence, in (3.8), we may restrict the sum on $m_1, \dots, m_p \in \mathbb{N}_0$ to the sum on $0 \le m_1 < m_2 < \dots < m_p$ and an additional sum on $\tau \in \mathfrak{S}_p$ and write $m_{\tau(1)}, \dots, m_{\tau(p)}$ instead of $m_1, \dots, m_p$. This yields that

$$\frac{\mathbb{P}_x\big(T > t; X(t) \in \mathrm{d}y\big)}{\mathbb{P}_0\big(X(t) \in \mathrm{d}y\big)} = \prod_{i=1}^p f_t(x_i) \sum_{0 \le m_1 < \dots < m_p} \prod_{i=1}^p a_{m_i}(t) \det\Big[\big(y_i^{m_j}\big)_{i,j=1,\dots,p}\Big] \det\Big[\big(x_i^{m_j}\big)_{i,j=1,\dots,p}\Big]. \tag{3.10}$$

Now use that the two determinants may be written using the so-called *Schur function* [Ma79] as

$$\det\Big[\big(x_i^{m_j}\big)_{i,j=1,\dots,p}\Big] = h(x)\,\mathrm{Schur}_m(x), \tag{3.11}$$

where we abbreviated $m = (m_1, \dots, m_p)$. The Schur function $\mathrm{Schur}_m(x)$ is a certain multi-polynomial in $x_1, \dots, x_p$ whose coefficients are nonnegative integers and may be defined combinatorially. It is homogeneous of degree $m_1 + \dots + m_p - \frac{p}{2}(p-1)$ and has the properties

$$\begin{aligned} \mathrm{Schur}_m(1,\dots,1) &= \frac{h(m)}{\prod_{1 \le i < j \le p}(j-i)}, \\ \mathrm{Schur}_{(0,1,\dots,p-1)}(x) &= 1, \\ \mathrm{Schur}_m(0,\dots,0) &= \begin{cases} 1 & \text{if } m = (0,1,\dots,p-1), \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.12}$$

Using (3.11) in (3.8), we arrive at

$$\frac{\mathbb{P}_x\big(T > t; X(t) \in \mathrm{d}y\big)}{\mathbb{P}_0\big(X(t) \in \mathrm{d}y\big)} = h(x)h(y) \prod_{i=1}^p f_t(x_i) \sum_{0 \le m_1 < \dots < m_p} \mathrm{Schur}_m(x)\,\mathrm{Schur}_m(y) \prod_{i=1}^p a_{m_i}(t). \tag{3.13}$$

In order to derive (3.5), note that

$$\widehat{\mathbb{P}}_x(X(t) \in \mathrm{d}y) = \mathbb{P}_x(T > t; X(t) \in \mathrm{d}y) \frac{h(y)}{h(x)}, \tag{3.14}$$

multiply (3.13) by $\mathbb{P}_0(X(t) \in \mathrm{d}y) h(y)/h(x)$ and note that $\lim_{x \to 0} \mathrm{Schur}_m(x) = 0$ unless $m = (0, 1, \dots, p-1)$ in which case $\mathrm{Schur}_m(x) = 1$. Recall that $f_t(0) = 1$ to derive that (3.5) holds.

Let us derive the asymptotics of $\mathbb{P}_x(T > t)$. We multiply (3.13) by $\mathbb{P}_0(X(t) \in \mathrm{d}y)$ and integrate on $y \in W$ to obtain

$$\mathbb{P}_x(T > t) = h(x) \prod_{i=1}^p f_t(x_i) \sum_{0 \le m_1 < \cdots < m_p} \mathrm{Schur}_m(x) \prod_{i=1}^p a_{m_i}(t) \int_W \mathbb{P}_0(X(t) \in \mathrm{d}y)\, h(y)\, \mathrm{Schur}_m(y). \tag{3.15}$$

Because of the assumption that $\lim_{t\to\infty} a_{m+1}(t)/a_m(t) = 0$ for any $m \in \mathbb{N}_0$, it is clear that in the limit $t \to \infty$ only the term for $m = (0, 1, \dots, p-1)$ survives. Recall that $\lim_{t\to\infty} f_t(x) = 1$ to derive (3.6). □

The case of Brownian motion on $I = \mathbb{R}$ is a special case of Lemma 3.2 with $f_t(x) = e^{-x^2/(2t)}$ and $a_m(t) = t^{-m}/m!$. In (3.5) we recover Weyl's formula for the joint density of the eigenvalues of the Gaussian unitary ensemble (see [Me91]). The upper tail asymptotics given by (3.6) were previously obtained in [Gr00].

Let us check that the BESQ^d satisfies the assumptions of Lemma 3.2. The transition density is given [BS96] by

$$p_t(x,y) = \begin{cases} \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} e^{-(x+y)/(2t)} I_\nu\left(\frac{\sqrt{xy}}{t}\right), & \text{if } x > 0, \\ \frac{y^\nu}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-y/(2t)}, & \text{if } x = 0, \end{cases} \tag{3.16}$$

where $\nu = \frac{d}{2} - 1$ is the index of BESQ^d, and Γ denotes the Gamma function and

$$I_\nu(z) = \sum_{m=0}^\infty \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m!\,\Gamma(\nu+m+1)} \tag{3.17}$$

is the modified Bessel function of index ν. Hence, (3.4) is satisfied with $f_t(x) = e^{-x/(2t)}$ and $a_m(t) = \Gamma(\nu+1)[m!\Gamma(\nu+m+1)]^{-1}(2t)^{-2m}$. In particular

$$C_t = \frac{\Gamma(\nu+1)^p}{(2t)^{p(p-1)}} \prod_{i=1}^p \frac{1}{\Gamma(i)\Gamma(\nu+i)}. \tag{3.18}$$

Hence, we recover (1.6) from (3.5), with explicit identification of the normalisation constant.

The right hand side of (3.6) is identified as follows. Use (3.16) and make the change of variable $z = y/(2t)$ to get that

$$\mathbb{E}_0\left[h(X(t))\mathbb{1}\{X(t) \in W\}\right] = \frac{(2t)^{\frac{p}{2}(p-1)}}{\Gamma(\nu+1)^p} \int_W h(z) \prod_{i=1}^p \left[z_i^\nu e^{-z_i}\right] \mathrm{d}z. \tag{3.19}$$

Now use Selberg's integral (see (17.6.5) in [Me91]) to finally deduce that (3.6) reads

$$\mathbb{P}_x(T > t) \sim (2t)^{-\frac{p}{2}(p-1)} h(x) K, \qquad t \to \infty, \tag{3.20}$$

where

$$K = \frac{\int_W h(z) \prod_{i=1}^p \left[z_i^\nu e^{-z_i}\right] \mathrm{d}z}{\prod_{i=1}^p \left[\Gamma(i)\Gamma(\nu+i)\right]} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\frac{p}{2})} \frac{1}{p!\,\Gamma(\frac{3+p}{2})\Gamma(\frac{3}{2})} \prod_{j=1}^p \frac{\Gamma(\nu+1+\frac{p}{2})\Gamma(\frac{3+j}{2})}{\Gamma(j)\Gamma(\nu+j)}. \tag{3.21}$$

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