Maximal Effective Bandwidth under "Leaky Bucket" Constraints

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We investigate the worst possible behaviour of a stationary source if the traffic emanating from it is constrained to have a peak rate no greater than r and to obey a "leaky bucket" constraint with bucket size b and leak rate σ. The quantity to be maximised is the effective bandwidth function which governs the asymptotic loss rate when a large number of such sources pass through a single server queue. We conjecture the form of the worse case traffic in general and prove the conjecture for the special case when T, the timescale parameter of the effective bandwidth, is less than both $b/(r-\sigma)$ and b/σ, the times taken respectively to fill and empty the leaky bucket.

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Abstract

We investigate the worst possible behaviour of a stationary source if the traffic emanating from it is constrained to have a peak rate no greater than ρ and to obey a "leaky bucket" constraint with bucket size β and leak rate σ. The quantity to be maximised is the effective bandwidth function which governs the asymptotic loss rate when a large number of such sources pass through a single server queue. We conjecture the form of the worst case traffic in general and prove the conjecture for the special case when T, the timescale parameter of the effective bandwidth, is less than both $\beta/(\rho-\sigma)$ and β/σ, the times taken respectively to fill and empty the leaky bucket.

1 Motivation

Consider a multiplexor at which traffic arrives from several independent stationary sources. To guarantee sufficient quality of service, for example bounds on the cell delay or loss, the multiplexor must have a reasonable description of the characteristics of the offered traffic. This may be given at connection setup when a traffic contract is agreed. The user contracts that his traffic will conform to certain parameters, perhaps defining a policing algorithm that bounds properties of the traffic. In return the network contracts to carry this traffic with a particular quality of service. The traffic contract gives the network information that will bound the amount of network resources required to carry the call.

The network may have no additional information about user traffic and must therefore infer its characteristics from the descriptors specified in the contract. In any decisions the network makes concerning resource allocation or call acceptance, it would be prudent of the network to assume the worst case, that is that the user is adversarial to the maximum extent permitted by the policer.

These considerations motivate an optimisation problem over the space of traffic processes. We must choose a functional on this space to represent the performance of the network, and a constraint to represent the action of the policer.

A constraint common in the literature is the "leaky bucket" mechanism described first in [1]. This consists of a counter for each connection that is incremented whenever a cell from that connection arrives and is decremented periodically. If the counter exceeds a fixed threshold, the cell responsible is declared a non-conforming cell. These cells are discarded, delayed, or tagged as a low priority cells. The user specifies both the rate σ at which the counter is incremented and the threshold β. We say that a traffic stream conforms to the leaky bucket constraint (σ, β) if none of its cells are non-conforming.

To simplify analysis, we will work with a fluid model. There is an obvious continuous version of the leaky bucket regulator. Here continuous fluid flows into a buffer at a rate equal to that of the arriving traffic stream and is drained at a constant rate. Again, a stream is conforming if the fluid never exceeds a level β.

The leaky bucket is the standard policing mechanism defined by the ATM Forum [2] under the name Generic Cell Rate Algorithm. With a single constraint of the form $(\rho, 0)$, the worst case source is trivial: it is the source that transmits at a constant rate ρ. Our concern will be the next simplest case, that of two constraints: a peak rate constraint $(\rho, 0)$ and a mean rate constraint (σ, β).

There are many functionals that could represent the degree of congestion in a network. Mitra and Morrison [3] investigate the worst case loss rate in a bufferless server with constant service rate. They show that the loss rate is maximised by a periodic traffic source with uniformly distributed phase whose transmission rate is ρ during a fraction m/ρ of the time and 0 during the remainder.

Doshi [4] considers the worst case loss ratio for a finite number of independent sources. In addition to considering the situation where there are N identical sources (the homogeneous case) he also tackles the inhomogeneous case, that is where $N-1$ of the sources remain fixed and the maximum is taken with respect to the behaviour of the remaining source. He shows that when the buffer size is zero, the worst case source is always on-off, that is, alternates between transmitting at the peak rate and rate zero. He also demonstrates with a counterexample that the loss rate for two identical and independent sources is not maximised when they are on-off.

Oechslin [5] also considers the loss rate of a finite number of independent identically distributed sources. He specialises to the case where the service rate per source is equal to the leak rate σ and the buffer size per source is close (within ϵ) to β. He finds an upper bound on the expected loss rate experienced by on-off sources and a lower bound on that experienced by symmetric sources. By making ϵ small enough, he can make the lower bound exceed the upper bound and thus prove that there are cases where on-off traffic is not the worst case.

Lee [6] finds the worst case average queueing delay for a single source.

The functional we will consider in this paper is related to the asymptotic loss rate as the number of sources becomes large. Using the techniques of Large Deviation theory, this can been seen to depend on a functional called the *effective bandwidth* of the source, which will be defined in the next section.

2 The Problem

Let $\mathbb{D}$ be the set of right continuous, non-decreasing functions $\mathbb{R}^+ \to \mathbb{R}^+$. We interpret the elements of this set as possible realisations of a traffic source. That is, $b(t)$ is the amount of traffic arriving in the interval $[0, t]$. We denote by C those elements b of $\mathbb{D}$ for which

$$\begin{aligned} b(t_2) - b(t_1) &\leq (t_2 - t_1)\rho &(1)\\ \text{and} \quad b(t_2) - b(t_1) &\leq (t_2 - t_1)\sigma + \beta, &(2) \end{aligned}$$

for all $t_2 \geq t_1 \geq 0$. These are the realisations which obey the peak rate and leaky bucket constraints. Observe that the first condition implies that all members of C are Lipschitz with constant ρ.

We identify traffic sources with probability measures on $\mathbb{D}$. Realisations are shifted forward in time with the operator $\Theta_h : C \to C$ defined, for each $h \geq 0$, by $(\Theta_h b)(t) := b(t + h)$. When we talk of stationary sources we mean stationary with respect to this operator. In other words, a measure μ on $\mathbb{D}$ is stationary if $\mu[\Theta_h^{-1}\mathcal{B}] = \mu[\mathcal{B}]$ for all Borel sets $\mathcal{B}$ and shifts $h \geq 0$. We are interested in the set of stationary sources that meet the constraints almost surely. These are represented by the set $\mathcal{C}$ of stationary probability measures μ on $\mathbb{D}$ for which $\mu(C) = 1$.

We define the *effective bandwidth* of any source μ to be

$$\mathcal{E}_{\vartheta,T}(\mu) := \frac{1}{\vartheta T} \log \mathbb{E}_\mu e^{\vartheta X_T},$$

where $X_T : \mathbb{D} \to \mathbb{R}^+ : b \mapsto b(T)$ is the coordinate evaluation at T. Here $\vartheta \in \mathbb{R}$ and $T \in [0, \infty)$ are two parameters called the *space scale* and the *time scale* respectively. See [7] for discussion of their interpretation. The effective bandwidth is exactly what we need to calculate the behaviour of the loss rate as the number of sources becomes infinite. Denote by $\mathbb{L}(S, B, N)$ the loss rate when a single server queue with service rate S and buffer size B is fed by N independent sources, each distributed with the same law μ. As was shown in [8], $\mathbb{L}$ obeys the large deviations asymptotics

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{L}(sN, bN, N) = -\inf_{t>0} (f_t^{(\mu)})^*(b + sT).$$

where $f_t^{(\mu)}(\vartheta) := \vartheta t \mathcal{E}_{\vartheta,T}(\mu)$ and the *Legendre-Fenchel* transform of f is defined to be $f^*(y) := \sup_{x\in\mathbb{R}}[xy - f(x)]$.

These asymptotics are also obeyed by the probability that the queue length exceeds a level Nb in an infinite buffer and by the proportion of time buffer is full [9]. For an overview of the many applications of the effective bandwidth function see the paper of Kelly [7].

The asymptotics above motivate the optimisation program

$$\text{maximise } \mathcal{E}_{\vartheta,T}(\mu), \qquad \text{subject to } \mu \in \mathcal{C}. \tag{3}$$

For each γ in the range $[0, T]$, let w_γ be the source that transmits a periodic traffic pattern with uniformly distributed phase and whose sample paths are composed of following four sections:

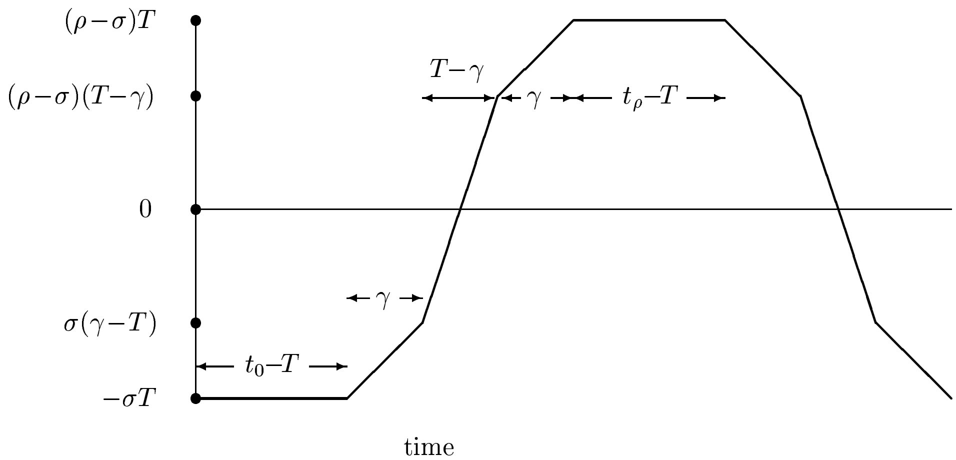

Figure 1: Graph of $z(t) := b(t+T)-b(t)-\sigma T$ in the region $[0, t_0+t_\rho+2\gamma]$

- a peak rate burst of duration $t_\rho := \beta/(\rho-\sigma)$,
- an interval in which the source transmits at rate σ, of duration γ,
- a silent interval of duration $t_0 := \beta/\sigma$,
- another interval at rate σ, again of duration γ.

Note that t_ρ and t_0 are, respectively, just long enough to fill the leaky bucket from empty and to empty it from full. A source of this type is illustrated in Figure 2. We make the following conjecture.

Conjecture 1 *For each ϑ and T, the stationary traffic source that maximises the effective bandwidth $\mathcal{E}_{\vartheta,T}$, under leaky bucket constraints (σ,β) and $(\rho,0)$, is one of the sources $\{w_\gamma\}_{0\le\gamma\le T}$.*

Note that this conjecture allows different sources to be worst case for different values of ϑ and T, If this conjecture is true, the problem of finding the worst case effective bandwidth for any given ϑ and T reduces to a one parameter optimisation over γ, which we now describe.

Consider the source w_γ. Let b be one of its realisations and let $z(t) := b(t+T)-b(t)-\sigma T$. Then

$$\mathcal{E}_{\vartheta,T}(w_\gamma) = \frac{1}{\vartheta T}\log\frac{1}{p}\int_0^p e^{\vartheta z(x)}\,dx + \sigma,$$

where $p := 2\gamma + t_0 + t_\rho$ is the period of the source. In this paper we consider the case when $T \le \min[t_0, t_\rho]$. For T in this range, z will be of the form shown in Figure 1. Using the identity

$$\int_{s_1}^{s_2} e^{\vartheta(c+mx)} = \frac{1}{m\vartheta}\left[e^{\vartheta(c+ms_2)} - e^{\vartheta(c+ms_1)}\right],$$

we may write down an explicit expression for the effective bandwidth of w_γ:

$$\mathcal{E}_{\vartheta,T}(w_\gamma) = \frac{1}{\vartheta T}\log\frac{1}{p}\sum_{n=0}^{3}\alpha_i e^{\vartheta A_i},$$

where

$$\begin{aligned} \alpha_0 &= (t_0 - T) - \frac{2}{\vartheta\sigma}, & \alpha_1 &= \frac{2}{\vartheta\sigma} - \frac{2}{\vartheta\rho}, \\ \alpha_2 &= \frac{2}{\vartheta\rho} - \frac{2}{\vartheta(\rho-\sigma)}, & \alpha_3 &= \frac{2}{\vartheta(\rho-\sigma)} + (t_\rho - T), \end{aligned}$$

and

$$\begin{aligned} A_0 &= 0, & A_1 &= \gamma\sigma, \\ A_2 &= \gamma\sigma + \rho(T-\gamma), & A_3 &= \rho T. \end{aligned}$$

The one parameter optimisation involved in maximising this expression over γ is easy to perform numerically.

Our main result is the following theorem.

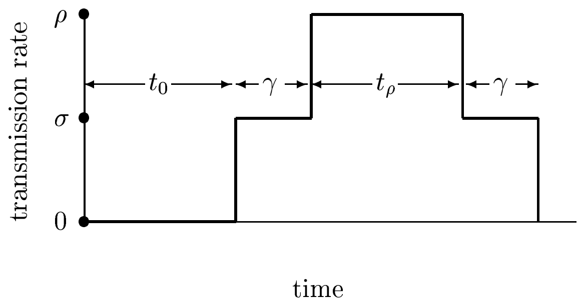

Figure 2: The "wedding cake" pattern, conjectured to have the worst case effective bandwidth.

Theorem 1 *In the case when $T \leq \min[\beta/(\rho-\sigma), \beta/\sigma]$, the supremum of $\mathcal{E}_{\vartheta,T}$ over all sources in $\mathcal{C}$ is attained by a source w_γ for some γ in the range $0 \leq \gamma \leq T$.*

The proof of this theorem will form the remainder of the paper.

3 A Useful Bound

For any $b \in \mathbb{D}$, let

$$U_b(t) := \sup_{s\in[0,t]} \Big(\sigma s - b(s)\Big).$$

This may be interpreted as the amount of fluid that could have leaked from the leaky bucket up to time t in excess of that which actually did. Intuitively, if $b \in C$ and we add $U_b(t)$ to b, the combined realisation will still meet the leaky bucket constraints.

Lemma 1 *Let $b \in C$ and define $d := b + U_b$. Then*

- *$d \in C$,*
- *for all $t \geq s \geq 0$, we have that $d(t) - d(s) \geq (t-s)\sigma - \beta$ and that $d(s) \geq s\sigma$,*
- *$U_d = 0$.*

Proof. Clearly, $d(t) \geq t\sigma$ for all $t \geq 0$. Condition (2) implies that $d(t) \leq \beta + \sigma t$ for all $t \geq 0$. Condition (1) implies $d(t) \leq (t-s)\rho + \sigma s$ for all $t \geq s \geq 0$. Subtracting, we deduce the first two statements.

To prove the third, we note that $U_d(t) := \sup\{s\sigma - d(s) : s \in [0,t]\} \leq 0$ for all $t \geq 0$. Since U_d is clearly non-negative it must be identically zero. □

Define

$$C' := \{b \in C : U_b \text{ is identically zero}\}.$$

Note that by the lemma above, C' is exactly the set of realisations of the form $b + U_b$, where $b \in C$. We will see that, since each $b \in C$ is less than some member of C', we will be able to restrict our attention to this smaller set.

Given the nature of the effective bandwidth function, it will prove useful to represent the traffic realisations in the following form. For any $b \in C'$ we may define the function $z(t) := b(t+T) - b(t) - \sigma T$, which specifies the amount of traffic in excess of the leaky bucket rate in each block of length T. Let Z be the set of functions obtained in this way from elements of C', that is

$$Z := \{z \in \mathcal{C}[0,\infty) : z(t) = b(t+T) - b(t) - \sigma T \text{ for some } b \in C'\}.$$

Our second lemma will translate the constraints on the realisations in C' into a constraint on each $z \in Z$. This will take the form of a bound on the area under z in certain intervals which depends only on the values taken by z at the end points of the interval.

Lemma 2 *Suppose $z \in Z$. If $z(s_1) \geq 0$ and $z(s_2) \geq 0$, then*

$$\int_{s_1}^{s_2} z(t)\,dt \leq \beta T - \frac{[z(s_1)]^2 + [z(s_2)]^2}{2(\rho - \sigma)}.$$

If $z(s_1) \leq 0$ and $z(s_2) \leq 0$, then

$$\int_{s_1}^{s_2} z(t)\,dt \geq -\beta T + \frac{[z(s_1)]^2 + [z(s_2)]^2}{2\sigma}.$$

Proof. We will only prove the first statement since the proof of the second is similar. Let b be such that $b + \sigma t \in C'$ and $z(t) = b(t+T) - b(t)$. Define

$$\begin{aligned} m &:= \sup\Big\{b(s_1+T) - b(t) : t \in [s_1, s_1+T]\Big\}, \\ n &:= b(s_2) - b(s_1+T), \\ p &:= \sup\Big\{b(t) - b(s_2) : t \in [s_2, s_2+T]\Big\}. \end{aligned}$$

Since b is absolutely continuous,

$$I := \int_{s_1}^{s_2} z(t)\,dt = \int_{s_1}^{s_2} \left(\int_t^{t+T} b'(x)\,dx \right) dt.$$

We may rewrite this double integral as follows (see Figure 3):

$$I = \int_{s_1}^{s_1+T} \int_t^{s_1+T} b'(x)\,dx\,dt + \int_{s_1+T}^{s_2} \int_{x-T}^{x} b'(x)\,dt\,dx + \int_{s_2-T}^{s_2} \int_{s_2}^{t+T} b'(x)\,dx\,dt. \tag{4}$$

Define $V(t) := \min\big[m,\, (\rho-\sigma)(s_1+T-t)\big]$. We have that $\int_t^{s_1+T} b'(x)\,dx \leq V(t)$ for all $t \leq s_1 + T$, and so the first term on the right hand side of (4) is less than

$$\int_{s_1}^{s_1+T} V(t)\,dt = mT - \frac{m^2}{2(\rho - \sigma)}.$$

Similarly the third term is less than $nT - n^2/2(\rho - \sigma)$. The middle term is equal to pT. Thus

$$I \leq Tm - \frac{m^2}{2(\rho - \sigma)} + Tp + Tn - \frac{n^2}{2(\rho - \sigma)}.$$

But $m + n + p \leq \beta$ and $m \geq z(s_1)$ and $n \geq z(s_2)$, and so $I \leq \beta T - (z(s_1)^2 + z(s_2)^2)/2(\rho - \sigma)$. $\square$

Motivated by this lemma, we introduce the notation

$$\begin{aligned} \overline{f}(x,y) &:= \beta T - \frac{x^2 + y^2}{2(\rho - \sigma)} \\ \text{and} \qquad \underline{f}(x,y) &:= \beta T - \frac{x^2 + y^2}{2\sigma}. \end{aligned}$$

4 A Comparison Lemma

In what follows, the symbol I denotes the indicator function, in other words $\mathrm{I}_A(x)$ is 1 if $x \in A$ and 0 otherwise.

Lemma 3 *Suppose that $D_1, D_2 \subset \mathbb{R}$ are Borel sets and that $z_1 : D_1 \to \mathbb{R}^+$ and $z_2 : D_2 \to \mathbb{R}^+$ are non-negative measurable functions such that*

$$\int_{D_1} z_1(t)\,\mathrm{I}_{z_1(t) \leq l}\,dt \geq \int_{D_2} z_2(t)\,\mathrm{I}_{z_2(t) \leq l}\,dt$$

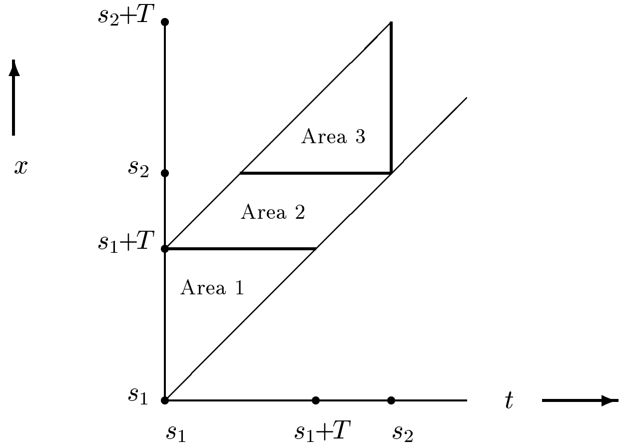

Figure 3: The range of integration of the double integral in the proof of Lemma 2.

for all $l \geq 0$, *and* $\int_{D_1} z_1(t)\,dt = \int_{D_2} z_2(t)\,dt$. *Suppose also that Leb*$\{t \in D_2 : z_2(t) = 0\} = 0$ *and that* $\zeta \geq 1$. *Then*

$$\int_{D_1} \left(e^{\vartheta z_1(t)} - \zeta\right) dt \leq \int_{D_2} \left(e^{\vartheta z_2(t)} - \zeta\right) dt.$$

Proof. Since $\int_{D_1} (e^{\vartheta z_1(t)} - \zeta)\,\mathrm{I}_{z_1(t)=0}\,dt = (1-\zeta)\,\mathrm{Leb}\{t \in \mathbb{R} : z_1(t) = 0\} \leq 0$, we may assume without loss of generality that $\mathrm{Leb}\{t \in \mathbb{R} : z_1(t) = 0\} = 0$. For each $l > 0$, let

$$G_i(l) := \int_{D_i} z_i\,\mathrm{I}_{z_i(t)\leq l}\,dt,$$

where i can be either 0 or 1. Then G_i is a distribution function on $(0,\infty)$ with an associated measure μ_i. By hypothesis, $G_1 \geq G_2$. Equivalently, $\mu_1 \leq \mu_2$ in the usual stochastic ordering (although μ_1 and μ_2 will not necessarily be probability measures, we have that $\mu_1(0,\infty) = \mu_2(0,\infty)$ and so there is no difficulty in comparing them). The function

$$k : \mathbb{R}^+ - \{0\} \to \mathbb{R} : x \mapsto \frac{e^{\vartheta x} - \zeta}{x}$$

is non-decreasing and therefore $\int k\,d\mu_1 \leq \int k\,d\mu_2$ since both integrals exist. But, for any function $f : \mathbb{R} \to \mathbb{R}$, we have that $\int f\,d\mu_i = \int z_i(t).(f \circ z_i)(t)\,dt$. The conclusion follows after applying this to k. □

The proof of Theorem 1 will involve comparing each section of $z \in Z$ that lies either entirely above or entirely below the x-axis with one of a set of functions which are in some sense optimal. We now introduce these functions.

5 An Important Family of Realisations

On the interval $[0, T + t_\rho]$, define the function

$$\overline{m}(x) := (\rho - \sigma)\min\left(x,\, T,\, T + t_\rho - x\right).$$

Note that if $z(t) := b(t+T) - b(t) - \sigma T$, where b is a realisation of the source w_T, then $\overline{m}$ is identical in shape to that part of z that lying above the x-axis. Also, the area under $\overline{m}$ is βT.

We wish to define a two-parameter family of functions $\{\overline{z}_{u_0,u_1} : u_0, u_1 \in [0, \beta T], u_0 \geq u_1\}$ such that the following hold:

- $\overline{z}_{u_0,u_1}$ agrees with $\overline{m}$ on some interval $[\overline{x}_0, \overline{x}_1]$,
- $\overline{z}_{u_0,u_1}$ has slope ρ to the left of $\overline{x}_0$ and slope $-\rho$ to the right of $\overline{x}_1$,

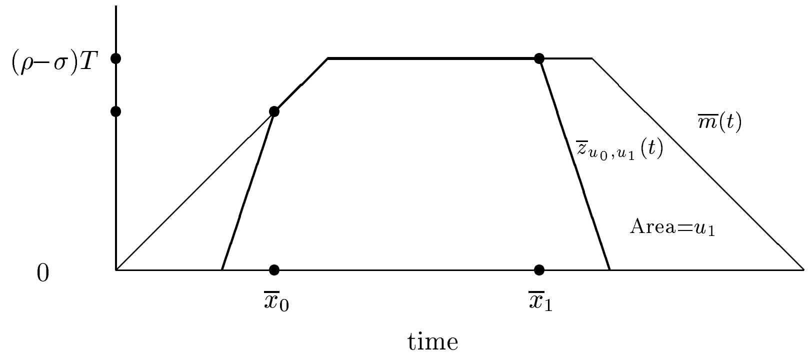

Figure 4: Graph of $\overline{z}_{u_0,u_1}$ when $\overline{x}_0 \leq \overline{x}_1$.

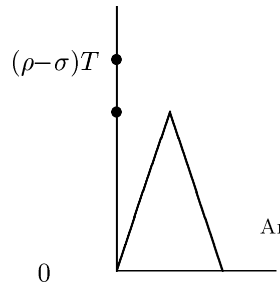

Figure 5: Graph of $\overline{z}_{u_0,u_1}$ when $\overline{x}_0 > \overline{x}_1$.

- the area of the region to the left of $\overline{x}_0$ bounded by $\overline{m}$, the x-axis, and $\overline{z}_{u_0,u_1}$, is $\beta T - u_0$,
- the area of the region to the right of $\overline{x}_1$ bounded by $\overline{m}$, the x-axis, and $\overline{z}_{u_0,u_1}$, is u_1.

More precisely, define $g(x) := \overline{m}[0,x] - \overline{m}^2(x)/2\rho$. Note that g is increasing, continuous, and that $g(0) = 0$ and $g(T + t_\rho) = \beta T$. For all $u \in [0, \beta T]$, define

$$\begin{array}{rrcl} & \overline{x}_0(u) & := & g^{-1}(\beta T - u) \\ \text{and} & \overline{x}_1(u) & := & T + t_\rho - g^{-1}(u). \end{array}$$

Let $u_0, u_1 \in [0, \beta T]$ be such that $u_0 \leq u_1$. If $\overline{x}_0(u_0) \leq \overline{x}_1(u_1)$, define the interval

$$\overline{D}(u_0,u_1) := [\overline{x}_0(u_0) - \overline{m}(\overline{x}_0(u_0))/\rho,\ \overline{x}_1(u_1) - \overline{m}(\overline{x}_1(u_1))/\rho]$$

and the function

$$\overline{z}_{u_0,u_1} : \overline{D}(u_0,u_1) \to \mathbb{R}^+ : x \mapsto \min\Big[\overline{m}(x),\ \overline{m}(\overline{x}_0(u_0)) + (x - \overline{x}_0(u_0))\rho,\ \overline{m}(\overline{x}_1(u_1)) - (x - \overline{x}_1(u_1))\rho\Big].$$

A function of this type is shown in Figure 4.

If $\overline{x}_0(u_0) > \overline{x}_1(u_1)$, then it will be impossible to define a function meeting all the requirements above. In this case we define the interval

$$\overline{D}(u_0,u_1) := [0,\ 2\sqrt{(u_1 - u_0)/(\rho - \sigma)}]$$

and the function

$$\overline{z}_{u_0,u_1} : \overline{D}(u_0,u_1) \to \mathbb{R}^+ : x \mapsto \min\Big[x\rho, (2\sqrt{(u_1 - u_0)/(\rho - \sigma)} - x)\rho\Big].$$

A function of this type is shown in Figure 5.

We also wish to define $\underline{z}_{u_0,u_1}$ which will be a non-positive function. In fact this function will be the negative of $\overline{z}_{u_0,u_1}$ defined above but using $\rho-\sigma$ as the leaky bucket rate. If a superscript denotes the leaky bucket rate to be used in the previous construction, then

$$\begin{aligned}
\underline{m}(x) &:= -\overline{m}^{(\rho-\sigma)}(x) \\
\underline{x}_0(u_0) &:= \overline{x}_0{}^{(\rho-\sigma)}(u_0) \\
\underline{D}(u_0,u_1) &:= \overline{D}^{(\rho-\sigma)}(u_0,u_1) \\
\underline{z}_{u_0,u_1} &:= \overline{z}^{(\rho-\sigma)}_{u_0,u_1}.
\end{aligned}$$

For every lemma concerning $\overline{z}_{u_0,u_1}$ we will have a dual lemma concerning $\underline{z}_{u_0,u_1}$.

The purpose of the functions defined above will become apparent when we state our next lemma. For now observe that if $z(t) = b(t+T) - b(t) - \sigma T$ where b is a realisation of w_γ, then the area under the positive section of z over a single period is $a = \beta T - (T-\gamma)^2(\rho-\sigma)\sigma/\rho$. Moreover, this section is identical to $\overline{z}_{u_a,v_a}$ with $u_a := (\beta T + a)/2$ and $v_a := (\beta T - a)/2$. Similarly, the negative section of z is identical to $\underline{z}_{v_a,u_a}$

This motivates the following definitions. For $a \in [0, \beta T]$, define

$$p(a) := \operatorname{Leb} \overline{D}(u_a, v_a) + \operatorname{Leb} \underline{D}(v_a, u_a) \tag{5}$$

$$\text{and} \qquad r(a) := \int_{\overline{D}(u_0,u_1)} e^{\vartheta \overline{z}_{u_a,v_a}(t)}\, dt + \int_{\underline{D}(u_0,u_1)} e^{\vartheta \underline{z}_{v_a,u_a}(t)}\, dt, \tag{6}$$

where u_a and v_a depend on a in the manner described above.

For the case above, $p(a) = t_\rho + t_0 + 2\gamma$ is the period of z, and $r(a) = \int_0^{p(a)} e^{\vartheta z(t)}\, dt$.

We set

$$\lambda := \sup_{a\in[0,\beta T]} \frac{r(a)}{p(a)}.$$

As $a \to 0$, $r(a)/p(a) \to 1$ and so $\lambda \geq 1$. Let a^* be where the maximum is attained. Let

$$\begin{aligned}
\overline{h} &:= \overline{m}(\overline{x}_0((\beta T - a^*)/2)) \\
\text{and} \qquad \underline{h} &:= \underline{m}(\underline{x}_0((\beta T - a^*)/2))
\end{aligned}$$

be the height of the kinks in $\overline{z}_{u_a,v_a}$ and $\underline{z}_{v_a,u_a}$ respectively. The following relation between $\overline{h}$, $\underline{h}$, and λ will be useful. Differentiating $r(a)/p(a)$ and setting the derivative equal to zero, we find that $p(a^*)r'(a^*) = r(a^*)p'(a^*)$. But $r'(a^*) = \exp(\vartheta\overline{h})/\overline{h} + \exp(-\vartheta\underline{h})/\underline{h}$ and $p(a^*) = 1/\overline{h} + 1/\underline{h}$. Thus

$$\lambda = \frac{r(a^*)}{p(a^*)} = \frac{\underline{h}e^{\vartheta\overline{h}} + \overline{h}e^{-\vartheta\underline{h}}}{\underline{h} + \overline{h}}. \tag{7}$$

The following properties of $\overline{z}_{u_0,u_1}$ are important. Suppose that $l > 0$ and define

$$\begin{aligned}
y_0 &:= \inf\{x : \overline{z}_{u_0,u_1}(x) \geq l\} \\
\text{and} \qquad y_1 &:= \sup\{x : \overline{z}_{u_0,u_1}(x) \geq l\}.
\end{aligned}$$

If $\overline{x}_0(u_0) \leq \overline{x}_1(u_1)$, let $h_0 := \overline{m}(\overline{x}_0(u_0))$ and $h_1 := \overline{m}(\overline{x}_1(u_1))$. Alternatively, if $\overline{x}_0(u_0) > \overline{x}_1(u_1)$, then let $h_0 := h_1 := \sqrt{(\rho-\sigma)(u_0-u_1)} = \max \overline{z}_{u_0,u_1}$. In both cases h_0 is the value taken by $\overline{z}_{u_0,u_1}$ when its slope changes from $\rho-\sigma$ and h_1 is the value it takes when its slope changes to $-(\rho-\sigma)$.

We have that, for each $u_0, u_1 \in [0, \beta T]$,

- the area under $\overline{z}_{u_0,u_1}$ over its domain $\overline{D}(u_0,u_1)$ is $u_0 - u_1$,
- $\overline{z}_{u_0,u_1}$ is unimodal,
- if $l \leq h_0$ then the area under $\overline{z}_{u_0,u_1}$ to the left of y_0 is $l^2/2\rho$. If $h_0 < l < (\rho-\sigma)T$ then the area under $\overline{z}_{u_0,u_1}$ to the right of y_0 is $\overline{f}(l,0) - u_1$,
- similarly, if $l \leq h_1$ then the area under $\overline{z}_{u_0,u_1}$ to the right of y_1 is $l^2/2\rho$. If $h_1 < l < (\rho-\sigma)T$ then the area under $\overline{z}_{u_0,u_1}$ to the left of y_1 is $\overline{f}(0,l) - \beta T + u_0$,

- if $\overline{x}_0(u_0) < \overline{x}_1(u_1)$ and $l \geq \max(h_0, h_1)$, then the area under $\overline{z}_{u_0,u_1}$ over the interval $[y_0, y_1]$ is $\overline{f}(l,l)$.

We use these properties to prove the following lemma. For any real valued function z, define z^+ to be the function $z^+(t) := \max[z(t), 0]$ and z^- to be the function $z^-(t) := \min[z(t), 0]$. We will also use the following notation: if z is a real valued function on $[0,\infty)$, then $z[s_1, s_2] := \int_{s_1}^{s_2} z$.

Lemma 4 *Suppose $z \in Z$ is such that $z(s_1) = z(s_2) = 0$ and $z[s_1, s_2] \geq 0$. Then, letting $a := z[s_1, s_2]$ and $u_1 := \inf_{t \leq s_2} \left(\overline{f}(z(t), 0) - z[t, s_2]\right)$, we have that*

$$\int_{s_1}^{s_2} z^+(t)\, \mathrm{I}_{z^+(t) \leq l}\, dt \geq \int_{\overline{D}(u_1+a,u_1)} \overline{z}_{u_1+a,u_1}(t)\, \mathrm{I}_{\overline{z}(t) \leq l}\, dt,$$

for any $l > 0$.

Proof. We first establish two inequalities which the constraints and the hypothesis impose on z. Firstly, let $\epsilon > 0$. Then there exists $t \leq s_2$ such that $\overline{f}(z(t), 0) - z[t, s_2] \leq u_1 + \epsilon$. Thus, for each $s \in [s_1, s_2]$, we have that

$$\begin{aligned} z[s_1, s] &= z[t, s] + z[s_1, s_2] - z[t, s_2] \\ &\leq \overline{f}(z(t), z(s)) + a + u_1 + \epsilon - \overline{f}(z(t), 0). \end{aligned}$$

Using the identity $\overline{f}(z(t), 0) - \overline{f}(z(t), z(s)) = \beta T - \overline{f}(z(s), 0)$, we conclude that $z[s_1, s] \leq \overline{f}(z(s), 0) - \beta T + u_1 + a + \epsilon$. Since ϵ is arbitrary, we have that

$$z[s_1, s] \leq \overline{f}(z(s), 0) - \beta T + u_1 + a$$

for all $s \in [s_1, s_2]$.

Secondly, we have the inequality $z[t, s_2] \leq \overline{f}(z(t), 0) - u_1$ for all $t \in [s_1, s_2]$.

We need only prove the result for $0 < l \leq \max_{x \in \overline{D}} \overline{z}(x)$. If z never rises above the level l in the interval $[s_1, s_2]$ then we are done since all the area lies below l. If z does rise above l then define

$$\begin{aligned} r &:= \inf\{x \in [s_1, s_2] : z(x) \geq l\} \\ \text{and} \qquad s &:= \sup\{x \in [s_1, s_2] : z(x) \geq l\}. \end{aligned}$$

We also define y_0 and y_1 as before and write the interval $\overline{D}(u_1 + a, u_1)$ over which $\overline{z}_{u_1+a,u_1}$ is defined as $[d_0, d_1]$. Suppose $l < h_0$. Then $\overline{z}[d_0, y_0] = l^2/2\rho \leq z[s_1, r] \leq z^+[s_1, r]$, where the first inequality uses the Lipschitz condition. On the other hand, if $l \geq h_0$, then $z^+[r, s_2] + z^-[s_1, s_2] \leq z[r, s_2] \leq \overline{f}(l, 0) - u_1 = \overline{z}[y_0, d_1]$. But $z[s_1, s_2] = \overline{z}[d_0, d_1] = a$. We thus have, again, that $\overline{z}[d_0, y_0] \leq z^+[s_1, r]$.

In a similar manner we can show that $\overline{z}[y_1, d_1] \leq z^+[s, s_2]$.

Now

$$\begin{aligned} \int_{s_1}^{s_2} z^+(t)\, \mathrm{I}_{z^+(t) \leq l}\, dt &\geq z^+[s_1, r] + z^+[s, s_2], \\ \text{and} \qquad \int_{d_0}^{d_1} \overline{z}(t)\, \mathrm{I}_{\overline{z}(t) \leq l}\, dt &= \overline{z}[d_0, y_0] + \overline{z}[y_1, d_1]. \end{aligned}$$

The conclusion follows. □

For $z \in Z$ we define

$$Q(z) := \limsup_{t \to \infty} \int_0^t (e^{\vartheta z(x)} - \lambda)\, dx.$$

For a function z defined only over some finite interval D, for example $\overline{z}_{u_0,u_1}$ or $\underline{z}_{u_0,u_1}$, we define $Q(z) := \int_D (\exp(\vartheta z(x)) - \lambda)\, dx$.

Corollary 1 *Let $z \in Z$. Suppose that $z(s_1) = z(s_2) = 0$ and that z is non-negative on (s_1, s_2). With a and u_1 defined as above, we have that $\int_{s_1}^{s_2} \left(e^{\vartheta z(t)} - \lambda\right) dt \leq Q(\overline{z}_{u_1+a,u_1})$.*

Proof. This is a simple application of Lemma 3. $\square$

The following lemma and corollary are dual to those above and are proved in an identical manner.

Lemma 5 *Suppose $z \in Z$ is such that $z(s_1) = z(s_2) = 0$ and $z[s_1, s_2] \leq 0$. Then, letting $a := z[s_1, s_2]$ and $u_1 := \inf_{t \leq s_2} \left(\overline{f}(z(t), 0) - z[t, s_2]\right)$, we have that*

$$\int_{s_1}^{s_2} z^+(t)\, \mathrm{I}_{z^+ \leq l}\, dt \geq \int_{d_0}^{d_1} \underline{z}_{u_1+a,u_1}(t)\, \mathrm{I}_{\underline{z} \leq l}\, dt,$$

for any $l > 0$.

Corollary 2 *Let $z \in Z$. Suppose that $z(s_1) = z(s_2) = 0$ and that z is non-positive on (s_1, s_2). With a and u_1 defined as above, we have that $\int_{s_1}^{s_2} \left(e^{\vartheta z(t)} - \lambda\right) dt \leq Q(\underline{z}_{u_1+a,u_1})$.*

To use Corollaries 1 and 2, we will break up $z \in Z$ into pieces lying either entirely above the x-axis or entirely below.

6 Markov Decision Procedures

Recall that a deterministic Markov Decision Procedure (MDP) is a set of states along with a reward function on the transitions between states. If x and y are states then we often write the transition (x, y) as $x \to y$. We define the following MDP:

- state space $S := \{(u, d) \subset [0, \beta T]^2 : u + d \leq \beta T\}$,
- if $x := (u_0, d_0)$ and $y := (u_1, d_1)$ are two states then the reward for the transition $x \to y$ is

$$W(x \to y) := \begin{cases} Q(\overline{z}_{u_1+d_1-d_0,u_1}) & \text{if } u_0 + d_0 \geq u_1 + d_1 \text{ and } d_0 \leq d_1, \\ Q(\underline{z}_{d_1+u_1-u_0,d_1}) & \text{if } u_0 + d_0 \geq u_1 + d_1 \text{ and } u_0 \leq u_1, \\ -\infty, & \text{if } u_0 + d_0 < u_1 + d_1, \\ 0, & \text{if } u_0 > u_1 \text{ and } d_0 > d_1. \end{cases}$$

We denote this MDP by $\mathcal{M} := (S, W)$. It is customary to call a sequence (finite or infinite) of states in an MDP a path. For any path x through $\mathcal{M}$, the long term average reward is defined to be

$$\mathcal{W}(x) := \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} W(x_i, x_{i+1}).$$

The motivation for introducing the MDP above comes from the following lemma. We define the constant $D := \left(\exp[\vartheta T(\rho - \sigma)] - \lambda\right) t_\rho$.

Lemma 6 *For each $z \in Z$, there exists a path x through $\mathcal{M}$ such that $Q(z) \leq \mathcal{W}(x) + D$.*

Proof. Our first step will be to decompose the positive time axis into intervals $\{[t_n, t_{n+1}]\}_{n \in \mathbb{N}}$ in such a way that, for each $n \in \mathbb{N}$, both $z(t_n) = 0$ and one of the following hold:

(A) z is non-negative on $[t_n, t_{n+1}]$,

(B) z is non-positive on $[t_n, t_{n+1}]$,

(C) $\underline{h} \leq z \leq \overline{h}$ on $[t_n, t_{n+1}]$, and moreover $z^+[t_n, t_{n+1}] \leq \beta T$ and $z^-[t_n, t_{n+1}] \geq -\beta T$.

We do this in the following way. Define

$$t_1 := \sup\{t \geq 0 : \ \underline{h} < z(x) < \overline{h} \text{ for all } x \in [0, t],\ z(t) = 0,\ z^+[0, t] \leq \beta T,\ z^-[0, t] \geq -\beta T\ \}.$$

and

$$t_2 := \inf\{t \geq t_1 : z(t) = 0\}.$$

Clearly the interval $[0,t_1]$ is of type (C) while $[t_1,t_2]$ is of type either (A) or (B). Also, either $z(x)$ lies outside $[\underline{h},\overline{h}]$ for some $x\in[t_1,t_2]$ or else $z^+[0,t_2]>\beta T$ or $z^-[0,t_2]>\beta T$. Thus $t_2>\min\left[2\overline{h}/\rho,-2\underline{h}/\rho,\beta T/\overline{h},-\beta T/\underline{h},\right]$. So, although either of the intervals may have zero length, the sum of their lengths is bounded below by this constant. Thus it is clear that we may perform the required decomposition using induction and that there will be a finite number of intervals in any finite time.

Let

$$
\begin{aligned}
u_n &:= \inf_{t\le t_n}\Big(\overline{f}\big(z(t),0\big)-z[t,t_n]\Big)\\
\text{and}\qquad d_n &:= \inf_{t\le t_n}\Big(\underline{f}\big(z(t),0\big)-z[t,t_n]\Big)\wedge z[0,t_n],
\end{aligned}
$$

where $\wedge$ denotes the minimum of two numbers.

Suppose z obeys condition (A) in the interval $[t_{n-1},t_n]$. Then $a:=z[t_{n-1},t_n]$ is non-negative and $u_n\le u_{n-1}-a$ and $d_n=d_{n-1}+a$. Applying Corollary 1, we have that

$$
\begin{aligned}
\int_{t_{n-1}}^{t_n}(e^{\vartheta z(x)}-\lambda)\,dx &\le Q(\overline{z}_{u_n+a,u_n})\\
&= W[(u_{n-1},d_{n-1})\to(u_n,d_n)].
\end{aligned}
$$

A similar argument shows that the same result is true if z obeys condition (B) in the interval.

Now let $[t_{n-1},t_n]$ be one of the intervals for which Condition (C) holds. We consider first the case when $a:=z[t_{n-1},t_n]\ge 0$. Define $D_1:=\{t\in[t_{n-1},t_n]:curv(t)<0\}$. Let

$$
\zeta:=\inf\Big\{l>0:\int_{t_{n-1}}^{t_n}z^+(t)\,\mathrm{I}_{z^+\le l}\,dt\ge a\Big\}.
$$

Let $m:=\int_{t_{n-1}}^{t_n}z^+(t)\,\mathrm{I}_{z^+\le\zeta}\,dt$. Then the subset of $[t_{n-1},t_n]$ on which z takes the value ζ has measure greater than $a-m$. Choose a subset of this set of measure $a-m$ and take its union with $\{t\in[t_{n-1},t_n]:0<z(t)<\zeta\}$ to form a set D_2. Let $D_3:=[t_{n-1},t_n]-D_2-D_1$.

We thus arrive at a decomposition of $[t_{n-1},t_n]$ into sets D_1, D_2, and D_3 in which z takes values in the ranges $[-\underline{h},0)$, $[0,\zeta]$, and $[\zeta,\overline{h}]$ respectively. Furthermore $z[D_2]=a$. Let $c:=-z[D_1]=z[D_3]$.

We compare z on D_3 with the function $\tilde{z}:[0,c/\overline{h}]\to\{\overline{h}\}$. The conditions of Lemma 3 are satisfied and we deduce that

$$
\int_{D_3}(e^{\vartheta z(t)}-\lambda)\,dt\le\int_0^{c/\overline{h}}(e^{\vartheta\tilde{z}(t)}-\lambda)\,dt=(e^{\vartheta\overline{h}}-\lambda)c/\overline{h}.
$$

Similarly,

$$
\int_{D_1}(e^{\vartheta z(t)}-\lambda)\,dt\le(e^{\vartheta\underline{h}}-\lambda)c/\underline{h}.
$$

Therefore the sum of these terms is non-positive, using the expression for λ given in (7).

For $l\ge\zeta$, we have $\int_{D_2}z(t)\,\mathrm{I}_{z\le l}\,dt=a\ge\int_{d_0}^{d_1}\overline{z}_{u_n+a,u_n}(t)\,\mathrm{I}_{\overline{z}\le l}\,dt$. For $l\in(0,\zeta)$, we apply Lemma 4 and obtain $\int_{D_2}z(t)\,\mathrm{I}_{z(t)\le l}\,dt\ge\int_{\overline{D}}\overline{z}(t)\,\mathrm{I}_{\overline{z}_{u_n+a,u_n}(t)\le l}\,dt$. Since $z(t)\ge 0$ for all $t\in D_2$, all the conditions of Lemma 3 are satisfied and we deduce that

$$
\int_{D_2}\left(e^{\vartheta z(t)}-\lambda\right)dt\le Q(\overline{z}_{u_n+a,u_n})
$$

Since $[t_n,t_{n+1}]$ is a disjoint union of D_1, D_2, and D_3, we have that

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}}\left(e^{\vartheta z(t)}-\lambda\right)dt &\le Q(\overline{z}_{u_n+a,u_n})\\
&= W\left[(u_{n-1},d_{n-1})\to(u_n,d_{n-1}+a)\right]\\
&= W\left[(u_{n-1},d_{n-1})\to(u_n,d_{n-1}+a)\right]+W\left[(u_n,d_{n-1}+a)\to(u_n,d_n)\right].
\end{aligned}
$$

The last equality holds since $d_n\le d_{n-1}-a$ and so $W[(u_n,d_{n-1}-a)\to(u_n,d_n)]=0$. The case when (C) holds for $[t_n,t_{n+1}]$ and $a:=z[t_{n-1},t_n]<0$ is handled similarly.

We construct the required path x inductively by appending, for each $n \in \mathbb{N}$, either the transition $(u_{n-1}, d_{n-1}) \rightarrow (u_n, d_n)$ if the interval $[t_n, t_{n+1}]$ obeys Conditions (A) or (B), or the transitions $(u_{n-1}, d_{n-1}) \rightarrow (u_n, d_{n-1}+a) \rightarrow (u_n, d_n)$ if it obeys Conditions (C). ¿From the considerations above, we have that, for all $n \in \mathbb{N}$,

$$\int_0^{t_n} (e^{\vartheta z(x)} - \lambda)\, dx \leq \sum_{i=0}^{n+c(n)} W(x_i, x_{i+1}),$$

where $c(n)$ is the number of intervals of type (C) before $[t_{n-1}, t_n]$. Let $t \in [t_n, t_{n+1}]$. By comparing z^+ on the interval $[t_n, t]$ with the function that takes value $T(\rho - \sigma)$ on the interval $[0, t_\rho]$, and using Lemma 3, we conclude that

$$\int_{t_n}^{t} (e^{\vartheta z(x)} - \lambda)\, dx \leq \left(e^{\vartheta T(\rho-\sigma)} - \lambda\right) t_\rho.$$

The conclusion follows. $\square$

If we can calculate $\sup \mathcal{W}(x)$ over all paths x through $\mathcal{M}$ then, using the lemma above, we will have an upper bound on $\sup Q(z)$ over all $z \in Z$. The following results from the theory of Markov Decision Procedures are well known [10]. Suppose the state space S of a MDP is compact and the reward function R is jointly continuous in the starting and final states and nowhere takes the value $-\infty$. Let P be the set of all paths and K be the set of cyclic paths. Let

$$\mathcal{R}(x) := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} R(x_{i-1}, x_i).$$

Then

$$\sup_{x\in P} \mathcal{R}(x) = \sup_{x\in K} \mathcal{R}(x) =: \Gamma.$$

Moreover, there exists a continuous function F on the state space such that

$$\sup_{y\in S} \big(R(x,y) + F(y)\big) = F(x) + \Gamma.$$

Such a function is called an (additive) eigenfunction of the MDP. In general, these functions are not unique. The value Γ is known as either the *eigenvalue* or the *maximal cycle mean* of the MDP.

Unfortunately, the reward function W does not satisfy these assumptions, in particular it can take the value $-\infty$ for some transitions. However, the following MDPs do satisfy the assumptions. For each $\alpha \in [0, \beta T]$ we define the Markov Decision Procedure $\mathcal{M}_\alpha$ to be

- *State Space.* $S_\alpha := [0, \alpha]$,
- *Reward.* The transition $u \rightarrow v$ has reward

$$W_\alpha(u \rightarrow v) := \begin{cases} Q(\overline{z}_{u,v}), & \text{if } u \geq v, \\ Q(\underline{z}_{\alpha-u,\alpha-v}), & \text{if } u < v. \end{cases}$$

Note that $u \rightarrow v$ in $\mathcal{M}_\alpha$ has the same reward as

$$\begin{pmatrix} u \\ \alpha - u \end{pmatrix} \rightarrow \begin{pmatrix} v \\ \alpha - v \end{pmatrix}$$

in $\mathcal{M}$. For each $\alpha \in [0, \beta T]$, S_α is compact and W_α is continuous and so the results above hold for $\mathcal{M}_\alpha$. The following lemma bounds the maximal cycle mean of these MDPs.

Lemma 7 *The maximal cycle mean of $\mathcal{M}_\alpha$ is non-positive for $\alpha = \beta T$. It is strictly negative for $\alpha < \beta T$.*

Proof. Our first objective is to show that the mean reward along any any path will be dominated by that over a cycle of length two.

For $v < u$, we have that

$$g_u(v) := \frac{\partial W_\alpha(u,v)}{\partial v} = \frac{e^{h_1(u,v)} - \lambda}{h_1(u,v)},$$

where

$$h_1(u,v) := \begin{cases} \overline{m}(\overline{x}_0(u)), & \text{if } \overline{x}_0(u) \le \overline{x}_1(v), \\ \sqrt{(u-v)(\rho-\sigma)}, & \text{otherwise.} \end{cases}$$

Since $h_1(u,v)$ is nondecreasing in u for fixed v, so too is $g_u(v)$.

Suppose that x, w, y, z are states such that $z < y \le w < x$. Then

$$W_\alpha(x,z) = W_\alpha(x,y) + \int_y^z g_x(t)\, dt$$

and

$$W_\alpha(w,z) = W_\alpha(w,y) + \int_y^z g_w(t)\, dt$$

Subtracting the second equation from the first and using the isotonicity of $g_u(v)$ in u, we have that $W_\alpha(x,z) + W_\alpha(w,y) \ge W_\alpha(x,y) + W_\alpha(w,z)$.

Now let c be a cycle in $\mathcal{M}_\alpha$. Without loss of generality, we may assume that transitions are alternately positive and negative and that the first jump is negative. This means we may write $c = u_1, v_1, u_2, v_2, \ldots, v_n$, with $u_i > v_i$ and $v_i > u_{i+1}$ for all $1 \le i \le n$. Here we are using the convention that $u_{n+1} := u_1$. Consider the set $J_0 := \{u_i \to v_i : 1 \le i \le n\}$ of negative transitions. We perform the following operation on J_0. If J_0 contains two transitions $x \to y$ and $w \to z$ such that $x > w \ge y > z$, then we replace them with the transitions $x \to z$ and $w \to y$. If there are no such pairs then we do nothing. Call the new set of transitions J_1. We iterate this procedure to form sets $J_2, J_3, \ldots$ and so on. For any set of transitions $K \subset S_\alpha \times S_\alpha$, define

$$\mathcal{G}(K) := \sum_{(x \to y) \in K} W_\alpha(x,y).$$

¿From the considerations above, we have that $\mathcal{G}(J_n) \le \mathcal{G}(J_{n+1})$. Since at each stage of the procedure the quantity

$$\sum_{(u \to v) \in J_n} (v-u)^2$$

can only increase, the procedure must terminate after a finite number of stages. Call the final set of transitions J. This set will not contain any pair of transitions $x \to y$ and $w \to z$ such that $x > w \ge y > z$. But there is a unique pairing of the starting states $\{u_i\}_{1 \le i \le n}$ and the destination states $\{v_i\}_{1 \le i \le n}$ that has this property. This shows that the end result of the algorithm described above does not depend on the order in which transitions are replaced.

If we now consider the set of downward transitions $I_0 := \{(v_i, u_i) : 1 \le i \le n\}$ we may employ a similar procedure to arrive at a set I for which $\mathcal{G}(I_0) \le \mathcal{G}(I)$. But again this set is independent of the order of substitution and in fact $I = \{v \to u : u \to v \in J\}$. In other words I is the same as J but with the direction of the transitions reversed. So,

$$\begin{aligned} \frac{1}{2n}\big(\mathcal{G}(J) + \mathcal{G}(I)\big) &= \frac{1}{2n} \sum_{(u \to v) \in J} \big[W(u \to v) + W(v \to u)\big] \\ &\le \frac{1}{2} \max_{(u \to v) \in J} \big[W(u \to v) + W(v \to u)\big]. \end{aligned}$$

This shows that every cycle has a lower mean reward than some cycle of length two.

Consider now the case when $\alpha = \beta T$. Then

$$W_{\beta T}(u,v) = \begin{cases} Q(\overline{z}_{u,v}) & \text{if } v \le u, \\ Q(\underline{z}_{\beta T - u, \beta T - v}) & \text{if } v > u. \end{cases}$$

For $u > v$, the partial derivatives are

$$\frac{\partial W_{\beta T}(u,v)}{\partial u} = -\frac{e^{\vartheta h_0(u,v)} - \lambda}{h_0(u,v)}, \qquad \text{and} \qquad \frac{\partial W_{\beta T}(u,v)}{\partial v} = \frac{e^{\vartheta h_1(u,v)} - \lambda}{h_1(u,v)},$$

where

$$h_0(u,v) := \begin{cases} \overline{m}(\overline{x}_1(v)), & \text{if } \overline{x}_0(u) \le \overline{x}_1(v), \\ \sqrt{(u-v)(\rho-\sigma)}, & \text{otherwise.} \end{cases}$$

So if we maximise W keeping $u-v$ constant, the maximum will only occur when $h_0 = h_1$. This will happen when $v = \beta T - u$. A similar argument holds for $u < v$ and we conclude that the optimum two-cycles are symmetric about $\beta T/2$. So consider the function

$$f(a) := \frac{1}{2} W_{\beta T}\left(\frac{\beta T - a}{2} \to \frac{\beta T + a}{2}\right) + \frac{1}{2} W_{\beta T}\left(\frac{\beta T + a}{2} \to \frac{\beta T - a}{2}\right)$$

defined on the interval $[0, \beta T]$. Its derivative is

$$f'(a) := \frac{e^{\vartheta \overline{q}(a)} - \lambda}{2\overline{q}(a)} + \frac{e^{-\vartheta \underline{q}(a)} - \lambda}{2\underline{q}(a)}$$

where

$$\overline{q}(a) := \begin{cases} \sqrt{(\rho-\sigma)a}, & \text{if } a < (\rho-\sigma)T^2/\rho, \\ \overline{m}(\overline{x}_0(a)), & \text{otherwise} \end{cases}$$

and

$$\underline{q}(a) := \begin{cases} \sqrt{(\rho-\sigma)a}, & \text{if } a < \sigma T^2/\rho, \\ \underline{m}(\underline{x}_0(a)), & \text{otherwise.} \end{cases}$$

This function is unimodal: it is non-decreasing in the interval $[0,\zeta]$, where $\zeta := \beta T - T^2(\rho-\sigma)\sigma/\rho$, and strictly decreasing in the interval $[\zeta, \beta T]$. Since $f'(a) \to -\infty$ both as $a \to 0$ and as $a \to \beta T$, f will attain its maximum value at a point a^* somewhere in the interval $(\zeta, \beta T)$. Since f' is strictly decreasing in this interval, this maximum will be unique.

Let p and r be defined as in (5) and (6). Since $\lambda := \sup\{r(a)/d(a) : 0 \le a \le \beta T\}$ and $f(a) = r(a) - \lambda p(a)$ we have that $f(a) \le 0$ for all a. This implies that the maximal cycle mean of $\mathcal{M}$ is non-positive. Also, from the discussion above, $f(a) < 0$ for $a \ne a^*$.

To prove the result for $\alpha < \beta T$ we will need to use the fact that $W_\alpha(u \to v)$ is independent of α for $v \le u$, and that $W_\alpha(u \to v) = W_{\alpha'}(\alpha' - \alpha + u \to \alpha' - \alpha + v)$. for $v \ge u$. Thus

$$\begin{aligned} W_\alpha(u \to v) + W_\alpha(v \to u) &= W_{\beta T}(u \to v) + W_{\beta T}(\beta T - \alpha + v \to \beta T - \alpha + u) \\ &\le W_{\beta T}\left(\frac{\beta T - v + u}{2} \to \frac{\beta T + v - u}{2}\right) + W_{\beta T}\left(\frac{\beta T + v - u}{2} \to \frac{\beta T - v + u}{2}\right) \\ &\le 0 \end{aligned}$$

Thus the maximal cycle mean of $\mathcal{M}_\alpha$ is also non-positive for $\alpha \in [0, \beta T)$. To prove strict inequality, note that the second inequality above is strict if $u - v \ne a^*$. Suppose therefore that $u - v = a^*$. To have equality we must have that $u = (\beta T + a^*)/2$ and that $\beta T - \alpha + v = (\beta T - a^*)/2$. We conclude that $\alpha = \beta T$ and thus the only MDP with maximal cycle mean zero is $\mathcal{M}$. □

We use the sup norm on $\mathbb{R}^2$, so that $||(u,d)|| := \max(u,d)$ for all $(u,d) \in S$.

Lemma 8 *For any path x through the Markov Decision Procedure $\mathcal{M}$, we have that $\mathcal{W}(x) < \infty$.*

Proof. If x contains a transition with reward $-\infty$ then we are done. Otherwise the sequence of states $x_i =: (u_i, d_i)$ must be such that $u_i + d_i$ is non-decreasing. Thus $u_i + d_i$ must converge to a limit α in the range $[0, \beta T]$.

Let $\epsilon > 0$. We have that W is finite on the set of transitions

$$T' := \{x \to y : u_0 + d_0 \ge u_1 + d_1, \text{ where } x =: (u_0, d_0) \text{ and } y =: (u_1, d_1)\}.$$

This set is compact. Moreover, on this set, $W(x,y)$ is jointly continuous in x and y, and hence uniformly continuous. Thus there exists $\delta > 0$ such that $|W(x \to y) - W(x' \to y')| \le \epsilon$ whenever both $||x - x'||$ and $||y - y'||$ are less than δ and $x \to y$ and $x' \to y'$ are both in T'.

Consider the mapping

$$\phi : S \to S : (u,d) \mapsto \frac{\alpha}{u+d}(u,d).$$

Clearly we can find ν such that if $\alpha \le u+d \le \alpha+\nu$ then $||\phi(u,d)-(u,d)|| < \delta$. But we have seen that $u_n + d_n$ converges to α from above, and so there is some $N \in \mathbb{N}$ such that $\alpha \le u_n + d_n \le \alpha + \nu$ for all $n > N$. Let $g_n := W(x_{n-1} \to x_n)$ and $g'_n := W(\phi(x_{n-1}) \to \phi(x_n))$. We conclude that $|g_n - g'_n| \le \epsilon$ when $n > N$. Thus we have that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g_n &= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=N+1}^{N+n} g_n \\
&\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=N+1}^{N+n} (g'_n + \epsilon) \\
&\le G_\alpha + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we conclude that $\limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n} g_n \le G_\alpha$.

Suppose that $\alpha < \beta T$. Then according to Lemma 7, $G_\alpha < 0$. Hence we can find N' such that $\sum_{i=1}^{n} g_n < 0$ for $n > N'$.

On the other hand, if $\alpha = \beta T$, then x is also path through the Markov decision procedure $\mathcal{M}_{\beta T}$. Its eigenfunction F is continuous and therefore bounded. So $\sum_{i=1}^{n} g_n \le F(u_0,d_0) - F(u_n,d_n)$ and again is bounded. □

7 The Main Result

We now have the tools necessary to prove the main result.

Proof of Theorem 1. Our first observation is that the ergodic theorem implies that, for any stationary source μ, the limit

$$\overline{f}(b) := \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{\vartheta(b(x+T)-b(x^-))}\,dx$$

exists μ-almost surely, and that $\overline{f}$ has the same expectation as $f(b) := \exp[\vartheta(b(T))]$. In other words, $(\vartheta T)^{-1}\log \mathbb{E}_\mu \overline{f} = \mathcal{E}_{\vartheta,T}(\mu)$ for all stationary sources μ.

Denote by L the set of realisations at which the function $\overline{f}$ is defined and let $b \in C \cap L$. Define $z(t) := b(t+T) - b(t) - \sigma T$. As before,

$$Q(z) := \limsup_{t\to\infty} \int_0^t (e^{\vartheta z(x)} - \lambda)\,dx.$$

Since $z \in Z$, we can apply Lemma 6 and deduce that there exists a path x in the Markov Decision Procedure $\mathcal{M}$ such that $Q(z) \le \mathcal{W}(x) + D$. But Lemma 8 says that $B := \mathcal{W}(x) < \infty$. Thus, for any $\epsilon > 0$, there exists $\tau > 0$ such that $\int_0^t e^{\vartheta z(x)}\,dx \le \lambda t + D + B + \epsilon$ for all $t > \tau$. We conclude that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t e^{\vartheta z(x)}\,dx \le \lambda.$$

Hence,

$$\overline{f}(b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{\vartheta(z(x)+\sigma T)}\,dx \le e^{\vartheta\sigma T}\lambda.$$

Since this is true for any $b \in C \cap L$ and we have that $\mu(C \cap L) = 1$, we conclude that $\mathbb{E}_\mu \overline{f} \le \lambda e^{\vartheta \sigma T}$. Hence $\mathcal{E}_{\vartheta,T}(\mu) \le \sigma + (\vartheta T)^{-1}\log\lambda = \mathcal{E}_{\vartheta,T}(w_{\gamma^*})$, which proves the theorem. □

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