



Maximally Entangled Mixed States and the Bell Inequality

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Maximally entangled mixed states and the Bell Inequality

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Recently a class of maximally entangled states has been proposed that has the maximum amount of entanglement for a given purity. We investigate how such states violate the conventional Bell inequality and discuss its implication.

Entanglement was recognized early as one of the key features of quantum mechanics[1]. The advantage offered by quantum entanglement relies on the crucial premise that it not be reproduced by any classical theory[2, 3]. Despite the fact that the possibility of quantum entanglement was acknowledged almost as soon as quantum theory was discovered, it is only in recent years that consideration has been given to finding methods to quantify it[4-6]. One of the previous techniques for investigating entanglement was the Bell inequality. The Bell inequality is known as a marker for entanglement in two qubits. If a state violates the Bell inequality then we know that entanglement is present. The reverse is well known not to be true. There are states that are entangled and do not violate such an inequality[7]. One example is the Werner state[8]. It has generally been found that it is only weakly entangled states that may not violate the Bell inequality (the Werner state is one such example). Strongly entangled states are expected to violate the inequality. Hence in this proceeding we investigate a class of states[9] that have the maximum amount of entanglement for a given mixture and the point at which they violate the Bell inequality. Do they have to be strongly entangled to violate the inequality.

Let us now define our measure of entanglement and the Bell inequality we will consider. In examining the degree of entanglement there are currently a number of measures available. These include the entanglement of distillation[4], the relative entropy of entanglement[10], but the canonical measure of entanglement is called the entanglement of formation (EOF)[4]. For an arbitrary two qubit system is simply given by[11],

$$E_F(\hat{\rho}) = h\left(\frac{1 + \sqrt{1 - \tau}}{2}\right), \quad (1)$$

where $h(x) = -x \log(x) - (1-x) \log(1-x)$ is Shannon's entropy function, and τ is the tangle[11] (concurrence squared) given by,

$$\tau = \mathcal{C}^2 = [\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}]^2, \quad (2)$$

where the λ 's are the square root of the eigenvalues in decreasing order of $\hat{\rho}\hat{\rho} = \hat{\rho} \otimes \sigma_y^A \otimes \sigma_y^B \hat{\rho}^* \otimes \sigma_y^A \otimes \sigma_y^B$. Here

$\hat{\rho}^*$ denotes the complex conjugation of $\hat{\rho}$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. For two qubits the tangle τ can be considered a measurement of entanglement and like the entanglement of formation ranges from zero for a separable state to one for a maximally entangled state. Next there are many Bell inequalities that could be investigated in this article but we will focus our attention on the *original* two qubit Bell inequality[2, 3],

$$\mathbf{B}_S = \left| \langle \hat{S}_1(\phi_1) \hat{S}_2(\phi_2) \rangle + \langle \hat{S}_1(\phi_1) \hat{S}_2(\phi_2') \rangle + \langle \hat{S}_1(\phi_1') \hat{S}_2(\phi_2) \rangle - \langle \hat{S}_1(\phi_1') \hat{S}_2(\phi_2') \rangle \right| \leq 2, \quad (3)$$

where

$$\hat{S}_i(\phi_i) = \cos \phi_i [|0\rangle\langle 0| - |1\rangle\langle 1|] + \sin \phi_i \left[e^{i\bar{\phi}_i} |0\rangle\langle 1| + e^{-i\bar{\phi}_i} |1\rangle\langle 0| \right]. \quad (4)$$

The inequality (3) is violated if $\mathbf{B}_S > 2$.

Given our measure of entanglement and the form of the Bell inequality to be investigated it is now time to specify exactly the form of the maximally entangled mixed states[9]. This state has the form

$$\hat{\rho} = \begin{pmatrix} g(\gamma) & 0 & 0 & \frac{\gamma}{2} \\ 0 & 1 - 2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\gamma}{2} & 0 & 0 & g(\gamma) \end{pmatrix}, \quad (5)$$

where,

$$g(\gamma) = \begin{cases} \gamma/2 & \gamma \geq 2/3 \\ 1/3 & \gamma < 2/3 \end{cases}, \quad (6)$$

and has been shown to have the maximal amount of entanglement for a certain degree of mixture (as measured by the linear entropy) [12], or vice versa. This state is entangled for all nonzero γ , and in fact it has been shown that the tangle simply is given by

$$\tau = \gamma^2. \quad (7)$$

For a given degree of mixture, the maximally entangled mixed state is generally significantly more entangled than the Werner state[8] at the same degree of mixture. How well does this state violate the Bell inequality? What degree of entanglement is required?

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In Fig. (1) we plot the maximum value of \mathbf{B}_S (optimizing the analyzer settings to maximize the violation) versus the degree of entanglement (as measured by the tangle) for two different classes of states. The first is the nonmaximally entangled pure state (curve a) specified by

$$|\Psi_{\text{non}}\rangle = \cos\theta|00\rangle + e^{i\chi}\sin\theta|11\rangle \quad (8)$$

and the second is our state (5). This results show very clearly that the maximally entangled mixed state and the non-maximally entangled pure state violate the Bell inequality by significantly different amounts for the same degree of entanglement. For these two different classes of entangled states there is a clear region where one of the states (the non-maximally entangled pure state) violates the Bell inequality. In fact our Bell inequality for the maximally entangled mixed state is only violated if $\tau > 0.5$ ($EOF > 0.6$) (compared with an $EOF > 0.44229$ for the Werner state). This is a significant degree of entanglement given that a Bell state has $\tau = 1.0$ ($EOF = 1.0$) and a separable state has $\tau = 0.0$ ($EOF = 0.0$).

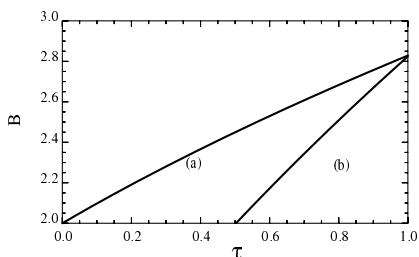


FIG. 1: Plot of the maximum violation of the spin Bell inequality versus the degree of entanglement (tangle τ) for the non-maximally entangled pure state (curve a) and the maximally entangled mixed state curve b). A violation of the spin Bell inequality is achieved when $|\mathbf{B}_S| > 2$.

The above results also tentatively indicates that the more mixture contained in a state, the higher the degree of entanglement required to violate the two qubit Bell inequality. To investigate this we will consider a modification of the maximally entangled mixed state given by

$$\hat{\rho}_m(\gamma, \xi) = (1 - \gamma)|0\rangle|1\rangle\langle 0|\langle 1| + \gamma|\Psi_{\text{non}}\rangle\langle\Psi_{\text{non}}|. \quad (9)$$

where $|\Psi_{\text{non}}\rangle$ is given by (8). This is simply a mixture of the non-maximally entangled pure state and the diagonal density matrix element $|0\rangle|1\rangle\langle 0|\langle 1|$. Choosing the parameters γ and ξ such that (9) just satisfies the Bell inequality (that is $\mathbf{B}_S = 2$) we vary the parameters γ, ξ such that we increase the degree of mixture in the system while maintaining $\mathbf{B}_S = 2$. For these γ and ξ values we then determine the degree of entanglement and mixture.

In Figure (2) we plot on the tangle-linear entropy plane, the boundary curve where $\mathbf{B}_S = 2$ for both states.

Figure (2) confirms for this state our idea that as the state becomes more mixed, more entanglement is required to violate the Bell inequality.

To summarize, in this article we have investigated the extent to which the maximally entangled mixed state violates Bell inequality. For this state a tangle $\tau = 0.54$ (an $EOF = 0.6$) is required to violate the Bell inequality.

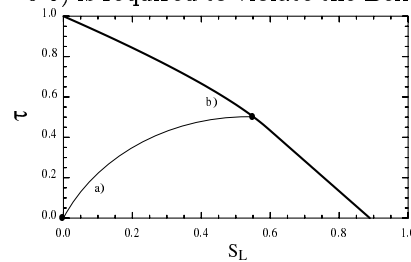


FIG. 2: Plot of the degree of entanglement versus linear entropy for (5). Curve a) traces out the curve for the state (9) where γ and ξ are chosen such that $\mathbf{B}_S = 2$.

This is a significant degree of entanglement and dispels the impression that only the weakly entangled states do not violate the Bell inequality. Our results indicate that the more mixed a system is made the more entanglement is generally required to violate the original Bell inequality to the same degree.

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- [12] The linearised entropy of a two qubit state $\hat{\rho}$ is given by

$$S_L = \frac{4}{3} \{1 - \text{Tr} [\rho^2]\} \quad (10)$$

The 4/3 normalisation for S_L ensures that for a general two qubit density matrix S_L ranges between 0 and 1.